

# Gromov-Witten invariants and Algebraic Geometry

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# Introduction

- ① Gromov-Witten invariants is an integral part of Mirror Symmetry conjecture;
- ② investigated in mathematics from algebraic geometry, symplectic geometry, representation theory, etc.;
- ③ still in its early development, after three decades;
- ④ has impacted greatly several branches in mathematics;
- ⑤ several research groups in China contributed to its development.

# Introduction

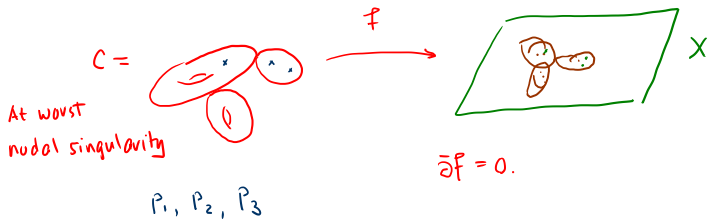
- ① Today I will approach this topic
  - from the angle of algebraic geometry;
  - based on my personal experience;
- ② GW invariants of quintics by algebraic geometry;
- ③ three examples of GW invariants techniques impact algebraic geometry;

# Introduction

$X$  a smooth projective variety,  $d \in H_2(X, \mathbb{Z})$ ,  $g, n \in \mathbb{Z}$

$$\overline{\mathcal{M}}_{g,n}(X, d) = \{f: C \rightarrow X, p_i \in C : f_*([C]) = d, f \text{ stable}\} / \sim$$

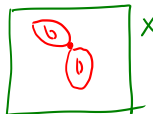
moduli of genus  $g$ ,  $n$ -pointed stable morphisms to  $X$  of class  $d$ .



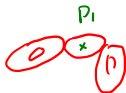
# Introduction

$f : C \rightarrow X$  stable means that  $\text{Aut}(f)$  is finite.

e.g. stable v. unstable maps:



$f$  unstable



The middle component  
contracted to a point.

$f'$  stable.

# Introduction

$\overline{\mathcal{M}}_{g,n}(X, d)$ : a DM stack, admits an obvious perfect obstruction theory, has virtual fundamental cycle

$$[\overline{\mathcal{M}}_{g,n}(X, d)]^{\text{vir}} \in A_{\nu} \overline{\mathcal{M}}_{g,n}(X, d),$$

(or  $\in H_{2\nu}(\overline{\mathcal{M}}_{g,n}(X, d), \mathbb{Q})$  as a homology class.)

$$\nu = \text{vir. dim} = (g - 1)(3 - \dim X) + d \cdot c_1(X) + n$$

- When  $X$  is a Calabi-Yau threefold, i.e.  $c_1(X) = 0$  and  $\dim X = 3$ , choose  $n = 0$ , then  $\nu = 0$ .

Gromov-Witten invariants of a smooth projective variety:

$$\langle \alpha_1, \dots, \alpha_n \rangle_X^{g,d} = \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}}} \text{ev}_1^* \alpha_1 \cdots \text{ev}_n^* \alpha_n \in \mathbb{Q}$$

where

$$\alpha_1, \dots, \alpha_n \in H^*(X, \mathbb{Q}).$$

**Problem:** *The structure of the collection of invariants*

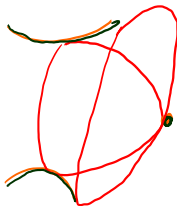
$$\{\langle \alpha_1, \dots, \alpha_n \rangle_X^{g,d}\}.$$

# Introduction

Gromov-Witten invariants is a counting problem, counting (algebraic) curves subject to constraints:

Constraints:

Curves of  
fixed class, genus



Two curves  
subject to  
the constraints.

Using stable maps is to compactify the moduli space.

# First Example: explicit formula by Candelas et. al.

Calabi-Yau threefold  $X$ :

- a smooth three-dimensional complex manifold,
- $H_1(X, \mathbb{Q}) = 0$  and  $c_1(X) = 0$ .

