Renormalizations and wandering Jordan curves of rational maps *

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January 21, 2015

Abstract

We realize a dynamical decomposition for a post-critically finite rational map admitting a combinatorial decomposition. We split the Riemann sphere into two completely invariant subsets. One is a subset of the Julia set consisting of uncountably many Jordan curve components with most of them being wandering. The other consists of components that are pullbacks of finitely many renormalizations, together with possibly uncountably many points. The quotient action on the decomposed pieces is encoded by a dendrite dynamical system. Independently, we introduce a surgery procedure to produce postcritically finite rational maps with wandering Jordan curves and prescribed renormalizations.

Nous réalisons une décomposition dynamique d'une fraction rationnelle postcritiquement fini admettant une dcomposition combinatoire. La sphére de Riemann sera divisée en deux sous ensembles totalement invariants. L'un est un sous ensemble de Julia avec un nombre non-dénombrable de composantes connexes, toutes courbes de Jordan, majoritairement errantes; l'autre posséde comme composantes connexes les preimages d'un nombre fini de renormalizations, avec éventuellement un nombre non-dénombrable de points. La dynamique induite sur l'espace quotient de ces composantes sera encodée par une dendrite dynamique. Indépendamment, nous introduisons une chirurgie de combinaison afin d'obtenir des fractions rationnelles ayant des courbes de Jordan errantes ainsi que des renormalizations prescrites.

1 Introduction

A rational map of one complex variable acts on the Riemann sphere and generates a dynamical system by iteration. One general principal in analyzing the iterated dynamical system is to decompose it into invariant sub-systems such that some of them have simpler dynamics. The Riemann sphere is canonically decomposed into the disjoint union of the Fatou set and the Julia set, defined by whether the iterated sequence forms a normal

*2010 Mathematics Subject Classification: 37F10, 37F20

†supported by the NSFC grant no. 11125106 and 11471317.

‡supported by the NSFC grant no. 11101402, 11471317 and 11231009.

§supported by ANR LAMBDA and Geanpyl Pays de la Loire

family in a neighborhood of the point. While the dynamics on the Fatou set is relatively tame, the dynamics on the Julia set is wild and chaotic.

It may happen that on a periodic subset of the Julia set the first return map behaves like another rational map on its own Julia set. The dynamics on the periodic subset is usually called a *renormalization*.

A continuum is a compact connected subset of the sphere with uncountably many points. It may also happen that a continuum in the Julia set wanders around and form an orbit of pairwise disjoint continua. The continuum is usually called a *wandering continuum*.

Our main contribution in this work is to show that the dynamics of a post-critically finite rational map can be decomposed into two completely invariant parts, under a purely combinatorial condition, namely the existence of a Cantor multicurve. In this case, the decomposed pieces have a fairly simple dynamical description: on one side, we get mostly wandering continua, and on the other side, we get mainly renormalizations. Furthermore, the induced action on the quotient space can be effectively encoded by an expanding map on a metric dendrite.

A Cantor multicurve is a natural refinement of a stable multicurves introduced by Thurston. Many existing decomposition results can be described as cutting along stable multicurves. Let us give an account of some of them.

For polynomials with a disconnected Julia set such that only finitely many filled Julia components contain post-critical points, one can extract a stable multicurve from a finite pullback of an equipotential curve in the basin of infinity. This stable multicurve forms a puzzle under pullback. The pioneering work of Branner-Hubbard on cubic polynomials uses a tableau to analyze the puzzle, and subsequently shows that the filled Julia set consists of components that are either pullbacks of a renormalization or points [4]. Later on DeMarco and Pilgrim use tableau and an infinite tree to encode the combination of these points and renormalization copies [13].

For polynomials with connected Julia sets there are similar decomposition results. Often there are more than one periodic external rays landing at a common periodic point. These periodic external rays cut the Julia set into pieces. Together with equipotential curves, these rays play the role of a stable multicurve and form again puzzles (called Yoccoz puzzles). As before the return dynamics on a periodic puzzle piece is a renormalization. And in various expanding cases, the remaining part of the Julia set, after extracting the renormalized copies and their pullbacks, consists of uncountable many point components. See for example [10, 20, 37], among others.

Let us turn now to non-polynomial rational maps. If a sub-hyperbolic rational map has a disconnected Julia set, one can also extract a stable multicurve in the multiply-connected Fatou domains and obtain a canonical decomposition of the Julia set, just as what one did in the basin of infinity of a polynomial with disconnected Julia set. Detailed analysis can be found in [8, 26].

The situation of a rational map with a connected Julia set is actually much harder: there is no multicurves in the Fatou set that separate the Julia set, and there is no completely invariant Fatou domains whose rays cut apart the Julia set. In some particular cases such as the Newton's method for cubic polynomials, or quadratic rational maps with a 2-periodic critical point, rays from distinct Fatou basins may joint together and form a cutting (see e.g. [19, 29, 33]). But this is by far the general case. For instance many Julia sets are homeomorphic to a Sierpinski carpet, where any two distinct Fatou components have disjoint closures.

A rational map is said to be post-critically finite if its critical orbits contain only finitely many points. For such a map, the Julia set is automatically connected. Conversely, any sub-hyperbolic rational map with a connected Julia set is quasi-conformally conjugate to a post-critically finite in a neighborhood of the Julia set. Our setting in the present work is precisely about post-critically finite rational maps. And we aim to develop a general decomposition procedure without invoking any particular topology of the Julia sets.

There is already a well-known combinatorial decomposition for these maps, in case that the map has a fully invariant Jordan curve up to homotopy. Cutting along this curve will decompose the rational map (or its second iteration if necessary) into two polynomials. The rational map is called the mating of the two polynomials. See for example [31, 32]. However, the decomposed polynomials can not be considered as renormalizations in the usual sense, as in general none of the small Julia sets is embedded in the original big Julia set. Many tips of a small Julia set are glued together under mating [30].

In this work we will establish a new type of decomposition procedure for post-critically finite rational maps. The first challenge is to find a natural class of multicurves to cut along. Our key concept, *Cantor multicurves*, is introduced precisely for this purpose. Contrary to the equator of a mating, a Cantor multicurve is a multicurve whose consecutive pullbacks will generate a strictly increasing number of curves in each homotopy class. See §2 for the definition.

The stability of multicurves is only measured up to homotopy. It is thus impossible to literally cut along a stable multicurve to obtain exact invariant pieces. The crucial step in our study is to promote a Cantor multicurve to a multi-annulus such that it is exactly invariant in certain sense. We will call such a dynamical system an *exact annular system* (see §3 for the definition)[1]. It will play the role of multiply-connected Fatou components in the disconnected case, and will allow us to decompose the Julia set into pairwise disjoint pieces.

Cantor multicurves and exact annular systems appear naturally in the study of rational maps with disconnected Julia sets, starting from McMullen's example of Cantor set of circles (see e.g. [14, 22, 26], among others). In the connected Julia set case, these concepts appear also in the flexible Lattès examples, and in Haissinsky-Pilgrim's example with 4 postcritical points [15]. However they have never been applied to general post-critically finite rational maps.

Actually our construction of the exact annular system from a Cantor multicurve is somewhat indirect, after several failed attempts with more direct approaches. We were led to modify the rational map to a branched covering in the Thurston equivalence class such that it has a topological exact annular system. Then applying a theorem of Rees and Shishikura, we obtain a semi-conjugacy from the branched covering to the rational map. Finally a careful analysis of the semi-conjugacy shows that the exact annular system for the branched covering descends to one for the rational map.

Once an exact annular system is found, we are naturally led to analyze the induced decomposition. The first step is to study the Julia set of the exact annular system. This is actually quite simple since it is expanding, as a sub-system of a post-critically finite rational map. As in the case of rational maps with disconnected Julia sets we will prove

[1] This is to be compared to the monograph [25], where stable multicurves are promoted to invariant Jordan curves in the setting of branched coverings.

that each component of its Julia set is a Jordan curve, there are uncountably many of them and all but countably many of them are wandering.

The next step is to analyze the complement of the grand orbit of the Julia set of the exact annular system. It has zero or uncountably many point components, and countably many continuum components which are pullbacks of finitely many renormalizations.

Finally, one should encode the relation between these decomposed components, namely study the induced action on the quotient space. We will show that this action is an expanding dynamical system on a dendrite. This completes the decomposition study along a Cantor multicurve of post-critically finite rational maps.

We will present further a result about wandering continua. For a post-critically finite rational map, the existence of a non-simply connected wandering continuum is equivalent to the existence of a Cantor multicurve. Moreover, the wandering continuum must be a Jordan curve.

It is known that for a polynomial with connected Julia set and without irrational indifferent cycles, it has no wandering continua if and only if its Julia set is locally connected. See [3, 16, 17, 18, 36]. For rational maps, besides known examples with disconnected Julia sets, Lattés example and Haissinky-Pilgrim's example with 4 post-critical points [15], we are not aware of a sufficient and necessary condition for the existence of a non-simply connected wandering continuum as we have developed here.

Our final, and yet somewhat independent task is to construct post-critically finite rational maps with Cantor multicurves and prescribed renormalizations. For this purpose we will introduce a new surgery procedure which we call *foldings*, to construct a branched covering from polynomials. The resulting map has a Cantor multicurve consisting of a single curve. We will show that under certain conditions the branched covering is Thurston equivalent to a rational map. Consequently, this rational map has a Cantor multicurve and hence wandering Jordan curves, and with a renormalization which is the polynomials we started with.

Perspectives. Here are some remarks and several problems related to this work.

(1) If a renormalization piece in our decomposition admits again a Cantor multicurve, one can make a further decomposition. Or one can join it to the original one and then make only decompositions for Cantor multicurves that are maximal under inclusions. In this way we can declare that the renormalizations are post-critically finite rational maps without Cantor multicurves. Maximal Cantor multicurves might not be unique. Therefore our decomposition needs not to be canonical. This phenomenon occurs also in realizing rational maps as matings of polynomials.

(2) The existence of simply-connected wandering continua is a very interesting problem and remains largely open, except for flexible Lattès maps, see for example [7]. We suspect that these are the only exceptions. One possible strategy to address the problem is to prove first that, in the presence of a Cantor multicurve, a wandering continuum must lie in a renormalization, and then rule out the latter possibility.

(3) The examples we constructed have a quotient dendrite that is the simplest case, namely a segment. We wish to construct examples with branching points in the future. The combinatorial construction is quite easy. The hard part is to show its rational realization. There are some techniques around that might be to apply. For example the iterated monodromy groups developed by Bartholdi-Nekrashevych [5] and the direct criterion developed by Dylan Thurston [35].

(4) Our final example provides a rational map with a Sierpinski carpet Julia set and a Cantor multicurve. One may use our examples to test McMullen's conjecture: the hyperbolic component of rational maps whose Julia sets are Sierpinski carpets has compact closure in the parameter space.

(5) Expanding Thurston type branched coverings have attracted many attentions in the recent years. We hope that some of the techniques developed in this work can be adapted to these expanding maps as well.

Definitions and statements. The following notations and definitions will be used throughout this paper.

• Let $U, V \subset \widehat{\mathbb{C}}$ be open sets in the Riemann sphere. We denote by $U \Subset V$ if $\overline{U} \subset V$.

• Let $A \subset \widehat{\mathbb{C}}$ be an annulus and $E \subset A$ be a connected open or closed set. We say that E is contained in A **essentially** if E separates the boundary ∂A.

• A continuum $E \subset \widehat{\mathbb{C}}$ is called n**-connected** with $n \in \mathbb{N} \cup \{\infty\}$ if $\widehat{\mathbb{C}} \backslash E$ has exactly n components.

• Let f be a rational map. Denote by $\mathcal{J}_f$ the **Julia set** of f and $\mathcal{F}_f$ the **Fatou set** of f. Refer to [2, 6, 23, 24] for the definitions and basic properties. The **post-critical set** of f is denoted by $\mathcal{P}_f$, refer to §2.1 for its definition.

• By **a multi-annulus** we mean a finite disjoint union of open annuli in $\widehat{\mathbb{C}}$ with finite modulus.

Definition 1. *Let $\mathcal{A}^1, \mathcal{A} \subset \widehat{\mathbb{C}}$ be two multi-annuli such that each component of $\mathcal{A}^1$ is contained in a component of $\mathcal{A}$ essentially. A map $g : \mathcal{A}^1 \to \mathcal{A}$ is called an* **annular system** *if*

(1) for each component A^1 of $\mathcal{A}^1$, its image $g(A^1)$ is a component of $\mathcal{A}$ and the map $g : A^1 \to g(A^1)$ is a holomorphic covering;

(2) there is an integer $n \geq 1$ such that for each component A of $\mathcal{A}$, the set $g^{-n}(\mathcal{A}) \cap A$ is non-empty and disconnected.

The **Julia set** *of g is defined by $\mathcal{J}_g := \bigcap_{n\geq 0} g^{-n}(\mathcal{A})$. An annular system $g : \mathcal{A}^1 \to \mathcal{A}$ is called* **proper** *if $\mathcal{A}^1 \Subset \mathcal{A}$; or* **exact** *if for every component A of $\mathcal{A}$, each of the two components of ∂A is also a component of $\partial(A \cap \mathcal{A}^1)$.*

Convention. Let f be a post-critically finite rational map. We say that an annulus $A \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ is homotopic rel $\mathcal{P}_f$ to a Jordan curve γ (or an annulus A') in $\widehat{\mathbb{C}} \backslash \mathcal{P}_f$ if essential Jordan curves in A are homotopic to γ (or essential curves in A') rel $\mathcal{P}_f$; and a multi-annulus $\mathcal{A}$ is homotopic rel $\mathcal{P}_f$ to a multicurve Γ (or a multi-annulus $\mathcal{A}'$) if each component of $\mathcal{A}$ is homotopic to a curve in Γ (or a component of $\mathcal{A}'$)) rel $\mathcal{P}_f$ and each curve in Γ (or each component of $\mathcal{A}'$) is homotopic to a component of $\mathcal{A}$.

Here are the main statements that we shall prove:

Theorem 1.1. (from a Cantor multicurve to an annular system) *Let f be a post-critically finite rational map with a Cantor multicurve Γ. There exists a unique multi-annulus $\mathcal{A} \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ homotopic rel $\mathcal{P}_f$ to Γ such that $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \to \mathcal{A}$ is an exact annular system, where $\mathcal{A}^1$ is the union of components of $f^{-1}(\mathcal{A})$ that are homotopic rel $\mathcal{P}_f$ to curves in Γ. Moreover, $\mathcal{J}_g \subset \mathcal{J}_f$, there are uncountably many components in $\mathcal{J}_g$, each is a Jordan curve. All but countably many of these components are wandering.*

Theorem 1.2. (from a wandering continuum to a Cantor multicurve) *Let f be a post-critically finite rational map. If K is a non-simply connected wandering continuum* *in $\mathcal{J}_f$, then K and all its forward iterates are Jordan curves and their homotopy classes rel. $\mathcal{P}_f$ form a Cantor multicurve.*

A connected subset $E \subset \widehat{\mathbb{C}}$ is called **of simple type** (w.r.t. $\mathcal{P}_f$) if there exists either a simply-connected domain $U \subset \widehat{\mathbb{C}}$ such that $E \subset U$ and U contains at most one point in $\mathcal{P}_f$, or an annulus $A \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ such that $E \subset A$; and **of complex type** (w.r.t. $\mathcal{P}_f$) otherwise.

Theorem 1.3. (renormalization) *Let f be a post-critically rational map with a stable Cantor multicurve Γ. Denote by $\mathcal{J}(\Gamma)$ the union of the grand orbit of the Julia set of the annular sub-system derived from Theorem 1.1. Set $\mathcal{K}(\Gamma) = \widehat{\mathbb{C}} \backslash \mathcal{J}(\Gamma)$. Then every component of $\mathcal{K}(\Gamma)$ is either a single point or a continuum that is eventually periodic. There are only finitely many periodic continuum components and each of them is either the closure of a quasi-disk or a complex type continuum. The former is the closure of a periodic Fatou domain, while the latter is the filled Julia set of a renormalization. There are at most $\#\Gamma + 1$ renormalizations.*

Theorem 1.4. (coding) *Let f be a post-critically rational map with a stable Cantor multicurve Γ. There exist an expanding finite dendrite map $\tau : \mathcal{T} \to \mathcal{T}$ and a continuous semi-conjugacy Θ from f to τ such that for each point $t \in \mathcal{T}$, the fiber $\Theta^{-1}(t)$ is a component of either $\mathcal{J}(\Gamma)$ or $\mathcal{K}(\Gamma)$.*

Refer to §5 and §6 for the definitions of renormalization and finite dendrite maps.