Example: (Fermat) quintic threefold

$$X = (x_1^5 + \cdots + x_5^5 = 0) \subset \mathbb{CP}^4.$$

# First Example

GW-invariants of Calabi-Yau threefolds reduce to

$$N_g(d) = \deg[\overline{\mathcal{M}}_g(X, d)]^{\text{vir}},$$

*↑ "X Calabi-Yau  
it is a zero cycle."*

generating function

$$F_X = \sum_{g \geq 0} \left( \sum_{d \in H_2} N_g(d) \cdot q^d \right) \lambda^{2g-2} = \sum F_{X,g} \cdot \lambda^{2g-2}.$$

*Clemens conjecture:  
finite rational curves of  
each degree.*

# First Example

Shocking: for quintic Calabi-Yau threefolds  $X$ , Candelas et. al. derived the formula (1991)

$$\tilde{C}_{ttt} = \sum_{d \geq 0} N_0(d) d^3 q^d$$

where  $\tilde{C}_{ttt}$  can be explicitly “calculated” (guessed more precisely) by performing the following:

# First Example

- 1 calculate  $C_{zzz} = \int_{\check{X}_z} \Omega_z \wedge \frac{d^3}{dz^3} \Omega_z$ ;  $\check{X}$  the mirror of  $X$ .
- 2 normalize it to  $\tilde{C}_{zzz}$ ;
- 3 make a mirror transformation

$$t(z) = \frac{1}{2\pi i} \log z + \frac{5}{2\pi i \phi_0(z)} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left( \sum_{j=n+1}^{5n} \frac{1}{j} \right) z^n$$

moduli of Calabi-Yau  
of type  $\hat{X}$ , 1-dimensional

- 4  $\tilde{C}_{ttt} = \left(\frac{dz}{dt}\right)^3 \tilde{C}_{zzz}$ ;
- 5  $\tilde{C}_{ttt} = \sum_{d \geq 0} N_0(d) d^3 q^d$ , where  $q = e^{2\pi i t}$ .

$\check{X}_z, \Omega_z$   
(3,0)-form  
on  $\check{X}_z$ .

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# First Example

Candelas et. al. derived a closed formula for  $\tilde{C}_{tty}$ ;

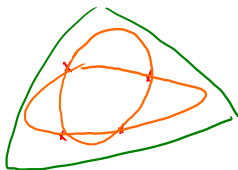
The formula is verified by Lian-Liu-Yau, Givental.

Why this computation is true remains a mystery.

## Second Example: explicit formula by Kontsevich

A classical problem in enumerative (algebraic) geometry is to enumerate the number  $N_d$  of degree  $d$  rational curves in  $\mathbb{P}^2$  subject to (the right number of) constraints.

Like passing through two points there is only one degree one rational curve in  $\mathbb{P}^2$ .



$N_d = \#$  degree  $d$  rational curves  
in  $\mathbb{P}^2$ , subject to constraints

## Second Example

Kontsevich used moduli of stable maps

$$\overline{\mathcal{M}}_{0,m}(\mathbb{P}^2, d)$$

to obtain a recursion formula, thus solving the problem (1993):

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left[ d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right].$$

# Second Example

Idea of proof: using a recursion formula,

- envisioned by Super-String theorists (the WDVV equation);
- proof using an identity of cycles

$$[\pi^{-1}(\xi_1)] = [\pi^{-1}(\xi_2)] \in A_* \overline{\mathcal{M}}_{0,m}(\mathbb{P}^2, d)$$

- by looking at the forgetful map

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,m}(\mathbb{P}^2, d) & & [\pi^{-1}(\xi_1)] \sim_{eq} [\pi^{-1}(\xi_2)] \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1 & \rightarrow & \xi_1 = \left\{ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right\} \\ & & \Downarrow \\ & & \xi_2 = \left\{ \begin{array}{c} 1 \\ 3 \\ 2 \\ 4 \end{array} \right\} \end{array}$$

# Constructing Gromov-Witten invariants

Influenced by progress via symplectic geometry in Mirror Symmetry Conjecture, the main challenge to algebraic geometers:

To construct the virtual cycles of the moduli of stable maps

$$[\overline{\mathcal{M}}_{g,n}(X, d)]^{\text{vir}} \in A_* \overline{\mathcal{M}}_{g,n}(X, d)$$

## 2. Construction of Gromov-Witten invariants

### Theorem 1 (L-Tian (96))

*Let  $X$  be a smooth projective manifold and  $d \in H_2(X, \mathbb{Z})$  be an algebraic class. Then the moduli of stable maps  $\overline{\mathcal{M}}_{g,n}(X, d)$  admits a virtual cycle*

$$[\overline{\mathcal{M}}_{g,n}(X, d)]^{\text{vir}} \in A_* \overline{\mathcal{M}}_{g,n}(X, d),$$

*constructed by using virtual normal cone based on its perfect obstruction theory. Further, the cycle is constant in the deformation class of  $X$ .*

# Construction

GW invariants is a counting problem, ...

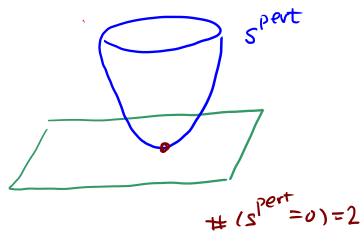
A counting when all parameters are ideal, (Sard's theorem ...)

With analysis, try to perturb the almost complex structures, ...

In algebraic geometry, the parallel topological construction is MacPherson's deformation to normal cone construction.

# Construction

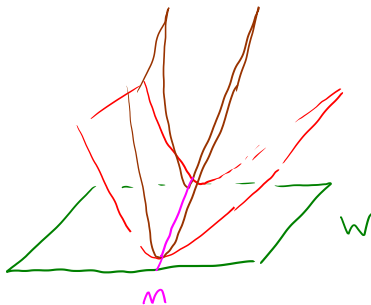
Toy model:  $M = (s = 0) \subset W$ ,  $E \rightarrow W$  v.b./smooth,  $s \in \Gamma(E)$ , ...



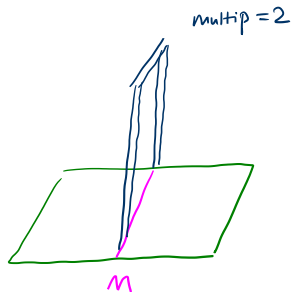
By perturbation: use  $s^{\text{pert}}$ ; counting is  $\#(s^{\text{pert}} = 0)$ .

# Construction

Same toy model:  $M = (s = 0) \subset W$ ,  $E \rightarrow W$ ,  $s \in \Gamma(E)$ , ...



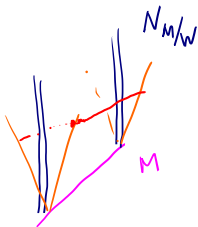
$$\lim_{t \rightarrow 0} \Gamma_{s/t} =$$



The normal cone:  $N_{M/W} := \lim_{t \rightarrow 0} \Gamma_{t^{-1}s} \subset E|_M$ , ...

# Construction

Same toy model:  $M = (s = 0) \subset W$ ,  $E \rightarrow W$ ,  $s \in \Gamma(E)$ , ...



Gysin map is  
via intersecting  
a perturbation  
of the zero section.

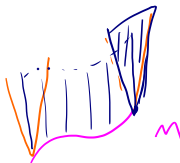
$$[M]^{\text{wt}} = 2 \cdot (pt)$$

The virtual cycle:  $[M]^{\text{vir}} = 0_E^! [N_{M/W}] \in A_* M$ . ( $0_E^!$ : Gysin map.)

# Construction

For  $M = \overline{\mathcal{M}}_{g,n}(X, d)$ , no canonical  $(s, E, W)$ , ....

Instead construct virtual normal cone  $N \subset E$  and  $E \rightarrow M$ ,



$N$  can have many components, of various shapes.

Use Gysin map:  $[\overline{\mathcal{M}}_{g,n}(X, d)]^{\text{vir}} = 0_E^![N]$ .

The contributions of this work:

- ① use normal cone construction;
- ② use virtual normal cone;
- ③ perfect obstruction theory  $\implies$  virtual normal cone.

(moduli spaces come with obstruction theories, some are perfect.)