Outline of the paper. This paper is organized as follows. In §2, we recall Thurston's theory and give the definition of a Cantor multicurve. Some equivalent conditions in the irreducible case are given here. In §3, we introduce the notion of exact annular systems and show that every component of their Julia set is a Jordan curve if they are expanding. Theorem 1.1 is proved in §4. In §5, we will study the decomposition pieces and prove Theorem 1.3. In §6, we introduce the definition of finite dendrite maps and prove Theorem 1.4. Theorem 1.2 is proved in §7. Our construction of rational maps with Cantor multicurves and prescribed renormalizations is contained in the final section §8. Precise statements will be given there.

2 Multicurves and Cantor multicurves

In this section, we will recall Thurston's characterization theorem, introduce the notion of Cantor multicurves, and establish some equivalent conditions.

Let F be a branched covering of the Riemann sphere $\widehat{\mathbb{C}}$. We always assume $\deg F \geq 2$ in this paper. Denote by Ω_F the set of critical points of F. The **post-critical set** of F is defined by

$$\mathcal{P}_F = \overline{\bigcup_{n \geq 1} F^n(\Omega_F)}.$$

The map F is called **post-critically finite** if $\mathcal{P}_F$ is finite.

A Jordan curve γ in $\widehat{\mathbb{C}} \backslash \mathcal{P}_F$ is **null-homotopic** (resp. **peripheral**) if one of its complementary components contains zero (resp. one) point of $\mathcal{P}_F$; or is **essential** otherwise, i.e. if each of its two complementary components contains at least two points of $\mathcal{P}_F$.

A **multicurve** Γ is a non-empty and finite collection of disjoint Jordan curves in $\widehat{\mathbb{C}}\setminus\mathcal{P}_F$, each essential and no two homotopic rel $\mathcal{P}_F$. We will say that Γ is **stable** if each essential curve in $F^{-1}(\beta)$ for $\beta \in \Gamma$ is homotopic rel $\mathcal{P}_F$ to a curve in Γ; and **pre-stable** if each curve $\gamma \in \Gamma$ is homotopic rel $\mathcal{P}_F$ to a curve in $F^{-1}(\beta)$ for some curve $\beta \in \Gamma$. A pre-stable multicurve Γ is called **irreducible** if for each pair $(\gamma, \beta) \in \Gamma \times \Gamma$, there exists a sequence $\{\delta_0 = \gamma, \delta_1, \cdots, \delta_n = \beta\}$ of curves in Γ such that $F^{-1}(\delta_k)$ has a component homotopic to δ_{k-1} rel $\mathcal{P}_F$ for $1 \le k \le n$.

Let Γ be a multicurve. Its **transition matrix** $M_\Gamma = (a_{\gamma\beta})$ is defined by:

$$a_{\gamma\beta} = \sum_{\alpha} \frac{1}{\deg(f : \alpha \to \beta)},$$

where the summation is taken over components α of $F^{-1}(\beta)$ which are homotopic to γ rel $\mathcal{P}_F$. Denote by λ_Γ the leading eigenvalue of its transition matrix M_Γ. A stable multicurve Γ is called a **Thurston obstruction** if $\lambda_\Gamma \ge 1$.

Two post-critically finite branched coverings F and G are called **Thurston equivalent** if there is a pair of homeomorphisms $(\phi, \psi) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that ϕ is isotopic to ψ rel $\mathcal{P}_F$ and $\phi \circ F \circ \psi^{-1} = G$.

Theorem 2.1. (Thurston's characterization theorem) *Let F be a post-critically finite branched covering of $\widehat{\mathbb{C}}$ with hyperbolic orbifold. Then F is Thurston equivalent to a rational map f if and only if F has no Thurston obstruction. Moreover, the rational map f is unique up to holomorphic conjugation.*

Refer to [12] or [23] for the definition of hyperbolic orbifold.

Lemma 2.2. *Let F be a branched covering of $\widehat{\mathbb{C}}$. For any pre-stable multicurve Γ_0, there is a stable and pre-stable multicurve Γ such that $\Gamma \supset \Gamma_0$ and hence $\lambda_{\Gamma_0} \le \lambda_\Gamma$. Conversely, for any stable multicurve Γ with $\lambda_\Gamma > 0$, there is an irreducible multicurve $\Gamma_0 \subset \Gamma$ such that $\lambda_{\Gamma_0} = \lambda_\Gamma$.*

Refer to [23] for the second part of the lemma. We only prove the first part.

Proof. Let $\tilde{\Gamma}_n$ be the collection of essential curves in $F^{-n}(\Gamma_0)$ for $n \ge 1$. Let Γ_n be a sub-collection of $\tilde{\Gamma}_n$ such that no two curves in Γ_n are homotopic rel $\mathcal{P}_F$ and any curve in $\tilde{\Gamma}_n$ is homotopic rel $\mathcal{P}_F$ to a curve in Γ_n. Then Γ_n is a pre-stable multicurve and each curve in Γ_n is homotopic to a curve in Γ_{n+1} for $n \ge 1$. Thus $\#\Gamma_n \le \#\Gamma_{n+1}$. Since for any multicurve Γ, $\#\Gamma \le \#\mathcal{P}_F - 3$, there is an integer $N \ge 0$ such that $\#\Gamma_N = \#\Gamma_{N+1}$. Thus Γ_N is a stable and pre-stable multicurve. □

Convention. Let Γ be a collection of curves in $\widehat{\mathbb{C}}$, we also use Γ to denote the union of curves in Γ as a subset of $\widehat{\mathbb{C}}$ if there is no confusion.

Let Γ be a multicurve of F. For each $\gamma \in \Gamma$, denote by $\Gamma(1, \gamma)$ the collection of curves in $F^{-1}(\Gamma)$ homotopic rel $\mathcal{P}_F$ to γ and $\Gamma(1, \Gamma) := \bigcup_{\gamma\in\Gamma} \Gamma(1, \gamma)$. Inductively, for $n \ge 1$, denote by $\Gamma(n+1, \gamma)$ the collection of curves in $F^{-1}(\Gamma(n, \Gamma))$ homotopic rel $\mathcal{P}_F$ to γ and $\Gamma(n+1, \Gamma) := \bigcup_{\gamma\in\Gamma} \Gamma(n+1, \gamma)$. Notice that $\Gamma(n, \Gamma)$ is contained in, but may not be equal to, the collection of curves in $F^{-n}(\Gamma)$ homotopic rel $\mathcal{P}_F$ to curves in Γ. Denote by

$$\kappa_n(\gamma) = \#\Gamma(n, \gamma) \text{ for each } \gamma \in \Gamma.$$

Definition 2. *A multicurve Γ is called a* **Cantor multicurve** *if it is pre-stable and $\kappa_n(\gamma) \to \infty$ as $n \to \infty$ for all $\gamma \in \Gamma$.*

A stable Cantor multicurve is in particular both stable and pre-stable. In the classical construction of mating of polynomials, there is a Jordan curve whose pre-image is a single curve homotopic to itself rel the post-critical set. In this case the multicurve consisting of this single curve is not a Cantor multicurve. It is quite easy to give examples of maps without Cantor multicurves, for instance topological polynomials (branched coverings with a totally invariant point). Concrete examples of rational maps with Cantor multicurves will be constructed in §8.

Lemma 2.3. *Suppose that Γ is an irreducible multicurve. The following statements are equivalent:*
(1) $\#\Gamma(1,\Gamma) > \#\Gamma$.
(2) $\kappa_1(\gamma) \geq 2$ for some $\gamma \in \Gamma$.
(3) $\kappa_n(\gamma) \to \infty$ for some $\gamma \in \Gamma$.
(4) $\kappa_n(\gamma) \to \infty$ for all $\gamma \in \Gamma$, i.e., Γ is a Cantor multicurve.
(5) There is a curve $\beta \in \Gamma$ such that $F^{-1}(\beta)$ has at least two curves in $\Gamma(1,\Gamma)$.

Proof. (1) $\iff$ (2): Since Γ is pre-stable, $\Gamma(1,\gamma)$ is non-empty for each $\gamma \in \Gamma$. Thus $\#\Gamma(1,\Gamma) > \#\Gamma$ if and only if $\kappa_1(\gamma) \geq 2$ for some $\gamma \in \Gamma$.

(1) $\iff$ (3): Since Γ is irreducible, $F^{-1}(\gamma)$ has at least one curve contained in $\Gamma(1,\Gamma)$ for each $\gamma \in \Gamma$. Thus if $\#\Gamma(1,\Gamma) > \#\Gamma$, then $\#\Gamma(n+1,\Gamma) > \#\Gamma(n,\Gamma)$ for all $n \geq 1$. So $\#\Gamma(n,\Gamma) \to \infty$ as $n \to \infty$. Therefore $\kappa_n(\gamma) \to \infty$ for some $\gamma \in \Gamma$. Conversely, if $\#\Gamma(1,\Gamma) = \#\Gamma$, then $\#\Gamma(n+1,\Gamma) = \#\Gamma(n,\Gamma)$ for all $n \geq 1$. Therefore $\kappa_n(\gamma) = 1$ for all $\gamma \in \Gamma$ and $n \geq 1$.

(3) $\iff$ (4): Since Γ is irreducible, for each pair $(\gamma,\beta) \in \Gamma \times \Gamma$, there is an integer $n \geq 1$ such that $F^{-n}(\beta)$ has a component δ homotopic to γ rel $\mathcal{P}_F$ and $F^k(\delta)$ is homotopic to a curve in Γ for $1 \leq k < n$. Therefore $\delta \in \Gamma(n,\gamma)$ and hence $\kappa_{n+k}(\gamma) \geq \kappa_n(\beta)$. So $\kappa_n(\gamma) \to \infty$ if $\kappa_n(\beta) \to \infty$.

(1) $\iff$ (5): Since Γ is irreducible, $F^{-1}(\gamma)$ has at least one curve contained in $\Gamma(1,\Gamma)$ for each $\gamma \in \Gamma$. Therefore $\#\Gamma(1,\Gamma) > \#\Gamma$ if and only if there is a curve $\beta \in \Gamma$ such that $F^{-1}(\beta)$ has at least two distinct curves contained in $\Gamma(1,\Gamma)$. □

Let $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ be a multicurve of F. Its **reduced transition matrix** $M_{r,\Gamma} = (b_{ij})$ is define by $b_{ij} = k$ if there are k components of $F^{-1}(\gamma_j)$ homotopic to γ_i rel $\mathcal{P}_F$. This definition was introduced by Shishikura.

Lemma 2.4. *Let Γ be a pre-stable multicurve of F. Then the leading eigenvalue of its reduced transition matrix satisfies that $\lambda(M_{r,\Gamma}) \geq 1$. Moreover, $\lambda(M_{r,\Gamma}) > 1$ if Γ is a Cantor multicurve. Conversely, if Γ is irreducible and $\lambda(M_{r,\Gamma}) > 1$, then Γ is a Cantor multicurve.*

Proof. Note that $M_{r,\Gamma}\mathbf{v} \geq \mathbf{v}$ for the vector $\mathbf{v} = (1,\cdots,1)$ since Γ is pre-stable. Thus $\lambda(M_{r,\Gamma}) \geq 1$ by Lemma A.1 in [8]. If Γ is a Cantor multicurve, then there exists an integer $n \geq 1$ such that $M_{r,\Gamma}^n\mathbf{v} \geq 2\mathbf{v}$. Thus $\lambda(M_{r,\Gamma})^n = \lambda(M_{r,\Gamma}^n) > 1$. Conversely, if Γ is irreducible and $\lambda(M_{r,\Gamma}) > 1$, then there exists at least one column of the matrix such that the summation of the entries of this column is bigger than one. Thus Γ is a Cantor multicurve by Lemma 2.3 (2). □

3 Annular systems

In this section we will show that every component of the Julia set of an expanding exact annular system is a Jordan curve.

Let $\mathcal{A}^1, \mathcal{A} \subset \widehat{\mathbb{C}}$ be two multi-annuli such that each component of $\mathcal{A}^1$ is contained in a component of $\mathcal{A}$ essentially. Recall that a map $g : \mathcal{A}^1 \to \mathcal{A}$ is an annular system if

(1) for each component A^1 of $\mathcal{A}^1$, its image $g(A^1)$ is a component of $\mathcal{A}$ and the map $g : A^1 \to g(A^1)$ is a holomorphic covering;

(2) there is an integer $n \geq 1$ such that for each component A of $\mathcal{A}$, the set $g^{-n}(\mathcal{A}) \cap A$ is non-empty and disconnected.

The Julia set of g is defined by $\mathcal{J}_g := \bigcap_{n\geq 0} g^{-n}(\mathcal{A})$. An annular system $g : \mathcal{A}^1 \to \mathcal{A}$ is proper if $\mathcal{A}^1 \Subset \mathcal{A}$; or exact if for every component A of $\mathcal{A}$, each of the two components of ∂A is also a component of $\partial(A \cap \mathcal{A}^1)$.

Remark. The definition of the Julia set $\mathcal{J}_g$ of an annular system g is misleading. At first, $\mathcal{J}_g$ need not to be compact. Secondly, $g^{-1}(\mathcal{J}_g) = \mathcal{J}_g$ and $g(\mathcal{J}_g) \subset \mathcal{J}_g$ from the definition but $g(\mathcal{J}_g)$ need not to be equal to $\mathcal{J}_g$ since we do not require the map g to be onto.

3.1 Basic properties

Proposition 3.1. *Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an annular system. There is an integer $N \geq 1$ such that $\deg(g^N|_A) \geq 2$ for each component A of $g^{-N}(\mathcal{A})$.*

Proof. Let $m \geq 1$ be the number of components of $\mathcal{A}$. By contradiction we assume that there is a component A of $g^{-m}(\mathcal{A})$ such that $\deg(g^m|_A) = 1$. There exist integers $0 \leq k < k+p \leq m$ such that both $g^k(A)$ and $g^{k+p}(A)$ are contained in the same component A^0 of $\mathcal{A}$. So $g^p(g^k(A)) \subset A^0$. Let $A^p \subset A^0$ be the component of $g^{-p}(A^0)$ containing $g^k(A)$. Since $g^p : A^p \to A^0$ is a covering between annuli and $g^k(A)$ is contained essentially in A^p, we have

$$\deg\left(g^p : A^p \to A^0\right) = \deg\left(g^p : g^k(A) \to g^{k+p}(A)\right) \leq \deg(g^m|_A) = 1.$$

Thus the moduli of the annuli A^p and A^0 are equal and hence $A^p = A^0$. It follows that $A^0 = g^p(A^0)$. Therefore $A^0 \cap g^{-np}(\mathcal{A}) = A^0$ for all integers $n \geq 1$. Since $g^{-n}(\mathcal{A}) \subset g^{-n+1}(\mathcal{A})$ for all $n \geq 1$, we conclude that $A^0 \cap g^{-n}(\mathcal{A}) = A^0$ for all $n \geq 1$. This contradicts the condition that $A^0 \cap g^{-n}(\mathcal{A})$ is disconnected for some $n \geq 1$. So $\deg(g^m|_A) \geq 2$ for each component A of $g^{-m}(\mathcal{A})$. $\square$

Proposition 3.2. *Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an exact annular system. Let $\{A^n\}$ be a nested sequence of annuli of $g^{-n}(\mathcal{A})$, i.e. the annulus A^n is a component of $g^{-n}(\mathcal{A})$ and $A^{n+1} \subset A^n$. Then either $\bigcap_{n\geq 1} A^n = \emptyset$ or for every $n \geq 1$, there is an integer $m > n$ such that $A^m \Subset A^n$.*

Proof. By exactness, either for any $n \geq 1$, there is an integer $m > n$ such that $A^m \Subset A^n$, or there are an integer $N \geq 1$ and a component L of ∂A^N such that $L \subset \partial A^n$ for $n \geq N$.