Current State of virtual cycle construction:

- ① Use cotangent complex, cone-stack, and bundle stack (Behrend-Fantechi);
- ② Attempt to use derived algebraic geometry, for more general obstruction theories;
- ③ Symmetric obstruction theories (Behrend); ...

# Genus zero invariants of quintics

Take  $Q_5 \subset \mathbb{P}^4$  a smooth quintic Calabi-Yau threefold, say

$$Q_5 = (x_1^5 + \cdots + x_5^5 = 0) \subset \mathbb{P}^4.$$

Kontsevich's formula of genus zero GW invariants of quintics (1994):

$$N_0(d) = \deg[\overline{\mathcal{M}}_0(Q_5, d)]^{\text{vir}} \in \mathbb{Q}.$$

$$= \int_{\overline{\mathcal{M}}_0(\mathbb{P}^4, d)} c_{\text{top}}(\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)).$$

# Genus zero invariants

Quintics:  $\varphi = x_1^5 + \cdots + x_5^5$ ,  $Q_5 = (s = 0) \subset \mathbb{P}^4$ ;

Consider  $\overline{\mathcal{M}}_0(\mathbb{P}^4, d)$ ;

Universal family of  $\overline{\mathcal{M}}_0(\mathbb{P}^4, d)$ :

- 1 family of curves  $\pi : \mathcal{C} \rightarrow \mathcal{M}_0(\mathbb{P}^4, d)$ ;
- 2 universal map:  $f : \mathcal{C} \rightarrow \mathbb{P}^4$ ;

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- 2 universal map:  $f : \mathcal{C} \rightarrow \mathbb{P}^4$ ;

$$\mathcal{C} = \text{two loops} \xrightarrow{f} \mathbb{P}^4$$

$$\text{shaded region} = \overline{\mathcal{M}}_0(\mathbb{P}^4, d)$$

# Genus zero invariants

Moduli spaces  $\mathcal{M}_0(C \rightarrow \mathbb{P}^4)$  if  $\varphi \circ f = 0$ .  $\mathcal{M}_0(\mathbb{P}^4, d)$

$$\overline{\mathcal{M}}_0(Q_5, d) = (\pi_* f^* \varphi = 0) \subset \overline{\mathcal{M}}_0(\mathbb{P}^4, d);$$

compare:

$$M = (s = 0) \subset W, \quad \text{toy model.}$$

and  $s = \pi_* f^* \varphi \in \Gamma(E = \pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5))$ .

This is the “toy model”, thus

$$[\overline{\mathcal{M}}_0(X_5, d)]^{\text{vir}} = \text{Euler.class}(\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5))$$

# Genus zero invariants

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# Genus zero invariants

- 1 Kontsevich's formula can be calculated using torus localization;
- 2 GW invariants of quintic explicit expressed in huge sums;
- 3 Prove genus zero Mirror Symmetry Conjecture possible.

Lian-Liu-Yau, and Givental, independently, last (giant) step  
*proved the genus zero Mirror Symmetry Conjecture*

# Genus one invariants of quintics

For genus one invariants of  $Q_5$ , the Kontsevich like formula fails.

$$[\overline{\mathcal{M}}_1(Q_5, d)]^{\text{vir}} \neq e(R^\bullet \pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5))$$

It is due to that  $W = \overline{\mathcal{M}}_1(\mathbb{P}^4, d)$  is not smooth.

# Genus one invariants

issues are

- ①  $\overline{\mathcal{M}}_1(\mathbb{P}^4, d) = \mathcal{M}^0 \cup \mathcal{M}^1 \cup \mathcal{M}^2 \cup \mathcal{M}^3$  has four components (pretty bad);
- ②  $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)$  is regular away from the intersections (not too bad);
- ③  $\pi_* f^* \mathcal{O}(5)$  is singular (pretty bad).