We only need to show that $\bigcap_{n\geq 1} A^n = \emptyset$ in the latter case. Since $\mathcal{A}$ has only finitely many components, there are a component B^0 of $\mathcal{A}$ and integers $i > j > k \geq N$ such that $g^i(A^i) = g^j(A^j) = g^k(A^k) = B^0$. As B^0 has exactly two boundary components, there are a boundary component L' of B^0, and two of them, say i and j, such that $g^i(L) = g^j(L) = L'$ (This formula means that as z tends to L in A^i, both $g^i(z)$ and $g^j(z)$ tends to L').

Denote by $B^n = g^j(A^{n+j})$ for $n \geq 0$, then $\{B^n\}$ is a nested sequence of annuli which have a common boundary component L'. Moreover, $g^p(B^p) = B^0$ and $g^p(L') = L'$ for $p = i - j$. It follows that $g^p(B^{np}) = B^{(n-1)p}$ for $n \geq 1$. Note that $B^p \neq B^0$. Otherwise $B^n = B^0$ for all $n \geq 1$ and thus contradicts the condition that $B^0 \cap g^{-n}(\mathcal{A})$ is disconnected for some $n \geq 1$.

Let U be the component of $\widehat{\mathbb{C}} \setminus L'$ containing B^0 and ϕ be a conformal map from U onto the unit disk $\mathbb{D}$. Then $h := \phi \circ g^p \circ \phi^{-1}$ is a holomorphic covering from $\phi(B^p)$ to $\phi(B^0)$, which can be extended continuously to the unit circle. By the symmetric extension principle, h can be extended to a holomorphic covering map from the annulus V_1 to V, where V_1, V are the unions of $\phi(B^p)$, $\phi(B^0)$ with its reflection and the unit circle, respectively. Since $V_1 \Subset V$, h is expanding under the hyperbolic metric of V. So $\bigcap_{n>0} h^{-n}(V) = \partial\mathbb{D}$ and hence $\bigcap_{n>0} h^{-n}(\phi(B^0)) = \emptyset$. Note that $\phi(B^{np}) = h^{-n}(\phi(B^0))$. Therefore $\bigcap_{n>0} B^{np} = \emptyset$ and hence $\bigcap_{n>0} A^n = \emptyset$. $\square$

Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an annular system and K be a connected component of $\mathcal{J}_g$. Then for each $n \geq 0$, there is a unique component of $g^{-n}(\mathcal{A})$, denoted by $A^n(K)$, such that $K \subset A^n(K)$. Consequently, $K \subset \bigcap_{n\geq 1} A^n(K)$.

Proposition 3.3. *Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an exact annular system.*

(1) For any component K of $\mathcal{J}_g$, K is a 2-connected continuum contained essentially in each $A^n(K)$ and $K = \bigcap_{n\geq 1} A^n(K)$.

(2) For each component A of $\mathcal{A}$ and any point $z \in A$, there exist components K_1, K_2 of $\mathcal{J}_g \cap A$ such that the annulus bounded by K_1 and K_2 contains the point z.

Proof. (1) For any $n \geq 0$, there is an integer $m > n$ such that $A^m(K) \Subset A^n(K)$ by Proposition 3.2. Since $A^{n+1}(K)$ is contained essentially in $A^n(K)$ for every $n \geq 0$, $\bigcap_{n\geq 0} A^n(K)$ is a 2-connected continuum contained essentially in each $A^n(K)$. By definition it is contained in $\mathcal{J}_g$ and hence is equal to K.

(2) Let A_1^n, $A_2^n \subset A$ be the components of $g^{-n}(\mathcal{A})$ such that they share a common boundary component with A. Then $\bigcap_{n\geq 0}(A_1^n \cup A_2^n) = \emptyset$ by Proposition 3.2. Thus there exists an integer $m \geq 1$ such that $z \notin (\overline{A_1^m} \cup A_2^m)$. Notice that there exists a component K_i of $\mathcal{J}_g$ contained essentially in A_i^m $(i = 1, 2)$. Thus the annulus bounded by K_1 and K_2 contains the point z. $\square$

By Proposition 3.3, for each component K of $\mathcal{J}_g$, $g(K)$ is a component of $\mathcal{J}_g$ and each component of $g^{-1}(K)$ is also a component of $\mathcal{J}_g$. We will say that a component K of $\mathcal{J}_g$ is **periodic** if there is an integer $p \geq 1$ such that $g^p(K) = K$; or **pre-periodic** if $f^k(K)$ is periodic for some integer $k \geq 1$; or **wandering** otherwise.

Proposition 3.4. *Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an exact annular system. Then any pre-periodic component K of $\mathcal{J}_g$ is a quasicircle.*

Proof. We only need to consider periodic components of $\mathcal{J}_g$ since each component of their pre-images is also a quasicircle. Let K be a periodic component of $\mathcal{J}_g$ with period $p \geq 1$. Then $g^p(A^p(K)) = A^0(K)$ and $A^p(K) \Subset A^0(K)$ by Proposition 3.2. Now applying quasiconformal surgery, we have a quasiconformal map ϕ of $\widehat{\mathbb{C}}$ such that $\phi \circ g^p \circ \phi^{-1} = z^d$ in a neighborhood of $\phi(K)$, where $|d| = \deg(g^p|_{A^p(K)}) \geq 2$. Thus K is a quasicircle. $\square$

3.2 Semi-conjugacy to linear systems

Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an exact annular system. In this sub-section, we want to characterize its dynamics by a linear system as the following. Denote by $A_1, \cdots, A_n$ the components of $\mathcal{A}$ and $A_1^1, \cdots, A_m^1$ the components of $\mathcal{A}^1$. Let

$$\mathcal{I} = I_1 \cup \cdots \cup I_n \quad \text{and} \quad \mathcal{I}^1 = I_1^1 \cup \cdots \cup I_m^1$$

be disjoint unions of open intervals on $\mathbb{R}^1$ such that
(a) $\mathcal{I}^1 \subset \mathcal{I}$ and $I_i^1 \subset I_j$ whenever $A_i^1 \subset A_j$, and
(b) for each I_i, $\partial I_i \subset \partial(\mathcal{I}^1 \cap I_i)$.
Define $\sigma : \mathcal{I}^1 \to \mathcal{I}$ by $\sigma(I_i^1) = I_j$ if $g(A_i^1) = A_j$ and σ is linear on each I_i^1. Set

$$\mathcal{I}^n = \sigma^{-n}(\mathcal{I}) \text{ for } n > 1 \text{ and } \mathcal{J}_\sigma = \bigcap_{n \geq 1} \mathcal{I}^n.$$

Proposition 3.5. *The linear system $\sigma : \mathcal{I}^1 \to \mathcal{I}$ is expanding and the closure of $\mathcal{J}_\sigma$ in $\mathbb{R}$ is a Cantor set.*

Proof. To prove the expanding property, we only need to show that there is an integer $n \geq 1$ such that for any $x \in \mathcal{I}^n$, $|(\sigma^n)'(x)| > 1$. For each $k \geq 1$, let l_k, L_k be the minimum and maximum of the length of the components of $\mathcal{I}^k$, respectively. Then $L_{k+1} \leq L_k$ for any $k \geq 1$. To prove $|(\sigma^n)'| > 1$, it is sufficient to show that there is an integer $n \geq 1$ such that $L_n < l_0$.

We will prove that $L_k \to L = 0$ as $k \to \infty$. Assume $L > 0$ by contradiction. Then for each $k \geq 1$, there is a component of $\mathcal{I}^k$ whose length is at least L. Therefore, there exists a sequence $\{I^k\}_{k \geq 1}$ with I^k a component of $\mathcal{I}^k$, such that $I^k \supset I^{k+1}$ and $|I^k| \geq L$.

Denote by $I^\infty = \bigcap_k I^k$. Then $|I^\infty| \geq L$ and $|I^k| \to |I^\infty|$ as $k \to \infty$. In particular, there exists an integer $k_0 \geq 0$ such that as $k \geq k_0$,

$$\frac{|I^k|}{|I^\infty|} < \frac{L_1 + l_1}{L_1}.$$

Since g is an annular system, there exists an integer $k_1 \geq k_0$ such that I^{k_1} contains another component I of $\mathcal{I}^{k_1+1}$ distinct from I^{k_1+1}. Thus

$$\frac{|I|}{|I^{k_1+1}|} \leq \frac{|I^{k_1}| - |I^{k_1+1}|}{|I^{k_1+1}|} < \frac{l_1}{L_1}.$$

Since σ^{k_1} is linear on I^{k_1}, we have

$$\frac{|\sigma^{k_1}(I)|}{|\sigma^{k_1}(I^{k_1+1})|} < \frac{l_1}{L_1}.$$

By the definition, $|\sigma^{k_1}(I)| \geq l_1$ and $|\sigma^{k_1}(I^{k_1+1})| \leq L_1$. So

$$\frac{|\sigma^{k_1}(I)|}{|\sigma^{k_1}(I^{k_1+1})|} \geq \frac{l_1}{L_1}.$$

This is a contradiction.

Now each component of $\mathcal{J}_\sigma$ is a single point since the linear system σ is expanding. It is easy to check that the closure of $\mathcal{J}_\sigma$ in $\mathbb{R}$ is a perfect set and hence a Cantor set. $\square$

For any point $x \in \mathcal{J}_\sigma$ and each $k \geq 1$, denote by $I^k(x)$ the component of $\mathcal{I}^k$ that contains the point x, then $\cap_{k\geq 1} I^k(x) = \{x\}$. For any two distinct points $x, y \in \mathcal{J}_\sigma$, either they are contained in different component of $\mathcal{I}^1$, or there exists an integer $k_0 \geq 2$ such that $I^{k_0}(x) \cap I^{k_0}(y) = \emptyset$ and $I^k(x) = I^k(y)$ for $1 \leq k < k_0$. In the latter case, $\sigma^{k_0-1}(I^{k_0}(x))$ and $\sigma^{k_0-1}(I^{k_0}(y))$ are different components of $\mathcal{I}^1$. Define the **itinerary** of a point $x \in \mathcal{J}_\sigma$ by $i(x) = (j_0, j_1, \cdots)$ if $\sigma^k(x) \in I^1_{j_k}$. Then $i(x) \neq i(y)$ if $x \neq y$.

Define the **itinerary** for each point $z \in \mathcal{J}_g$ by $i_*(z) = (j_0, j_1, \cdots)$ if $g^k(z) \in A^1_{j_k}$. Define a map $\Pi : \mathcal{J}_g \to \mathcal{J}_\sigma$ by $\Pi(z) = x$ if $i_*(z) = i(x)$. It is well-defined and surjective by Proposition 3.2.

Proposition 3.6. *The map $\Pi : \mathcal{J}_g \to \mathcal{J}_\sigma$ is continuous and $\sigma \circ \Pi = \Pi \circ g$ on $\mathcal{J}_g$. For each point $x \in \mathcal{J}_\sigma$, $\Pi^{-1}(x)$ is a component of $\mathcal{J}_g$.*

Proof. It is easy to check that $\sigma \circ \Pi = \Pi \circ g$ on $\mathcal{J}_g$, and $\Pi^{-1}(x)$ is a component of $\mathcal{J}_g$ for each point $x \in \mathcal{J}_\sigma$. Fix any point $x \in \mathcal{J}_\sigma$. The collection $\{I^k(x) \cap \mathcal{J}_\sigma\}_{k\geq 1}$ forms a basis of neighborhoods of the point x in $\mathcal{J}_\sigma$. Now $\Pi^{-1}(\{I^k(x) \cap \mathcal{J}_\sigma\}) = A^k(\Pi^{-1}(x)) \cap \mathcal{J}_g$ is open in $\mathcal{J}_g$ for every $k \geq 1$. So Π is continuous. □

Since the set of pre-periodic points is a countable set, we have:

Corollary 3.7. *There are uncountably many wandering components in $\mathcal{J}_g$.*

For any point $x \in \mathcal{J}_\sigma$, its **ω-limit set** $\omega(x)$ is defined to be the set of points $y \in \mathcal{J}_\sigma$ such that $\sigma^{k_n}(x)$ converges to y as $n \to \infty$ for a subsequence $k_n \to \infty$.

Proposition 3.8. *Let $x \in \mathcal{J}_\sigma$ be a wandering point. Then $\omega(x)$ is an infinite set.*

Proof. Assume that $\omega(x)$ is finite. Define $d(y_1, y_2)$ to be the Euclidean distance if y_1, y_2 are contained in the same component of $\mathcal{I}$, or infinity otherwise. There exists a constant $\delta > 0$ such that $d(y_1, y_2) > \delta$ for any two distinct points $y_1, y_2 \in \omega(x)$ and $d(y_1, y_2) > \delta$ if y_1, y_2 are contained in different components of $\mathcal{I}^1$. Take a constant $M \in (1, \infty)$ such that $|\sigma'(x)| < M$ for any point $x \in \mathcal{I}^1$. By the definition of $\omega(x)$, there exists a constant $N \geq 1$ such that for any $n \geq N$, there exists a unique point $y_n \in \omega(x)$ such that $d(\sigma^n(x), y_n) < \delta/(2M)$. Thus $d(\sigma^{n+1}(x), \sigma(y_n)) < \delta/2$. It follows that $y_{n+1} = \sigma(y_n)$ for $n \geq N$. This contradicts the fact that σ is expanding. □

3.3 Common boundary

Recall that each component of the Julia set of an exact annular system is a 2-connected continuum by Proposition 3.3.

Theorem 3.9. *Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an exact annular system and K be a component of $\mathcal{J}_g$. Let U and V be the two components of $\widehat{\mathbb{C}} \backslash K$. Then $\partial U = \partial V = K$.*

Proof. Assume that each component of $\mathcal{A}$ contains at least two components of $\mathcal{A}^1$ (otherwise we consider g^n for some $n \geq 2$ by the definition). Then $\|g'\| > 1$ under the hyperbolic metric of $\mathcal{A}$.

If K is eventually periodic, then K is a quasicircle by Proposition 3.4 and hence the theorem holds. Now we suppose that K is wandering. Let Π be a semi-conjugacy from $g : \mathcal{J}_g \to \mathcal{J}_g$ to a linear system $\sigma : \mathcal{I}^1 \to \mathcal{I}$ defined in Proposition 3.6. Then $x = \Pi(K)$

is a wandering point. Thus $\omega(x)$ is an infinite set by Proposition 3.8. In particular $\omega(x)$ contains a point $y \in \mathcal{I}$ such that $y \notin \partial\mathcal{I}$. It follows that there exists a component I^m of $\sigma^{-m}(\mathcal{I})$ such that $y \in I^m$ and I^m is contained in the interior of $\mathcal{I}$. Hence there exists an increasing sequence $\{n_k\}_{k\geq 1}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and $\sigma^{n_k}(x) \in I^m$.

Denote by A^m the component of $g^{-m}(\mathcal{A})$ corresponding to the interval I^m. Then $A^m \Subset \mathcal{A}$ and $g^{n_k}(K) \subset A^m$. For any component J of $\mathcal{J}_g$, denote by $A^n(J)$ the component of $g^{-n}(\mathcal{A})$ that contains J . Then we have

$$g^{n_k}(A^{m+n_k}(K)) = A^m(g^{n_k}(K)) = A^m.$$

For each annulus $W \Subset \mathcal{A}$, define

$$\mathrm{width}(W) = \sup_{z\in W}\Big\{d_W(z,\partial_+W) + d_W(z,\partial_-W)\Big\},$$

where $\partial_\pm W$ denotes the two boundary components of W and $d_W(z,\partial_\pm W)$ denotes the infimum of the length of arcs connecting z to $\partial_\pm W$ in W under the hyperbolic metric of $\mathcal{A}$.

Pick an annulus W_0 bounded by smooth curves such that $W_0 \Subset \mathcal{A}$ and $A^m \subset W_0$. Then $\mathrm{width}(W_0) < \infty$ and there exists a constant $\lambda > 1$ such that $\|g'(z)\| \geq \lambda > 1$ for every point $z \in g^{-1}(W_0)$.

Denote by W_k the component of $g^{-n_k}(W_0)$ that contains K. Then

$$A^{m+n_k-n_j}(g^{n_j}(K)) \subset g^{n_j}(W_k) \Subset A^{n_k-n_j}(g^{n_j}(K))$$

for $0 \leq j \leq k$ (set $n_0 = 0$). Note that $A^{n_k-n_j}(g^{n_j}(K)) \subset W_0$ if $n_k - n_j \geq m$. Thus $g^{n_j}(W_k) \subset W_0$ if $k - j \geq m$. Therefore $\|(g^{n_k})'(z)\| \geq \lambda^{k-m}$ for any point $z \in W_k$ since the finite orbit $\{z, g(z), \cdots, g^{n_k-1}(z)\}$ passes at least $k-m$ times through the set $g^{-1}(W_0)$ where $\|g'\| \geq \lambda$ and $\|g'\| > 1$ always. So

$$\mathrm{width}(W_k) \leq \lambda^{m-k}\mathrm{width}(W_0).$$

Hence $\mathrm{width}(W_k) \to 0$ as $k \to \infty$.

Clearly $\partial U \cup \partial V \subset K$. In order to prove that $\partial U = \partial V = K$ we only need to show that $K \subset \partial U$ by symmetry. Otherwise, assume $z \in K\backslash\partial U$. Then the spherical distance $d(z,\partial U) > 0$. Label the boundary components of W_k by $\partial_\pm W_k$ such that $\partial_+ W_k \subset U$. Then $d(z,\partial_+W_k) > d(z,\partial U) > 0$. Note that there exists a constant $M > 0$ such that $d_{W_k}(z,\partial_+W_k) \geq M\cdot d(z,\partial_+W_k)$ for all $k \geq 0$. Therefore

$$\mathrm{width}(W_k) \geq d_{W_k}(z,\partial_+W_k) \geq M\cdot d_U(z,\partial_\pm U) > 0.$$

This contradicts the fact that $\mathrm{width}(W_k) \to 0$ as $k \to \infty$. □

3.4 Local connectivity

In the appendix we will give an example constructed by X. Buff showing that an exact annular system may have a non-locally connected wandering Julia component. The next theorem gives a sufficient condition about the local connectivity of wandering components. The idea of the proof comes from [26].

Theorem 3.10. *Let $g : \mathcal{A}^1 \to \mathcal{A}$ be an exact annular system. Suppose that g is expanding, i.e. there exists a smooth metric ρ on $\mathcal{A}$ and a constant $\lambda > 1$ such that $\|g'\| \geq \lambda$. Then every component of $\mathcal{J}_g$ is a Jordan curve.*

Proof. Pick a pre-periodic component of $\mathcal{J}_g$ in each component of $\mathcal{A}$ and denote by Γ_0 the collection of them. It is a multicurve consisting of quasicircles. Denote by Γ_n the collection of curves in $g^{-n}(\Gamma_0)$. Then for any curve $\gamma \in \Gamma_n$ and any curve $\beta \in \Gamma_m$ with $m \neq n$, either they are disjoint or one coincides with another.

For each curve $\beta \in \Gamma_1$, there is a unique curve $\gamma \in \Gamma_0$ such that β and γ are contained in the same component of $\mathcal{A}$. If $\beta \neq \gamma$, there is a homotopy $\Phi_\beta : S^1 \times [0,1] \to \mathcal{A}$ from γ to β such that $\phi_t := \Phi_\beta(\cdot, t)$ is a homeomorphism for any $t \in [0,1]$, and in particular, $\phi_0(S^1) = \gamma$, $\phi_1(S^1) = \beta$ and $\phi_t(S^1)$ is a curve between β and γ. If $\beta = \gamma$, define $\Phi_\beta(\cdot, t) : S^1 \to \beta$ to be a homeomorphism independent on t.

Define the homotopic length of a path $\delta : [0,1] \to \mathcal{A}$ by

$$\text{h-length}(\delta) = \inf \Big\{ \text{length of } \alpha \text{ under metric } \rho \Big\},$$

where the infimum is taken over all the path α in $\mathcal{A}$ connecting $\delta(0)$ to $\delta(1)$ and homotopic to δ. Then

$$\text{h-length}(\tilde{\delta}) \leq \frac{1}{\lambda} \cdot \text{h-length}(\delta)$$

for any lift $\tilde{\delta}$ of δ under the map g since $\|g'\| \geq \lambda$.

For each $\beta \in \Gamma_1$ and any $s \in S^1$, $\Phi_\beta(s, \cdot)$ maps the interval $[0,1]$ to a path $\delta_{\beta,s}$ in the closed annulus $\Phi_\beta(S^1 \times [0,1])$ which connects two points in each of its boundary. So there is a constant $C > 0$ such that $\text{h-length}(\delta_{\beta,s}) < C$ for each $\beta \in \Gamma_1$ and any $s \in S^1$.

For each wandering component K of $\mathcal{J}_g$, let α_n be the unique curve of Γ_n with $\alpha_n \subset A^n(K)$. Then $g^n(\alpha_n) \in \Gamma_0$ and $\beta := g^n(\alpha_{n+1}) \in \Gamma_1$ are contained in the same component of $\mathcal{A}$. Now the homotopy Φ_β from $g^n(\alpha_n)$ to β defined above can be lifted to an homotopy from α_n to α_{n+1}, denote it by $\Psi_n : S^1 \times [0,1] \to A^n(K)$, by the following commutative diagram:

$$\begin{tikzcd} S^1 \times [0,1] \arrow[r, "\Psi_n"] \arrow[d, "P_d"'] & A^n(K) \arrow[d, "g^n"] \\ S^1 \times [0,1] \arrow[r, "\Phi_\beta"'] & g^n(A^n(K)), \end{tikzcd}$$

where $d = \deg(g|_{A^n(K)})$ and $P_d(s,t) = (s^d, t)$, i.e. $P(\cdot, t)$ is a covering of S^1 with degree d. Set $\psi_{n,t} := \Psi_n(\cdot, t)$. It is a homeomorphism for any $t \in [0,1]$, and in particular, $\psi_{n,0}(S^1) = \alpha_n$ and $\psi_{n,1}(S^1) = \alpha_{n+1}$. For any $s \in S^1$, $\Psi_n(s,t)(S^1)$ is a path connecting a point in α_n with a point in α_{n+1} whose homotopic length is less than $C\lambda^{-n}$.

These isotopies Ψ_n can be pasted together to a continuous map $\Psi : S^1 \times [0, \infty) \to \mathcal{A}$ as the following:

$$\Psi(s,t) = \begin{cases} \Psi_0(s,\, t) & \text{on } S^1 \times [0,1] \\ \Psi_1\Big(\psi_{1,0}^{-1} \circ \psi_{0,1}(s),\, t-1\Big) & \text{on } S^1 \times [1,2] \\ \quad\vdots & \quad\vdots \\ \Psi_n\Big(\psi_{n,0}^{-1} \circ \psi_{n-1,1} \circ \cdots \circ \psi_{1,0}^{-1} \circ \psi_{0,1}(s),\, t-n\Big) & \text{on } S^1 \times [n, n+1] \\ \quad\vdots & \quad\vdots \end{cases}$$

Set $h_t = \Psi(\cdot, t)$. Then $h_n(S^1) = \alpha_n$. For each $s \in S^1$ and any integers $m > n \geq 0$, the homotopic length of the path $\zeta_s(n,m) := \{\Psi(s,t) : n \leq t \leq m\}$ satisfies:

$$\text{h-length}\left(\zeta_s(n,m)\right) \leq C\lambda^{-n} + \cdots + C\lambda^{1-m} \leq \frac{C}{(\lambda-1)\lambda^{n-1}}.$$

Note that the two endpoints of $\zeta_s(n,m)$ are $h_n(s) \in \alpha_n$ and $h_m(s) \in \alpha_m$. The above inequality shows that $\{h_n\}$ is a Cauchy sequence and hence converges uniformly to a continuous map h. Since $\alpha_n \subset A^n(K)$, we have $h(S^1) \subset \bigcap_{n>1} A^n(K) = K$. Note that $h(S^1)$ separates the two components of $\widehat{\mathbb{C}}\backslash K$. Thus $h(S^1) = K$ by Theorem 3.9. Therefore K is locally connected and hence is a Jordan curve. □

4 From multicurves to annular systems

We will prove Theorem 1.1 in this section. We were unable to prove this theorem directly. Instead we will take a detour to the space of branched coverings of the sphere. We will first modify topologically the rational map f to a branched covering F in its Thurston equivalence class such that F has a topological exact annular system. We then apply a theorem of Rees and Shishikura (refer to Theorem A.1 in the appendix) to obtain a semi-conjugacy from F to f. Finally we show that the existence of an exact annular system is preserved under the semi-conjugacy.

Proof of Theorem 1.1. **Step 1. Topological modification**. Let f be a post-critically rational map with a Cantor multicurve Γ. There exists a multi-annulus $\mathfrak{C} \subset \widehat{\mathbb{C}}\backslash\mathcal{P}_f$ homotopic to Γ rel $\mathcal{P}_f$ such that its boundary $\partial\mathfrak{C}$ is a disjoint union of Jordan curves in $\widehat{\mathbb{C}}\backslash\mathcal{P}_f$. Let $\mathfrak{C}^*$ be the union of all the components of $f^{-1}(\mathfrak{C})$ which are homotopic to curves in Γ rel $\mathcal{P}_f$. Then for each $\gamma \in \Gamma$, there is at least one component of $\mathfrak{C}^*$ homotopic to γ rel $\mathcal{P}_f$ since Γ is pre-stable.

For each $\gamma \in \Gamma$, denote by $\mathfrak{C}^*(\gamma)$ the smallest annulus containing all the components of $\mathfrak{C}^*$ which are homotopic to γ rel $\mathcal{P}_f$. Then its boundary are two Jordan curves in $\widehat{\mathbb{C}}\backslash\mathcal{P}_f$ homotopic to γ rel $\mathcal{P}_f$. Set $\mathfrak{C}^*(\Gamma) = \cup_{\gamma\in\Gamma}\mathfrak{C}^*(\gamma)$. There exist a neighborhood U of $\mathcal{P}_f$ and a homeomorphism θ_0 of $\widehat{\mathbb{C}}$ such that θ_0 is isotopic to the identity rel U and $\theta_0(\mathfrak{C}) = \mathfrak{C}^*(\Gamma)$.

Set $F := f \circ \theta_0$ and $\mathfrak{C}^1 := \theta_0^{-1}(\mathfrak{C}^*)$, then $\mathcal{P}_F = \mathcal{P}_f$ and F is Thurston equivalent to f via the pair (θ_0, id). Moreover, the restriction $F|_{\mathfrak{C}^1} : \mathfrak{C}^1 \to \mathfrak{C}$ is a topological exact annular system.

Step 2. Semi-conjugacy. By Theorem A.1, there exist a neighborhood V of the critical cycles of F and a sequence $\{\phi_n\}$ $(n \geq 1)$ of homeomorphisms of $\widehat{\mathbb{C}}$ isotopic to the identity rel $\mathcal{P}_F \cup V$ such that $f \circ \phi_n = \phi_{n-1} \circ F$ and the sequence $\{\phi_n\}$ converges uniformly to a continuous onto map h of $\widehat{\mathbb{C}}$ and $f \circ h = h \circ F$.

Step 3. Survival of the annular system. This is the main step. Define

$$T = \{w \in \widehat{\mathbb{C}} : h^{-1}(w) \text{ crosses some component of } \mathfrak{C}\},$$

here we say a continuum E **crosses** an annulus C if E intersects the both boundary components of C. Then $T \subset \mathcal{J}_f$ by Theorem A.1 (3). It is easy to see that T is closed.

Lemma 4.1. *The set T is empty.*

This lemma is crucial. Here the property of Cantor multicurves is essential. It is not true for the equator curve in a mating of polynomials. We prove at first a purely topological result.

Lemma 4.2. *Let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_n$ be a finite disjoint union of Jordan curves on $\widehat{\mathbb{C}}$ and $L \subset \widehat{\mathbb{C}}$ be a compact subset. Then for any Jordan domain D containing L, there is an integer $N \geq 0$ such that for any two distinct points $z_1, z_2 \in L$, there exists a Jordan arc δ in D connecting z_1 with z_2 such that $\#(\delta \cap \Gamma) \leq N$.*

Proof. Set

$$\Lambda = \{\alpha : \alpha \text{ is a component of } \Gamma \cap D \text{ such that } \alpha \cap L \neq \emptyset\}.$$

Then $N := \#\Lambda < \infty$. In fact, let $\gamma : S^1 \times \{1, \cdots, n\} \to \Gamma$ be a homeomorphism. Then $\gamma^{-1}(\Gamma \cap L)$ is a compact subset, which is covered by the open intervals $\{\gamma^{-1}(\alpha), \alpha \in \Lambda\}$. Therefore Λ is finite.

For any two distinct points $z_1, z_2 \in L$, set

$$\Lambda(z_1, z_2) = \{\alpha \in \Lambda : \alpha \cup \partial D \text{ separates } z_1 \text{ from } z_2\}.$$

There exists a Jordan arc $\delta \subset D$ connecting z_1 with z_2 such that δ intersects each $\alpha \in \Lambda(z_1, z_2)$ on a single point and disjoint from other components of $\Gamma \cap D$. So $\#(\delta \cap \Gamma) \leq \#\Lambda(z_1, z_2) \leq N$. □

Proof of Lemma 4.1. Assume $T \neq \emptyset$ by contradiction. Then $f(T) \subset T$. In fact, suppose $w \in T$, i.e., $h^{-1}(w)$ crosses some component of $\mathfrak{C}$, then $h^{-1}(w)$ crosses some component C^1 of $\mathfrak{C}^1$. By Theorem A.1 (4), $h^{-1}(f(w)) = F(h^{-1}(w))$. So $h^{-1}(f(w))$ crosses $F(C^1)$ which is a component of $\mathfrak{C}$, so $f(w) \in T$. Set $T_\infty = \bigcap_{n \geq 0} f^n(T)$. Then T_∞ is a non-empty closed set and $f(T_\infty) = T_\infty$.

Pick one point $w_0 \in T_\infty$. Since $f(T_\infty) = T_\infty$, there exists a sequence of points $\{w_n\}_{n \geq 0}$ in T_∞ such that $f(w_{n+1}) = w_n$ (i.e. T_∞ contains a backward orbit). Either w_n is periodic for all $n \geq 0$ or there is an integer $n_0 \geq 0$ such that w_n is not periodic for all $n \geq n_0$. In the former case all the points w_n are not critical points of f since $w_n \in \mathcal{J}_f$. In the latter case, there exists an integer $n_1 \geq 0$ such that w_n are non-critical points of f for $n \geq n_1$. So in both cases, we have a sequence of points $\{w_n\}_{n \geq 0}$ in $T_\infty \backslash \Omega_f$ such that $f(w_{n+1}) = w_n$.

Set $L_n = h^{-1}(w_n)$. By Theorem A.1 (4), L_n is a component of $F^{-1}(L_{n-1})$ and there exists a Jordan domain $D_0 \supset L_0$ such that $F^n : D_n \to D_0$ is a homeomorphism for $n \geq 1$, where D_n is the component of $F^{-n}(D_0)$ containing L_n.

Pick an essential Jordan curve in every components of $\mathfrak{C}$. They form a Cantor multicurve Γ_0. By Lemma 4.2, there exists an integer $N \geq 0$ such that for any two distinct points $z_0, z_0' \in L_0$, there is a Jordan arc $\delta \subset D_0$ connecting z_0 with z_0', such that $\#(\delta \cap \Gamma_0) \leq N$.

On the other hand, since Γ_0 is a Cantor multicurve, there is an integer $m > 0$ such that for each component C of $\mathfrak{C}$, there are at least $N + 1$ components of $F^{-m}(\mathfrak{C})$ contained essentially in C. Since L_m crosses a component of $\mathfrak{C}$, there exist two distinct points $z_m, z_m' \in L_m$ such that $F^{-m}(\Gamma_0)$ has at least $N + 1$ components separating z_m from z_m'.