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# Genus one invariants

We derived a Kontsevich like formula for genus one invariants:

Definition 2 (Li-Zinger(04))

$$N_1^{\text{red}}(d) = \langle e(\text{de-sing}), [\text{main component of } \overline{\mathcal{M}}_1(\mathbb{P}^4, d)] \rangle.$$

We proved

Theorem 3 (Li-Zinger(04))

$$N_1(d) = N_1^{\text{red}}(d) + \frac{1}{12} N_0(d).$$

# Genus one Mirror Symmetry Conjecture

- 1 Via a desingularization constructed by Vakil-Zinger (05), one can evaluate  $N_1^{\text{red}}(d)$  via torus localization, in terms of a huge sum indexed by graphs;
- 2 Via a clever combinatorics manipulation, Zinger (07) proved the genus one Mirror Symmetry Conjecture.

# Genus two or higher invariants

Continue this line of argument, ..., no progress, and discouraging.

## Theorem 4 (Graber-Harris-Starr (01))

*Let  $K$  be a field of transcendence degree 1 over  $k$ , and  $X$  is a rationally connected variety over  $K$ , then  $X$  has a  $K$ -rational point (i.e.  $X(K) \neq \emptyset$ ).*

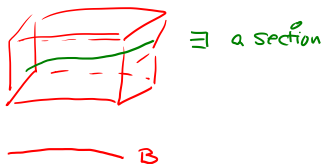
# GW techniques to Algebraic Geometry

Its algebraic geometric version

## Theorem 5 (Graber-Harris-Starr (01))

*Let  $f : X \rightarrow B$  be a non-constant map to a smooth curve  $B$ , such that the general fiber is rationally connected. Then  $f$  has a section.*

$X$  is rationally connected if any two general points  $p, q$  on  $X$  can be connected by a chain of rational curves.

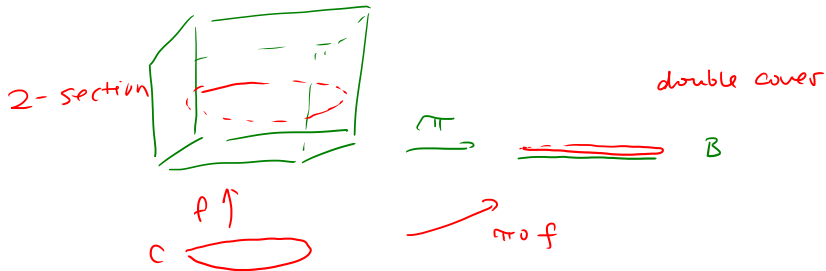


# GW techniques to Algebraic Geometry

The proof of this theorem is influenced by the GW theory:

- 1 for  $f : X \rightarrow B$ , using the moduli of stable maps

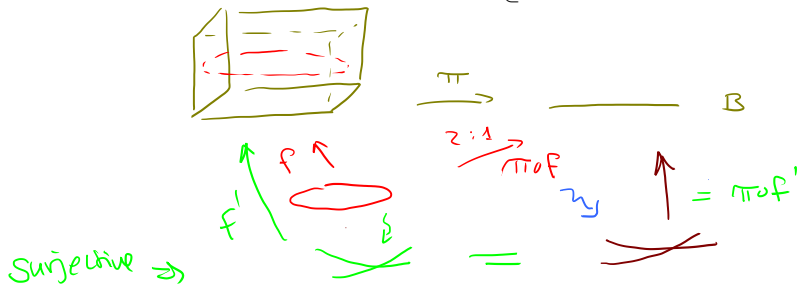
$$\overline{\mathcal{M}}_g(X, d) \longrightarrow \overline{\mathcal{M}}_g(B, d').$$



# GW techniques to Algebraic Geometry

Key is to show (for  $B = \mathbb{P}^1$ ), there is a component surjective

$$\begin{array}{ccc}
 \mathcal{W} & \xrightarrow{\quad \subset \quad} & \overline{\mathcal{M}}_g(X, d) \\
 \text{surjective} \downarrow & & \downarrow \\
 \text{the main component of } & \xrightarrow{\quad \subset \quad} & \overline{\mathcal{M}}_g(\mathbb{P}^1, d')
 \end{array}$$



# GW and AG, past and present II

Next lecture,

- ① I will explain the recent work toward higher genus invariants of quintics;
- ② I will show another example where GW technique applied to algebraic geometry.