Now $F^m(z_m), F^m(z_m') \in L_0$. So there exists a Jordan arc $\delta \subset D_0$ connecting $F^m(z_m)$ with $F^m(z_m')$ such that $\#(\delta \cap \Gamma_0) \leq N$. Let δ_m be the component of $F^{-m}(\delta)$ connecting z_m with z_m'. Then

$$\#(\delta_m \cap F^{-m}(\Gamma_0)) \leq N$$

since $F^m : \delta_m \to \delta$ is a homeomorphism. This contradicts the fact that $F^{-m}(\Gamma_0)$ has at least $N+1$ components separating z_m from z'_m. $\square$

Corollary 4.3. *For any $n \geq 0$ and any distinct components E_1, E_2 of $\widehat{\mathbb{C}} \backslash F^{-n}(\mathfrak{C})$, $h(E_1)$ is disjoint from $h(E_2)$.*

Proof. E_1 and E_2 are separated by a component A of $F^{-n}(\mathfrak{C})$. If $h(E_1) \cap h(E_2) \neq \emptyset$, pick a point $w \in h(E_1) \cap h(E_2)$, then $h^{-1}(w)$ crosses A. So $F^n(h^{-1}(w)) = h^{-1}(f^n(w))$ crosses $F^n(A)$ by Theorem A.1 (4). This contradicts Lemma 4.1. $\square$

Construction of the multi-annulus $\mathcal{A}$. Denote by $\widehat{E} = h^{-1}(h(E))$ for any subset $E \subset \widehat{\mathbb{C}}$. Then $\widehat{E}$ is also a continuum if E is a continuum by Theorem A.1 (5).

Denote by $\mathcal{E} = \widehat{\mathbb{C}} \backslash \mathfrak{C}$. Then $F^{-1}(\widehat{\mathcal{E}}) = \widehat{F^{-1}(\mathcal{E})}$ by Theorem A.1 (7). Thus if E^1 is a component of $F^{-1}(\mathcal{E})$, then $\widehat{E^1}$ is a component of $F^{-1}(\widehat{\mathcal{E}})$ by Corollary 4.3.

For any two disjoint continua $E_1, E_2 \subset \widehat{\mathbb{C}}$, we denote by $A(E_1, E_2)$ the unique annular component of $\widehat{\mathbb{C}} \backslash (E_1 \cup E_2)$. For each component C of $\mathfrak{C}$, there are two distinct components E_+, E_- of $\mathcal{E}$ such that $C = A(E_+, E_-)$. Define $\widetilde{C} := A(\widehat{E}_+, \widehat{E}_-)$. It is an annulus contained essentially in C by Corollary 4.3. We claim that the following statements hold:

(a) $h^{-1}(h(\widetilde{C})) = \widetilde{C}$.

(b) $\widetilde{C} \cap \widehat{E} = \emptyset$ for any subset $E \subset \widehat{\mathbb{C}}$ with $E \cap \widetilde{C} = \emptyset$.

(c) $h(\widetilde{C})$ is an annulus homotopic to C rel $\mathcal{P}_f$.

Proof. (a) For any point $z \in \widetilde{C}$, if $h^{-1}(h(z))$ is not contained in $\widetilde{C}$, then it must intersect $E_+ \cup E_-$. So $z \in \widehat{E}_+ \cup \widehat{E}_-$. This is a contradiction.

(b) If $z \in \widetilde{C} \cap \widehat{E}$, then $h^{-1}(h(z)) \subset \widetilde{C}$ and hence is disjoint from E. It contradicts the condition that $z \in \widehat{E}$.

(c) Let Q_+, Q_- be the two components of $\widehat{\mathbb{C}} \backslash \widetilde{C}$. Then both $\widehat{Q}_+$ and $\widehat{Q}_-$ are disjoint from $\widetilde{C}$ by (b). Moreover, they are also disjoint from each other since $h^{-1}(h(z))$ does not cross C for any point $z \in \widehat{\mathbb{C}}$ by Lemma 4.1. So $\widehat{\mathbb{C}} \backslash h(\widetilde{C})$ has exactly two components, $h(Q_+)$ and $h(Q_-)$. Therefore $h(\widetilde{C})$ is an annulus. Since h is homotopic to the identity rel $\mathcal{P}_f$, the annulus $h(\widetilde{C})$ is homotopic to C rel $\mathcal{P}_f$. $\square$

Now let $\widetilde{\mathfrak{C}}$ be the union of $\widetilde{C}$ for all the components C of $\mathfrak{C}$. Then $\widetilde{\mathfrak{C}} \subset \mathfrak{C}$ and it is a multi-annulus homotopic to Γ rel $\mathcal{P}_f$. Set $\mathcal{A}$ to be the union of $h(\widetilde{C})$ for all the components C of $\mathfrak{C}$. Since $h(\widetilde{C_1})$ is disjoint from $h(\widetilde{C_2})$ for distinct components C_1, C_2 of $\mathfrak{C}$ by (b), $\mathcal{A}$ is a multi-annulus and homotopic to Γ rel $\mathcal{P}_f$ by (c). Moreover, $\mathcal{A}$ is disjoint from a neighborhood of critical cycles of f since h is the identity in a neighborhood of critical cycles of f.

Construction of $\mathcal{A}^1$. For each component C^1 of $\mathfrak{C}^1$, there are two distinct components E^1_+, E^1_- of $F^{-1}(\mathcal{E})$ such that $C^1 = A(E^1_+, E^1_-)$. Define $\widetilde{C^1} := A(\widehat{E^1_+}, \widehat{E^1_-})$ as above. It is an annulus essentially contained in C^1. Moreover, the following statements hold:

(a1) $h^{-1}(h(\widetilde{C^1})) = \widetilde{C^1}$.

(b1) $\widetilde{C^1} \cap \widehat{E} = \emptyset$ for any subset $E \subset \widehat{\mathbb{C}}$ with $E \cap \widetilde{C^1} = \emptyset$.

(c1) $h(\widetilde{C^1})$ is an annulus homotopic to C^1 rel $\mathcal{P}_f$.

Set $\widetilde{\mathfrak{C}^1}$ to be the union of $\widetilde{C^1}$ for all the components C^1 of $\mathfrak{C}^1$. Set $\mathcal{A}^1$ to be the union of $h(\widetilde{C^1})$ for all the components C^1 of $\mathfrak{C}^1$. Then it is a multi-annulus essentially contained in $\mathcal{A}$.

Invariance of $\mathcal{A}$. Note that each component of $\widetilde{\mathfrak{C}}$ is a component of $\widehat{\mathbb{C}}\backslash\widehat{\mathcal{E}}$ and each component of $\widetilde{\mathfrak{C}^1}$ is a component of $\widehat{\mathbb{C}}\backslash F^{-1}(\widehat{\mathcal{E}}) = \widehat{\mathbb{C}}\backslash\widehat{F^{-1}(\mathcal{E})}$. So $F : \widetilde{\mathfrak{C}^1} \to \widetilde{\mathfrak{C}}$ is proper. Since $\widetilde{\mathfrak{C}} = h^{-1}(\mathcal{A})$ and $\widetilde{\mathfrak{C}^1} = h^{-1}(\mathcal{A}^1)$, the map $f : \mathcal{A}^1 \to \mathcal{A}$ is also proper.

For any component E of $\mathcal{E}$, there is a unique component E^1 of $F^{-1}(\mathcal{E})$ such that $\partial E \subset \partial E^1$. Moreover, $E^1 \subset E$ and $E\backslash E^1$ is a disjoint union of Jordan domains in E. We claim that $\widehat{E}\backslash E = \widehat{E^1}\backslash E$.

Since $E \supset E^1$, we have $\widehat{E} \supset \widehat{E^1}$. On the other hand, any component D of $\widehat{\mathbb{C}}\backslash E$ is a Jordan domain. Assume $z \in \widehat{E} \cap D$, then $h^{-1}(h(z))$ is a full continuum intersecting ∂E by Theorem A.1 (3). Thus $h^{-1}(h(z))$ intersects ∂E^1. Therefore $z \in \widehat{E^1}$ and hence $\widehat{E}\backslash E \subset \widehat{E^1}\backslash E$. The claim is proved.

By the claim, $\widetilde{\mathfrak{C}^1} \subset \widetilde{\mathfrak{C}}$ and each component of $\partial\widetilde{\mathfrak{C}}$ is a component of $\partial\widetilde{\mathfrak{C}^1}$. Hence $\mathcal{A}^1 \subset \mathcal{A}$ and each component of ∂A for any component A of $\mathcal{A}$ is a component of ∂A^1 for some component A^1 of $\mathcal{A}^1$ in A. So $f : \mathcal{A}^1 \to \mathcal{A}$ is an exact annular system.

Step 4. Uniqueness of $\mathcal{A}$. Suppose that $f : \mathcal{A}_1^1 \to \mathcal{A}_1$ is another exact annular system such that $\mathcal{A}_1$ is homotopic to Γ rel $\mathcal{P}_f$. Pick an essential Jordan curve in every components of $\mathcal{A}$ and $\mathcal{A}_1$, respectively. They form two multicurves $\Gamma_0 \subset \mathcal{A}$ and $\Gamma_1 \subset \mathcal{A}_1$. Both of them are homotopic to Γ rel $\mathcal{P}_f$. So there exist a neighborhood U of the critical cycles of f and a homeomorphism θ_0 of $\widehat{\mathbb{C}}$ such that $\theta_0(\Gamma_0) = \Gamma_1$ and θ_0 is isotopic to the identity rel $\mathcal{P}_f \cup U$. By Theorem A.1, there exist a neighborhood V of the critical cycles of f and a sequence $\{\theta_n\}$ $(n \geq 1)$ of homeomorphisms of $\widehat{\mathbb{C}}$ isotopic to the identity rel $\mathcal{P}_f \cup V$, such that $f \circ \theta_n = \theta_{n-1} \circ f$. Moreover, $\{\theta_n\}$ converges uniformly to a continuous map h of $\widehat{\mathbb{C}}$.

It is easy to see that h is the identity in the Fatou set of f. On the other hand, h is also the identity on the Julia set $\mathcal{J}_f$ since the closure of $\cup_{n\geq 0} f^{-n}(\mathcal{P}_f)$ contains $\mathcal{J}_f$ and θ_n is the identity on $f^{-n}(\mathcal{P}_f)$. So $\{\theta_n\}$ converges uniformly to the identity.

For each component A of $\mathcal{A}$, set $A(n, \Gamma_0)$ to be the closed annulus bounded by two curves in $A \cap (f|_{\mathcal{A}^1})^{-n}(\Gamma_0)$ such that $A \cap (f|_{\mathcal{A}^1})^{-n}(\Gamma_0) \subset A(n, \Gamma_0)$. Then $\theta_n(A(n, \Gamma_0)) \subset \mathcal{A}_1$ since $\theta_n(f^{-n}(\Gamma_0)) = f^{-n}(\Gamma_1)$. By Proposition 3.2, for any compact set $G \subset A$, $G \subset A(n, \Gamma_0)$ as n is large enough. So $A \subset \mathcal{A}_1$ since $\{\theta_n\}$ converges uniformly to the identity. It follows that $\mathcal{A} \subset \mathcal{A}_1$. By symmetry, we have $\mathcal{A} = \mathcal{A}_1$.

Step 5. Properties of $\mathcal{J}_g$. We want to prove that $\mathcal{J}_g \subset \mathcal{J}_f$. Assume by contradiction that there is a point $z \in \mathcal{J}_g\backslash\mathcal{J}_f$. Then $\{f^n(z)\}_{n\geq 0}$ converges to a super-attracting cycle of f as $n \to \infty$. But $f^n(z) \in g^n(\mathcal{J}_g) \subset \mathcal{J}_g$. Thus $\overline{\mathcal{A}}$ contains a critical cycle. This is a contradiction since $\mathcal{A}$ is disjoint from a neighborhood of critical cycles.

There is a singular conformal metric ρ on $\widehat{\mathbb{C}}$ where the singularities may occur at $\mathcal{P}_f$ such that f is strictly expanding on $(\widehat{\mathbb{C}}, \rho)$ except in a neighborhood of super-attracting cycles (e.g., the hyperbolic metric on the orbifold of f, refer to [10, 34, 36]). Applying Theorem 3.10, we see that every component of $\mathcal{J}_g$ is a Jordan curve. In particular, there exists a wandering Jordan curve on $\mathcal{J}_g$ by Corollary 3.7. $\square$

5 Decompositions and renormalizations

We will prove Theorem 1.3 in this section. At first, we want to introduce the definition of rational-like maps and prove a straightening theorem.

5.1 Rational-like maps

Definition 3. *Let $U \Subset V$ be two finitely-connected domains in $\widehat{\mathbb{C}}$. A map $g : U \to V$ is called a* **rational-like map** *if*

(1) g is holomorphic and proper with $\deg g \geq 2$,

(2) the orbit of every critical point of g (if any) stays in U, and

(3) each component of $\widehat{\mathbb{C}} \backslash U$ contains at most one component of $\widehat{\mathbb{C}} \backslash V$.

The **filled Julia set** *of g is defined by*

$$\mathcal{K}_g = \bigcap_{n>0} g^{-n}(V).$$

We say that a rational-like map $g : U \to V$ is a **renormalization** *of a rational map f if $g = f^p|_U$ for some integer $p \geq 1$ and $\deg g < \deg f^p$.*

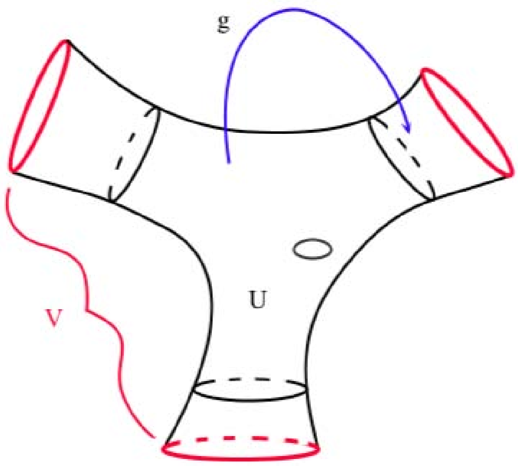

Figure 1. A rational-like map.

Remark. A rational-like map here is actually a repelling system of constant complexity in [8].

Proposition 5.1. *Let $g : U_1 \to U_0$ be a rational-like map. Then $g^{-n}(U_0)$ is connected for any $n \geq 1$. The filled Julia set $\mathcal{K}_g$ is a continuum.*

Proof. Pick a domain $V_0 \Subset U_0$ such that every component of $\widehat{\mathbb{C}} \backslash V_0$ contains exactly one component of $\widehat{\mathbb{C}} \backslash U_0$, $U_1 \Subset V_0$ and every component of ∂V_0 is a Jordan curve. Set $V_1 := g^{-1}(V_0)$. Then $V_1 \Subset V_0$, every component of $\widehat{\mathbb{C}} \backslash V_1$ contains at most one component of $\widehat{\mathbb{C}} \backslash V_0$ and each component of ∂V_1 is a Jordan curve.

Since every component of $\widehat{\mathbb{C}} \backslash V_1$ contains at most one component of $\widehat{\mathbb{C}} \backslash V_0$, each component W of $V_0 \backslash \overline{V_1}$ is either a disk or an annulus. In the latter case, one of the component of the boundary ∂W is a component of the boundary ∂V_0 and the other is a component of the boundary ∂V_1.

Denote by $V_n = g^{-n}(V_0)$ for $n > 1$. Then $V_{n+1} \Subset V_n$ for $n \geq 1$. Since all the critical orbits of g stay in U_1 and thus in $\mathcal{K}_g$, each component W of $V_1 \backslash \overline{V_2}$ is also either a disk or an annulus. In the latter case, one of the component of ∂W is a component of ∂V_1 and the other is a component of ∂V_2. Therefore, V_2 is also connected and every component of $\widehat{\mathbb{C}}\backslash V_2$ contains at most one component of $\widehat{\mathbb{C}}\backslash V_1$. Inductively, we have that V_{n+1} is connected and every component of $\widehat{\mathbb{C}}\backslash V_{n+1}$ contains at most one component of $\widehat{\mathbb{C}}\backslash V_n$. It follows that $\mathcal{K}_g$ is a connected compact set. $\square$

Similar to Douady-Hubbard's polynomial-like map theory [11], we may have a straightening theorem for rational-like maps with a slightly different proof.

Theorem 5.2. *Let $g : U \to V$ be a rational-like map, then there is a rational map f with $\deg f = \deg g$ and a quasiconformal map ϕ of $\widehat{\mathbb{C}}$ such that:*
(a) $f \circ \phi = \phi \circ g$ in a neighborhood of $\mathcal{K}_g$,
(b) the complex dilatation μ_ϕ of ϕ satisfying $\mu_\phi(z) = 0$ for a.e. $z \in \mathcal{K}_g$,
(c) $\mathcal{J}_f = \partial\phi(\mathcal{K}_g)$, and
(d) each component of $\widehat{\mathbb{C}}\backslash\phi(\mathcal{K}_g)$ contains at most one point of $\mathcal{P}_f$.
Moreover, the rational map f is unique up to holomorphic conjugation.

Proof. Pick a domain $V_1 \Subset V$ such that every component of $\widehat{\mathbb{C}}\backslash V_1$ contains exactly one component of $\widehat{\mathbb{C}}\backslash V$, $U \Subset V_1$ and every component of ∂V_1 is a quasicircle. Then $U_1 := g^{-1}(V_1) \Subset V_1$, every component of $\widehat{\mathbb{C}}\backslash U_1$ contains at most one component of $\widehat{\mathbb{C}}\backslash V_1$ and each component of ∂U_1 is a quasicircle.

Let $E_1, \cdots, E_m$ be the components of $\mathcal{E} := \widehat{\mathbb{C}}\backslash\overline{V_1}$. Let $B_1, \cdots, B_n$ be the components of $\mathcal{B} := \widehat{\mathbb{C}}\backslash\overline{U_1}$ such that $B_i \supset E_i$ for $1 \le i \le m$. Then $\mathcal{E} \Subset \mathcal{B}$. Define a map τ on the index set by $\tau(i) = j$ if $g(\partial B_i) = \partial E_j$.

Let $D_i \subset \mathbb{C}$ $(i = 1, \cdots, n)$ be round disks centered at a_i with unit radius such that their closures are pairwise disjoint. Denote their union by $\mathcal{D}$. Define a map Q on $\mathcal{D}$ by

$$Q(z) = r(z - a_i)^{d_i} + a_{\tau(i)},\ z \in D_i,$$

where $0 < r < 1$ is a constant and $d_i = \deg(g|_{\partial E_i})$. Then $Q(D_i) \Subset D_{\tau(i)}$. Denote by $D_{\tau(i)}(r) := Q(D_i)$ and $\mathcal{D}(r) = Q(\mathcal{D})$.

Let $\psi : \mathcal{E} \to \mathcal{D}(r)$ be a conformal map such that $\psi(E_i) = D_i(r)$. It can be extended to a quasiconformal map in a neighborhood of $\overline{\mathcal{E}}$ since the components of $\mathcal{E}$ are quasidisks with pairwise disjoint closures. Since $Q : \partial\mathcal{D} \to \partial\mathcal{D}(r)$ and $g : \partial\mathcal{B} \to \partial\mathcal{E}$ are coverings with same degrees on corresponding components, there is a homeomorphism $\psi_1 : \partial\mathcal{B} \to \partial\mathcal{D}$ such that $\psi \circ g = Q \circ \psi_1$.

Since each component of $\partial\mathcal{B}$ is a quasicircle, the conformal map $\psi : \mathcal{E} \to \mathcal{D}(r)$ can be extended to a homeomorphism $\psi : \overline{\mathcal{B}} \to \overline{\mathcal{D}}$ such that $\psi|_{\partial\mathcal{B}} = \psi_1$ and ψ is quasiconformal on $\mathcal{B}$. Define a map

$$G = \begin{cases} g & \text{on } U_1, \\ \psi^{-1} \circ Q \circ \psi & \text{on } \overline{\mathcal{B}}. \end{cases}$$

Then G is a quasiregular branched covering of $\widehat{\mathbb{C}}$. Set $\mathcal{O} := \psi^{-1}(\{a_1, \cdots, a_n\})$. Then

$$G(\mathcal{O}) \subset \mathcal{O} \text{ and } \mathcal{P}_G\backslash\mathcal{K}_g \subset \mathcal{O}$$

since no critical point of g escapes. Moreover, for each point $z \in \widehat{\mathbb{C}}\backslash\mathcal{K}_g$, its forward orbit $\{G^n(z)\}$ converges to the invariant set $\mathcal{O}$.

By Measurable Riemann Mapping Theorem, there is a quasiconformal map Φ of $\widehat{\mathbb{C}}$ such that its complex dilatation satisfies $\mu_\Phi = 0$ on U_1 and $\mu_\Phi = \mu_\psi$ on $\mathcal{B}$. Set $F := \Phi \circ G \circ \Phi^{-1}$. Then F is holomorphic in the interior of $\Phi(g^{-1}(U_1) \cup \mathcal{E})$.

For any orbit $\{F^n(z)\}_{n\geq 0}$, if z is not contained in the interior of $\Phi(g^{-1}(U_1)\cup\mathcal{E})$, then z is contained in either $\Phi(\overline{U_1})\backslash\Phi\circ g^{-1}(U_1)$ or the closure of $\Phi(\mathcal{B})\backslash\Phi(\mathcal{E})$. In the latter case, $F(z)\in\Phi(\mathcal{E})$ and thus $F^2(z)$ is contained in the interior of $\Phi(\mathcal{E})$. In the former case, $F(z)\in\Phi(\mathcal{B})\backslash\Phi(\mathcal{E})$ and thus $F^3(z)$ is contained in the interior of $\Phi(\mathcal{E})$. Thus $F^n(z)$ is contained in the interior of $\Phi(\mathcal{E})$ for $n\geq 3$ in both cases. This shows that every orbit of F passes through the closure of $\widehat{\mathbb{C}}\backslash\Phi(g^{-1}(U_1)\cup\mathcal{E})$ at most three times. Applying Shishikura's Surgery Principle (see Lemma 15 in [1]), there is quasiconformal map $\Phi_1:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $f=\Phi_1\circ F\circ\Phi_1^{-1}$ is a rational map. Moreover, $\mu_{\Phi_1}(z)=0$ for *a.e.* $z\in\Phi(\mathcal{K}_g)$. Set $\phi=\Phi_1\circ\Phi$. Then $f\circ\phi=\phi\circ g$ on U_1 and $\mu_\phi(z)=0$ for *a.e.* $z\in\mathcal{K}_g$.

For a compact set $E\subset\widehat{\mathbb{C}}\backslash\phi(\mathcal{K}_g)$, its forward orbit $\{f^n(E)\}$ converges to the invariant set $\phi(\mathcal{O})\subset\mathcal{F}_f$. Moreover, $\mathcal{P}_f\backslash\phi(\mathcal{K}_g)\subset\phi(\mathcal{O})$. So $\widehat{\mathbb{C}}\backslash\phi(\mathcal{K}_g)\subset\mathcal{F}_f$. Since $\phi(\mathcal{K}_g)$ is completely invariant under f, we have $\partial\phi(\mathcal{K}_g)=\mathcal{J}_f$.

If there is another rational map f_1 satisfying the conditions of the theorem, then there is a quasiconformal map θ of $\widehat{\mathbb{C}}$ such that $f_1\circ\theta=\theta\circ f$ in a neighborhood of $\phi(\mathcal{K}_g)$ and $\mu_\theta(z)=0$ for *a.e.* $z\in\phi(\mathcal{K}_g)$.

Let W be a periodic Fatou domain of f in $\widehat{\mathbb{C}}\backslash\phi(\mathcal{K}_g)$ with period $p\geq 1$. Then W is simply-connected and contains exactly one point $z_0\in\mathcal{P}_f$, which is a super-attracting periodic point. Therefore there is a conformal map η from W onto the unit disc $\mathbb{D}$ such that $\eta(z_0)=0$ and $\eta\circ f^p\circ\eta^{-1}(z)=z^d$ with $d=\deg_{z_0}f^p>1$. On the other hand, let $z_1\in\theta(W)$ be the the super-attracting periodic point of f_1, then there is a conformal map $\eta_1:\theta(W)\to\mathbb{D}$ such that $\eta_1(z_1)=0$ and $\eta_1\circ f_1^p\circ\eta_1^{-1}(z)=z^d$. Therefore

$$\eta_1\circ\theta\circ f^p\circ\theta^{-1}\circ\eta_1^{-1}(z)=z^d$$

in a neighborhood of $\partial\mathbb{D}$ in $\mathbb{D}$. This shows that $T=\eta_1\circ\theta\circ\eta^{-1}$ is a rotation on $\partial\mathbb{D}$ (see the commutative diagram below).

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "z\mapsto z^d"'] & W \arrow[l, "\eta"'] \arrow[r, "\theta"] \arrow[d, "f^p"'] & \theta(W) \arrow[r, "\eta_1"] \arrow[d, "f_1^p"] & \mathbb{D} \arrow[d, "z\mapsto z^d"] \\
\mathbb{D} & W \arrow[l, "\eta"] \arrow[r, "\theta"'] & \theta(W) \arrow[r, "\eta_1"'] & \mathbb{D}
\end{tikzcd}$$

Let $\theta_W=\eta_1^{-1}\circ T\circ\eta$. Then $\theta_W:W\to\theta(W)$ is holomorphic, $\theta_W=\theta$ on the boundary ∂W and $f_1\circ\theta_W=\theta_W\circ f$.

Define $\Theta_0:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ by $\Theta_0=\theta_W$ on all the super-attracting Fatou domains of f in $\widehat{\mathbb{C}}\backslash\phi(\mathcal{K}_g)$, and $\Theta_0=\theta$ otherwise. Then Θ_0 is a quasiconformal map and $\Theta_0\circ f=f_1\circ\Theta_0$ on the union of $\phi(\mathcal{K}_g)$ and all the super-attracting Fatou domains of f in $\widehat{\mathbb{C}}\backslash\phi(\mathcal{K}_g)$. Pullback Θ_0, we get a sequence of quasiconformal maps $\Theta_n:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $\Theta_0\circ f^n=f_1^n\circ\Theta_n$, in particular, the following diagram commutes.

$$
\begin{tikzcd}
\vdots \arrow[d, "f"'] & \vdots \arrow[d, "f_1"] \\
\widehat{\mathbb{C}} \arrow[r, "\Theta_2"] \arrow[d, "f"'] & \widehat{\mathbb{C}} \arrow[d, "f_1"] \\
\widehat{\mathbb{C}} \arrow[r, "\Theta_1"] \arrow[d, "f"'] & \widehat{\mathbb{C}} \arrow[d, "f_1"] \\
\widehat{\mathbb{C}} \arrow[r, "\Theta_0"] & \widehat{\mathbb{C}}_0
\end{tikzcd}
$$

It is easy to check that Θ_n converges uniformly to a holomorphic conjugacy from f to f_1. $\square$

5.2 Renormalizations

Let f be a post-critically rational map with a stable Cantor multicurve Γ. By Theorem 1.1, there is a unique multi-annulus $\mathcal{A} \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ homotopic rel $\mathcal{P}_f$ to Γ such that $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \to \mathcal{A}$ is an exact annular system, where $\mathcal{A}^1$ is the union of the components of $f^{-1}(\mathcal{A})$ homotopic rel $\mathcal{P}_f$ to curves in Γ. Moreover, $\mathcal{J}_g \subset \mathcal{J}_f$ and each component of $\mathcal{J}_g$ is a Jordan curve. Denote by

$$\mathcal{J}(\Gamma) = \bigcup_{n \geq 1} f^{-n}(\mathcal{J}_g).$$

Since $g^{-1}(\mathcal{J}_g) = \mathcal{J}_g$, we have $\mathcal{J}_g \subset f^{-1}(\mathcal{J}_g)$, each component of $\mathcal{J}_g$ is also a component of $f^{-1}(\mathcal{J}_g)$ and each component of $f^{-1}(\mathcal{J}_g)$ is a Jordan curve. Consequently, $f^{-1}(\mathcal{J}(\Gamma)) = \mathcal{J}(\Gamma)$ and each component of $\mathcal{J}(\Gamma)$ is a Jordan curve. By the definition of $\mathcal{J}_g$, we have:

$$\mathcal{J}(\Gamma) = \bigcup_{n \geq 1} \bigcap_{m \geq n} f^{-m}(\mathcal{A}).$$

Denote by $\mathcal{K}(\Gamma) = \widehat{\mathbb{C}} \backslash \mathcal{J}(\Gamma)$. It is completely invariant and

$$\mathcal{K}(\Gamma) = \bigcap_{n \geq 1} \bigcup_{m \geq n} (\widehat{\mathbb{C}} \backslash f^{-m}(\mathcal{A})).$$

Since $\partial \mathcal{A} \subset \partial f^{-1}(\mathcal{A})$, we have

$$\partial f^{-n+1}(\mathcal{A}) \subset \partial f^{-n}(\mathcal{A}) \subset \mathcal{K}(\Gamma) \text{ for } n \geq 1.$$

Recall that a connected subset $E \subset \widehat{\mathbb{C}}$ is of simple type (w.r.t. $\mathcal{P}_f$) if there exists either a simply-connected domain $U \subset \widehat{\mathbb{C}}$ such that $E \subset U$ and U contains at most one point in $\mathcal{P}_f$, or an annulus $A \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ such that $E \subset A$; and is of complex type (w.r.t. $\mathcal{P}_f$) otherwise. Since $f(\mathcal{P}_f) \subset \mathcal{P}_f$, for each simple type continuum $E \subset \widehat{\mathbb{C}}$, each component of $f^{-1}(E)$ is also simple type.

Set $\mathcal{B}^0 = \widehat{\mathbb{C}} \backslash \mathcal{A}$. It has $\#\Gamma + 1$ components and each of them is of complex type. For each component B of $\mathcal{B}^0$ and any component A of $f^{-n}(\mathcal{A})$ with $n \geq 1$, either $A \cap B = \emptyset$

or $A \subset B$ since $\partial\mathcal{A}_n \subset \partial\mathcal{A}_{n+1}$. In the latter case, the essential Jordan curves in A is either null-homotopic or peripheral since Γ is stable. Thus for each $n \geq 1$, $B \backslash f^{-n}(\mathcal{A})$ has exactly one complex type component.

Denote by $\mathcal{B}^n$ the union of complex type components of $\widehat{\mathbb{C}} \backslash f^{-n}(\mathcal{A})$ for $n \geq 1$. Then $\mathcal{B}^n \subset \mathcal{B}^{n-1}$ and it also has exactly $\#\Gamma + 1$ components since each component of $\mathcal{A} \backslash f^{-n}(\mathcal{A})$ is of simple type. Obviously, each component of $\mathcal{B}^n$ is also a component of $f^{-1}(\mathcal{B}^{n-1})$. Denote by

$$\mathcal{K}_c = \bigcap_{n \geq 0} \mathcal{B}^n.$$

Then $\mathcal{K}_c$ is compact and has exactly $\#\Gamma + 1$ components which are of complex type. Each of its components is also a component of $f^{-1}(\mathcal{K}_c)$. It is easy to verify that $\mathcal{K}_c \subset \mathcal{K}(\Gamma)$ and each component of $\mathcal{K}_c$ is also a component of $\mathcal{K}(\Gamma)$.

Each component of $\widehat{\mathbb{C}} \backslash \mathcal{K}_c$ is either a component of $\mathcal{A}$ or a simply-connected domain contains at most one point of $\mathcal{P}_f$. Therefore each component of $\mathcal{K}(\Gamma) \backslash \mathcal{K}_c$ is of simple type. In summary we have:

Proposition 5.3. *The compact set $\mathcal{K}_c$ is the union of complex type components of $\mathcal{K}(\Gamma)$. It has exactly $\#\Gamma + 1$ components and each of them is also a component of $f^{-1}(\mathcal{K}_c)$.*

By the above result, there exist periodic components in $\mathcal{K}_c$.

Theorem 5.4. *Let K be a periodic component of $\mathcal{K}_c$ with period $p \geq 1$. There exist domains $U \Subset V$ in $\widehat{\mathbb{C}}$ such that $K \subset U$ and $g = f^p|_U : U \to V$ is a renormalization of f with filled Julia set $\mathcal{K}_g = K$.*

Proof. Let $B_0, \cdots, B_{p-1}$ be the components of $\widehat{\mathbb{C}} \backslash \mathcal{A}$ such that $K \subset B_0$ and $f^i(K) \subset B_i$ for $0 < i < p$. Let $A_1, \cdots, A_n$ be the components of $\mathcal{A}$ whose boundary intersects B_0. Set

$$W' = B_0 \cup \bigcup_{i=1}^{n} A_i.$$

It is a finitely-connected domain. Let W_1' be the component of $f^{-p}(W')$ containing K. Then $W_1' \subset W'$ and each component of $W' \backslash \overline{W_1'}$ is either an annulus disjoint from $\mathcal{P}_f$ or a disk containing at most one point of $\mathcal{P}_f$ since Γ is stable.

Each A_i contains exactly one component of $W_1' \backslash K$, denoted by A_i^p, which is a component of $f^{-p}(\mathcal{A})$ and shares a common boundary component with A_i. By relabelling the index of A_i, we may assume that

$$f^p(A_1^p) = A_2, \cdots, f^p(A_{q-1}^p) = A_q \text{ and } f^p(A_q^p) = A_1,\ q \geq 1.$$

There is at least one of them, say A_i, such that $A_i^p \subsetneq A_i$. Otherwise $A_i^p = A_i$ for $1 \leq i \leq q$ and hence $f^{qp}(A_1) = A_1$. It contradicts the fact that $f : \mathcal{A}^1 \to \mathcal{A}$ is an annular system.

Assume that $A_1^p \subsetneq A_1$. There exists a Jordan curve γ_1 essentially contained in A_1 such that it is disjoint from A_1^p. Let γ_q' be the component of $f^{-p}(\gamma_1)$ in A_q^p. Then we can find a Jordan curve γ_q essentially contained in A_q such that γ_q' separates γ_q from K. Inductively, for $2 \leq i < q$, let γ_i' be the component of $f^{-p}(\gamma_{i+1})$ in A_i^p, we can find a Jordan curve γ_i essentially contained in A_i such that γ_i' separates γ_i from K. Since γ_1 is disjoint from A_1^p, the component of $f^{-p}(\gamma_2)$ in A_1 separates γ_1 from K as well.

Do this process for each cycle, we have a Jordan curve γ_i essentially contained in each periodic annulus A_i such that if $f^p(A_i^p) = A_j$, then the component of $f^{-p}(\gamma_j)$ in A_i separates γ_i from K. If A_i is not periodic but $A_j = f^p(A_i^p)$ is periodic, then there is always a Jordan curve γ_i essentially contained in A_i such that the component of $f^{-p}(\gamma_j)$ in A_i separates γ_i from K.

In summary, we have a Jordan curve $\gamma_i \subset A_i$ for each A_i such that if $f^p(A_i^p) = A_j$, then the component of $f^{-p}(\gamma_j)$ in A_i^p separates γ_i from K. Let $W \subset W'$ be the domain bounded by the curves γ_i defined above. Then $W_1 \Subset W$, where W_1 is the component of $f^{-p}(W)$ containing K, and each component of $W \backslash \overline{W_1}$ is either an annulus disjoint from $\mathcal{P}_f$ or a disk containing at most one point of $\mathcal{P}_f$.

Let W_n be the component of $f^{-np}(W)$ containing K for $n \geq 2$. Then $W_n \Subset W_{n-1}$ and each component of $W \backslash \overline{W_1}$ is either an annulus disjoint from $\mathcal{P}_f$, or a disk which contains at most one point of $\mathcal{P}_f$.

Since $\mathcal{P}_f$ is finite, there is an integer $N \geq 1$ such that $W_n \cap \mathcal{P}_f = W_N \cap \mathcal{P}_f$ for $n \geq N$. Set $U = W_{N+1}$, $V = W_N$ and $g := f^p|_U : U \to V$. Then every critical points of g stay in U. By Proposition 3.1, there is an integer $n \geq 1$ such that $\deg(f^n|_A) \geq 2$ for all the components A of $\mathcal{A}^n$. So we have $\deg g \geq 2$. Therefore $g : U \to V$ is a rational-like map.

Now we want to show that $\deg g < \deg f^p$. Otherwise $\mathcal{J}_f \subset \mathcal{K}_g$. But we know that the Julia set of the annular system $f : \mathcal{A}^1 \to \mathcal{A}$ is contained in $\mathcal{J}_f$. This is impossible. So $\deg g < \deg f^p$. It follows that g is a renormalization of f. $\square$

From Theorem 5.2, Theorem 5.4 and Theorem 2.1 in [34], we have:

Corollary 5.5. *Let K be a component of $\mathcal{K}_c$. For each component W of $\widehat{\mathbb{C}} \backslash K$, its boundary ∂W is locally connected.*

5.3 Topology of $\mathcal{K}(\Gamma)$

Recall that $\mathcal{K}_c$ is the union of complex type components of $\mathcal{K}(\Gamma)$. Each component of $f^{-n}(\mathcal{K}(\Gamma))$ for $n \geq 1$ is also a component of $\mathcal{K}(\Gamma)$. It is either a component of $\mathcal{K}_c$ or a simple type continuum which could be:

(a) (*disk-type*) a compact set contains exactly one point of $\mathcal{P}_f$; or

(b) (*annular-type*) a compact set disjoint from $\mathcal{P}_f$ but has exactly two complementary components contains points of $\mathcal{P}_f$; or

(c) (*trivial-type*) a compact set disjoint from $\mathcal{P}_f$ and has exactly one complementary components contains points of $\mathcal{P}_f$.

For each point $x \in \mathcal{P}_f \backslash \mathcal{K}_c$ (if exists), either $x \notin \cup_{n \geq 1} f^{-n}(\mathcal{K}_c)$ or there exists an integer $n_0(x) \geq 1$ such that $f^{-n_0(x)}(\mathcal{K}_c)$ has a disk-type component containing the point x. In both cases, there exists an integer $n_1(x) \geq 1$ such that $f^{-n_1(x)}(\mathcal{K}_c)$ has an annular-type component separating the point x from other points in $\mathcal{P}_f$. For each component A of $\mathcal{A}$, there exists an integer $n_2(A) \geq 1$ such that $f^{-n_2(A)}(\mathcal{K}_c)$ has an annular-type component contained in A essentially. Since $\mathcal{P}_f$ is finite and $\mathcal{A}$ has only finitely many components, there exists an integer $s \geq 1$ such that

$$s \geq \begin{cases} n_0(x) & \text{for } x \in (\cup_{n \geq 1} f^{-n}(\mathcal{K}_c)) \backslash \mathcal{K}_c, \\ n_1(x) & \text{for } x \in \mathcal{P}_f \backslash \mathcal{K}_c, \\ n_2(A) & \text{for each component } A \text{ of } \mathcal{A}. \end{cases}$$

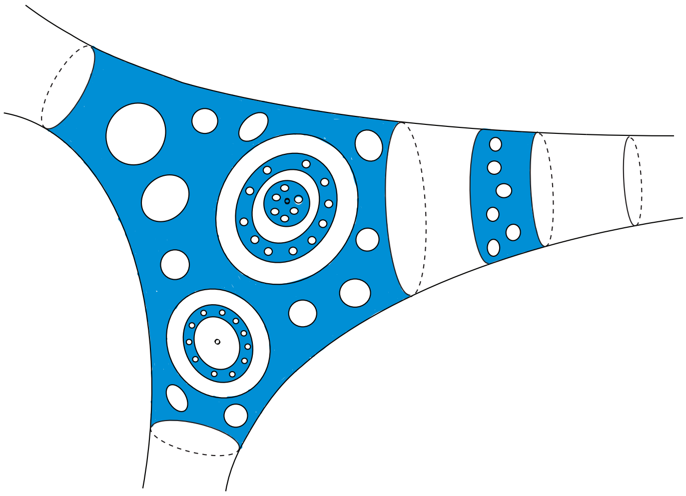

Figure 2. The components of $\mathcal{K}_e$.

Denote by $\mathcal{K}_e$ the union of components of $f^{-s}(\mathcal{K}_c)$ which are not trivial-type. Then $f(\mathcal{K}_e) \subset \mathcal{K}_e$. Denote by $\mathcal{U} = \widehat{\mathbb{C}} \setminus \mathcal{K}_e$. Then $f^{-1}(\mathcal{U}) \subset \mathcal{U}$. It usually has infinitely many components except the case that $\mathcal{K}_c \cup \mathcal{A} = \widehat{\mathbb{C}}$. Each component of $\mathcal{U}$ is either a simply-connected domain contains at most one point in $\mathcal{P}_f$, or a component of $f^{-s}(\mathcal{A})$ and hence is an annulus.

Denote by $\mathcal{U}_n = f^{-n}(\mathcal{U})$ for $n \geq 0$. Then $\mathcal{U}_{n+1} \subset \mathcal{U}_n$. Each component of $\mathcal{U}_n$ is either a simply-connected domain contains at most one point in $\mathcal{P}_f$, or a component of $f^{-n-s}(\mathcal{A})$.

Denote by $\mathcal{K}_r = \mathcal{K}(\Gamma) \setminus \cup_{n \geq 0} f^{-n}(\mathcal{K}_c)$. For each component K of $\mathcal{K}_r$, denote by $U_n(K)$ the component of $\mathcal{U}_n$ that contains K for each $n \geq 0$.

Lemma 5.6. *For each component K of $\mathcal{K}_r$ and each $n \geq 0$, there exists an integer $m > n$ such that $U_m(K) \Subset U_n(K)$.*

Proof. If $U_n(K)$ is an annulus, then it is a component of $f^{-n-s}(\mathcal{A})$. Pick a point $z \in K$. From Proposition 3.3 (2), there exist two components A_1, A_2 of $f^{-m-s}(\mathcal{A})$ for some $m > n$ such that both A_1 and A_2 are contained in $U_n(K)$ essentially and the 2-connected continuum between A_1 and A_2, denoted by E, contains the point z. Note that both ∂A_1 and ∂A_2 are contained in $f^{-m-s}(\mathcal{K}_c)$. Thus $U_m(K) \subset E$ and hence $U_m(K) \Subset U_n(K)$.

The same argument works when $U_n(K)$ is simply-connected. □

Lemma 5.7. *Let K be a component of $\mathcal{K}_r$ and $n \geq 0$ be an integer. If $U_n(K)$ is an annulus, then there exists an integer $m > n$ such that $U_m(K)$ is either simply-connected or an annulus but is not contained in $U_n(K)$ essentially.*

Proof. Otherwise, $\{U_m(K)\}$ are all annuli and $U_{m+1}(K)$ is contained in $U_m(K)$ essentially for all $m \geq n$. Then $\cap_{n \geq 0} U_n(K)$ is a component of $\mathcal{J}(\Gamma)$. Contradiction. □

Let K be a component of $\mathcal{K}_r$. By Lemma 5.7, we know that as n is large enough, either $U_n(K)$ is simply-connected, or $U_n(K)$ is an annulus and one component $E_n(K)$ of $\widehat{\mathbb{C}} \setminus U_n(K)$ contains at most one point of $\mathcal{P}_f$. Denote by $V_n(K) = U_n(K) \cup E_n(K)$ in the case or $V_n(K) = U_n(K)$ otherwise. Then $V_n(K)$ is a simply-connected domain contains at most one point of $\mathcal{P}_f$ as n is large enough and $V_{n+1}(K) \subset V_n(K)$ for all $n \geq 0$. Moreover, for each integer $n \geq 0$, there exists an integer $m > n$ such that $V_m(K) \subset U_n(K)$. Thus $\cap_{n>0} V_n(K) = \cap_{n>0} U_n(K)$ and it is disjoint from $\cup_{n\geq 0} f^{-n}(\mathcal{K}_c)$ and $\mathcal{J}(\Gamma)$. Thus we have

$$K = \bigcap_{n>0} U_n(K) = \bigcap_{n>0} V_n(K).$$

Corollary 5.8. *Each component of $\mathcal{K}(\Gamma)$ is compact.*

Proposition 5.9. *Let K be a periodic component of $\mathcal{K}_r$. Then K is either a single point or the closure of a quasi-disk, which is a periodic Fatou domain of f.*

Proof. Let K be a periodic component of $\mathcal{K}_r$ with period $p \geq 1$. Then as n is large enough, $V_n(K)$ is simply-connected and $f^p : V_{n+p}(K) \to V_n(K)$ is proper with at most one critical point. Moreover $V_{n+p}(K) \Subset V_n(K)$ by Lemma 5.6. If K contains no super-attracting periodic points of f, then

$$\deg(f^p : V_{n+p}(K) \to V_n(K)) = 1$$

as n is large enough. Thus K is a single point. Otherwise $f^p : V_{n+p}(K) \to V_n(K)$ is a polynomial-like map and K is the closure of a quasi-disk, which is a periodic Fatou domain of f. □

The set $\mathcal{U}$ can be decomposed into $\mathcal{U} = \mathcal{D} \sqcup \mathcal{G} \sqcup \mathcal{R}$ by the following:

(1) $\mathcal{D}$ consists of simply-connected components of $\mathcal{U}$ which are disjoint from $\mathcal{P}_f$ (hence are components of $\mathcal{K}_c$).

(2) $\mathcal{G}$ consists of simply-connected components of $\mathcal{U}$ which contains exactly one point of $\mathcal{P}_f$.

(3) $\mathcal{R}$ consists of annular components of $\mathcal{U}$ (each of them is component of $f^{-s}(\mathcal{A})$ which is either contained essentially in $\mathcal{A}$ or peripheral around a point $x \in \mathcal{P}_f \backslash \mathcal{K}_c$, i.e. it separates the point x from other points of $\mathcal{P}_f$).

Obviously, both $\mathcal{G}$ and $\mathcal{R}$ have only finitely many components, but $\mathcal{D}$ either is empty or has infinitely many points.

Lemma 5.10. *For each component G of $\mathcal{G}$, $f^{-1}(G) \cap \mathcal{R} = \emptyset$ and G contains exactly one component of $f^{-1}(\mathcal{G})$.*

Proof. Assume that $f^{-1}(G)$ has a component $W \subset \mathcal{R}$, then $G = f(W) \subset f^{-s+1}(\mathcal{A})$. But $f^{-s+1}(\mathcal{A})$ is disjoint from $\mathcal{P}_f$. Contradiction.

Denote by x be the unique point of $\mathcal{P}_f \cap G$. Then $f(x)$ is also contained in a component G_1 of $\mathcal{G}$. Thus $f^{-1}(G_1)$ has a component in G.

Let W_1 be the component of $\widehat{\mathbb{C}} \backslash \mathcal{K}_c$ that contains $f(x)$. Let W_0 be the component of $f^{-1}(W_1)$ that contains the point x. Then $G \subset W_0$. Since W_0 contains a unique point in $f^{-1}(P_f)$, so does G. Therefore G contains exactly one component of $f^{-1}(\mathcal{G})$. □

Split the set $\mathcal{D}$ into $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}''$ by their components according to whether or not they intersect $f^{-1}(\mathcal{P}_f \cup \mathcal{R})$. Since $f^{-1}(\mathcal{U}) \subset \mathcal{U}$ and $\mathcal{R}$ has only finitely many components, we know that $\mathcal{D}'$ has only finitely many components.

Lemma 5.11. *Let K be a wandering component of $\mathcal{K}(\Gamma)$. There exists an integer $n > 0$ such that $f^n(K) \subset \mathcal{D}'$.*

Proof. At first, we claim that there exists an integer $m > 0$ such that $f^m(K) \subset \mathcal{D}$. Assume by contradiction that $f^n(K) \subset \mathcal{R} \cup \mathcal{G}$ for all $n > 0$.

Suppose that there exists an integer $n_0 > 0$ such that $f^n(K) \subset \mathcal{R}$ for all $n \geq n_0$. There is a component A of $\mathcal{A}$ such that $f^{n_0+s}(K) \subset A$ since each component of $\mathcal{R}$ is a component of $f^{-s}(\mathcal{A})$ which is either contained essentially in $\mathcal{A}$ or peripheral around a point $x \in \mathcal{P}_f \backslash \mathcal{K}_c$. On the other hand, there are at least one component of $\mathcal{K}_e$ contained in A. Thus each component of $A \backslash \mathcal{K}_e$ is either a component of $\mathcal{D}$ or is contained in $\mathcal{A}^1$. Thus $f^{n_0+s}(K) \subset \mathcal{A}^1$ by the assumption. Therefore $f^{n+s}(K) \subset \mathcal{A}^1$ for all $n \geq n_0$. Consequently, $K \subset \mathcal{J}(\Gamma)$. Contradiction.

From the assumption, there exists an integer $k > 1$ such that $f^k(K) \subset \mathcal{G}$. Then $f^{k-1}(K) \subset \mathcal{G} \cup \mathcal{D}$ by Lemma 5.10 and hence $f^{k-1}(K) \subset \mathcal{G}$ by the assumption. Therefore $f^n(K) \subset \mathcal{G}$ for all $n > 0$. It follows that K is eventually periodic by Lemma 5.10. It is a contradiction. Now the claim is proved.

By the claim, there exists an integer $m > 0$ and a component W of $\mathcal{D}$ such that $f^m(K) \subset W$. If W is a component of $\mathcal{D}''$, then $f^{m+1}(K) \subset \mathcal{D}$ since W is disjoint from $f^{-1}(\mathcal{P}_f)$ and $f^{-1}(\mathcal{R})$. Assume by contradiction that $f^{n+m}(K) \subset \mathcal{D}''$ for all $n \geq 1$. Let W_n be the component of $\mathcal{D}''$ that contains $f^{n+m}(K)$. Since each of them is a component of $\mathcal{K}_c$, we have $f(W_n) = W_{n+1}$. Thus they are either eventually periodic or wandering. Since $\partial W_n \subset \mathcal{J}_f$, the former case is impossible since they are disjoint from $\mathcal{P}_f$. The latter case is also impossible by Sullivan's no wandering Fatou domain Theorem. □

Proposition 5.12. *Each wandering component of* $\mathcal{K}(\Gamma)$ *is a single point.*

Proof. For each simply-connected domain $D \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$, denote by

$$\text{h-diameter}(D) = \sup_{z,w \in D} \ell[\gamma(z,w)],$$

where $\gamma(z,w)$ is an arc in D connecting the points z and w, and $\ell[\gamma(z,w)]$ is the infimum of the length of arcs in $\widehat{\mathbb{C}} \backslash \mathcal{P}_f$ under the orbifold metric over all the arcs in $\widehat{\mathbb{C}} \backslash \mathcal{P}_f$ homotopic to $\gamma(z,w)$ rel $\mathcal{P}_f \cup \{z, w\}$.

It is easy to verify that if $\overline{D}$ is locally connected and disjoint from super-attracting periodic points of f, then h-diameter$(D) < \infty$.

For each component D of $\mathcal{D}$, $\overline{D}$ is locally connected by Corollary 5.5 and disjoint from super-attracting periodic points of f. Thus h-diameter$(D) < \infty$. Therefore there exists a constant $M < \infty$ such that h-diameter$(D) \leq M$ for each component D of $\mathcal{D}'$.

Since $\overline{\mathcal{D}'}$ is disjoint from superattracting cycles of f, there exists a constant $\lambda > 1$ such that $\|f'\| \geq \lambda$ on $f^{-1}(\mathcal{D}')$ under the orbifold metric.

Let K be a wandering component of $\mathcal{K}_r$. From Lemma 5.11, there exists an infinite increasing sequence $\{n_k\}$ of positive integers such that $f^{n_k}(K) \subset D_k$ and D_k are components of $\mathcal{D}'$. Let W_k be the component of $f^{-n_k}(D_k)$ that contains K, then h-diameter$(W_k) \leq M\lambda^{-k}$. Thus the diameter of W_k converges to zero as $k \to \infty$. It follows that K is a single point. □

Proof of Theorem 1.3. Combining Propositions 5.3, 5.9, 5.12 and Theorem 5.4, we obtain Theorem 1.3. □

6 Coding the component-wise dynamics

In this section we will prove Theorem 1.4, which is suggested by Pilgrim. At first we give some definitions.

Definition 4. *A* **dendrite** *is a locally connected and uniquely arc-wise connected continuum. Let $\mathcal{T} \subset \widehat{\mathbb{C}}$ be a dendrite. A continuous onto map $\tau : \mathcal{T} \to \mathcal{T}$ is called a* **finite dendrite map** *if there exists a finite tree $T_0 \subset \mathcal{T}$ such that the following statements hold.*

(1) For each $n \geq 1$, $T_n := \tau^{-n}(T_0)$ is also a finite tree with $v(T_n) = \tau^{-n}(v(T_0))$, where $v(\cdot)$ denotes the set of vertices of a tree.

(2) $T_n \subset T_{n+1}$ and $v(T_n) \subset v(T_{n+1})$.

(3) τ is a homeomorphism on each edge of T_n.

(4) $\cup_{n\geq 0} T_n$ is dense in $\mathcal{T}$.

(5) $\deg \tau := \sup\{\#\tau^{-1}(x), x \in \mathcal{T}\} < \infty$.

6.1 The tower of tree maps

Definition 5. *By* **a tower of tree maps** *we mean an infinite sequence of triples $\{T_n, \iota_n, \tau_n\}_{n\geq 0}$, where T_n are finite trees, $\iota_n : T_n \to T_{n+1}$ are inclusions and $\tau_n : T_{n+1} \to T_n$ are continuous onto maps such that:*
(1) $\iota_n(\mathcal{V}_n) \subset \mathcal{V}_{n+1}$, where $\mathcal{V}_n$ is the set of vertices of T_n;
(2) $\tau_n^{-1}(\mathcal{V}_n) = \mathcal{V}_{n+1}$; and
(3) the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\tau_n"] & T_n \arrow[r, "\tau_{n-1}"] \arrow[d, "\iota_n"'] & T_{n-1} \arrow[r, "\tau_{n-2}"] \arrow[d, "\iota_{n-1}"'] & \cdots \arrow[r, "\tau_1"] & T_1 \arrow[r, "\tau_0"] \arrow[d, "\iota_1"'] & T_0 \arrow[d, "\iota_0"] & \\
\cdots \arrow[r, "\tau_{n+1}"] & T_{n+1} \arrow[r, "\tau_n"] & T_n \arrow[r, "\tau_{n-1}"] & \cdots \arrow[r, "\tau_2"] & T_2 \arrow[r, "\tau_1"] & T_1 \arrow[r, "\tau_0"] & T_0
\end{tikzcd}$$

The degree of the tree map $\tau_n : T_{n+1} \to T_n$ is defined by

$$\deg \tau_n = \sup\{\#\tau_n^{-1}(y), y \in T_n\}.$$

Note that the sequence $\{\deg \tau_n\}$ is increasing. The degree of the tower is defined to be its limit as $n \to \infty$.

A tower of tree maps $\{T_n, \iota_n, \tau_n\}_{n\geq 0}$ is called **expanding** if there exist a constant $\lambda > 1$ and a linear metric on T_0 such that $(\tau_0 \circ \iota_0) : T_0 \to T_0$ is C^1 under this linear metric and the norm of its derivative is bigger than λ on T_0.

Theorem 6.1. *Let $\{T_n, \iota_n, \tau_n\}_{n\geq 0}$ be an expanding tower of tree maps. Suppose that its degree is bounded. There exist an expanding finite dendrite map $\tau : \mathcal{T} \to \mathcal{T}$ and inclusions $i_n : T_n \to \mathcal{T}$ for all $n \geq 0$ such that $i_n = i_{n+1} \circ \iota_n$ and the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r, "\tau_{n+1}"] & T_{n+1} \arrow[r, "\tau_n"] \arrow[d, "i_{n+1}"'] & T_n \arrow[r, "\tau_{n-1}"] \arrow[d, "i_n"'] & \cdots \arrow[r, "\tau_1"] & T_1 \arrow[r, "\tau_0"] \arrow[d, "i_1"'] & T_0 \arrow[d, "i_0"] \\
\cdots \arrow[r, "\tau"] & \mathcal{T} \arrow[r, "\tau"] & \mathcal{T} \arrow[r, "\tau"] & \cdots \arrow[r, "\tau"] & \mathcal{T} \arrow[r, "\tau"] & \mathcal{T}
\end{tikzcd}$$

Moreover the finite dendrite map $\tau : \mathcal{T} \to \mathcal{T}$ is unique up to topological conjugacy.

We will call $\tau : \mathcal{T} \to \mathcal{T}$ the **limit** of the tower of tree maps $\{T_n, \iota_n, \tau_n\}_{n\geq 0}$.

Proof. Let $|\cdot|$ be an expanding linear metric on T_0, i.e. there exists a constant $\lambda > 1$ such that $|(\tau_0 \circ \iota_0)'| > \lambda$ on T_0. Define a metric on $\iota_0(T_0) \subset T_1$ such that ι_0 is an isometry and a metric on $T_1 \backslash \iota_0(T_0)$ such that the norm of the derivative of τ_0 is a constant $\lambda_1 > \max\{\lambda, d\}$, where d is the degree of the tower. Then we get a metric on T_1 such that ι_0 is an isometry and $\lambda_1 \geq |\tau_0'| \geq \lambda$ on T_1.

Inductively, we can define a metric on each T_n with $n \geq 1$ such that ι_{n-1} is an isometry and $\lambda_1 \geq |\tau_{n-1}'| \geq \lambda$ on T_n.

Denote by $\widetilde{\mathcal{S}}$ the space consisting of left infinite sequences $(\cdots, t_1, t_0)$ with $t_n \in T_{n+k}$ for some integer $k \geq 0$, such that for any $n \geq 0$, if $t_n \in T_{n+k}$, then $t_{n+1} = \iota_{n+k}(t_n)$. Define an equivalent relation on $\widetilde{\mathcal{S}}$ by

$$(\cdots, s_1, s_0,) \sim (\cdots, t_1, t_0)$$

if there exists an integer k such that $s_n = t_{n+k}$ whenever $n, n+k \geq 0$. Let $\mathcal{S}$ be the quotient space $\widetilde{\mathcal{S}}/\sim$. For each point $(\cdots, t_1, t_0)$ in $\widetilde{\mathcal{S}}$, we denote by $[\cdots, t_1, t_0]$ representing its equivalence class in $\mathcal{S}$. Since ι_n is an isometry, there exists a metric ρ on $\mathcal{S}$ such that the inclusion $i_n : T_n \to \mathcal{S}$ defined by:

$$i_n(t) = [\cdots, \iota_{n+1} \circ \iota_n(t), \iota_n(t), t]$$

is an isometry. Clearly, $i_{n+1} \circ \iota_n = i_n$ on T_n.

Since $\tau_n \circ \iota_n = \iota_{n-1} \circ \tau_{n-1}$ on T_n, there exists a continuous onto map $\tau : \mathcal{S} \to \mathcal{S}$ such that $\tau \circ i_n = i_{n-1} \circ \tau_{n-1}$ on T_n. Moreover $\lambda_1 \geq |\tau'| \geq \lambda$.

Since $\lambda_1 > d$, $\mathcal{S}$ is bounded. Let $\mathcal{T}$ be the completion of $\mathcal{S}$. Then it is a dendrite. The map τ can be extended to be a continuous onto map on $\mathcal{T}$ since τ is uniformly continuous.

The proof of the uniqueness of $\tau : \mathcal{T} \to \mathcal{T}$ is direct. □

6.2 Coding the quotient action

Let f be a post-critically finite rational map with a stable Cantor multicurve Γ. We want to define a tower of tree maps from f and construct a semi-conjugacy from f to its limit.

Denote by Γ_n the collection of the curves in $f^{-n}(\Gamma)$ for $n \geq 0$. Then each curve in Γ_n is essential in $\widehat{\mathbb{C}} \backslash f^{-n}(\mathcal{P}_f)$ and no two of them are homotopic in $\widehat{\mathbb{C}} \backslash f^{-n}(\mathcal{P}_f)$.

For each curve $\gamma \in \Gamma_n$, denote by $\Gamma_{n+1}(\gamma)$ the curves in Γ_{n+1} homotopic to γ rel $f^{-n}(\mathcal{P}_f)$. Since Γ is pre-stable and stable, the next lemma is easy to check:

Lemma 6.2. *For each curve $\gamma \in \Gamma_n$, $\Gamma_{n+1}(\gamma) \neq \emptyset$ and any curve in $\Gamma_{n+1} \backslash \Gamma_{n+1}(\gamma)$ does not separate curves in $\Gamma_{n+1}(\gamma)$. Moreover, for any two curves γ_1 and γ_2 in Γ_n, if there is no curve in Γ_n separating γ_1 from γ_2, then each curve in $\Gamma_{n+1} \backslash (\Gamma_{n+1}(\gamma_1) \cup \Gamma_{n+1}(\gamma_2))$ does not separate curves in $\Gamma_{n+1}(\gamma_1) \cup \Gamma_{n+1}(\gamma_2)$.*

Dual trees. For any $n \geq 0$, let T_n be the dual tree of Γ_n defined by the following: There is a bijection between vertices of T_n and components of $\widehat{\mathbb{C}} \backslash \Gamma_n$. Two vertices are connected by an edge if their corresponding components of $\widehat{\mathbb{C}} \backslash \Gamma_n$ have a common boundary component, which is a curve in Γ_n. Thus there is a bijection between edges of T_n and curves in Γ_n. Denote by e_γ the edge of T_n corresponding to the curve $\gamma \in \Gamma_n$.

Inclusion maps. The homotopy rel $f^{-n}(\mathcal{P}_f)$ induces an inclusion $\iota_n : T_n \to T_{n+1}$ by the following: For each curve $\gamma \in \Gamma_n$, define

$$\iota_n : e_\gamma \to \bigcup_{\beta \in \Gamma_{n+1}(\gamma)} e_\beta \cup \{\text{common endpoints of } e_\beta\}$$

to be a homeomorphism such that it preserves the orientation induced by a choice of orientations on these curves.

By the definition, ι_n is continuous on each edge. The continuity of ι_n at vertices comes from Lemma 6.2. The injectivity comes from the fact that no two curves in Γ_n are homotopic rel $f^{-n}(\mathcal{P}_f)$.

Induced tree maps. Given any $n \geq 0$. A continuous map $\tau : T_{n+1} \to T_n$ is called an induced tree map if for each edge e_γ of T_{n+1} corresponding to a curve $\gamma \in \Gamma_{n+1}$, $\tau : e_\gamma \to e_{f(\gamma)}$ is a homeomorphism such that it preserves the orientation induced by the map $f : \gamma \to f(\gamma)$. It is easy to check that induced tree maps always exist.

Lemma 6.3. *There exists a linear metric ρ_1 on the tree T_1 and an induced tree map $\tau_0 : T_1 \to T_0$ such that $\iota_0 \circ \tau_0$ is linear on each edge of T_1 and $|(\iota_0 \circ \tau_0)'| \geq \lambda$ for some constant $\lambda > 1$.*

Proof. Let $\{e_1, \cdots, e_n\}$ be the edges of T_0. Let $M = (b_{ij})$ be the reduced transition matrix of Γ defined in §2. Then its leading eigenvalue $\lambda_0 > 1$ by Lemma 2.4 since Γ is a Cantor multicurve. Thus there exist a constant $\lambda \in (1, \lambda_0)$ and a positive eigenvector $\mathbf{v} = (v(e_i))$ such that $M\mathbf{v} > \lambda\mathbf{v}$ by Lemma A.1 in [8]. Define a linear metric ρ_1 on T_1 such that for each edge e of T_1, it has length $v(\tau_0(e))$. Then the length of $\iota_0(e_i)$ is:

$$|\iota_0(e_i)| = \sum_j b_{ij}\, v(e_j) > \lambda v(e_i).$$

Define $\tau_0 : T_1 \to T_0$ to be an induced tree map such that $\iota_0 \circ \tau_0$ is linear on each edge of T_1. Then $|(\iota_0 \circ \tau_0)'| > \lambda$. □

There exists an induced tree map $\tau_1 : T_2 \to T_1$ such that $\tau_1 \circ \iota_1 = \iota_0 \circ \tau_0$ on T_1. Inductively, for each $n \geq 2$, there exists an induced tree map $\tau_{n-1} : T_n \to T_{n-1}$ such that $\tau_{n-1} \circ \iota_{n-1} = \iota_{n-2} \circ \tau_{n-2}$ on T_{n-1}. Then $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$ is an expanding tower of tree maps with degree $\deg \tau_n \leq \deg f$. We call it the **induced tower of tree maps** of f with respect to the multicurve Γ.

Denote by $\tau_f : \mathcal{T}(\Gamma) \to \mathcal{T}(\Gamma)$ the limit of the induced tower of tree maps of f with respect to the multicurve Γ. Then it is an expanding finite dendrite map by Theorem 6.1. The next theorem is a more precise version of Theorem 1.4.

Theorem 6.4. *Let f be a post-critically rational map with a stable Cantor multicurve Γ. There exist an expanding finite dendrite map $\tau_f : \mathcal{T}(\Gamma) \to \mathcal{T}(\Gamma)$ and a continuous onto map $\Theta : \widehat{\mathbb{C}} \to \mathcal{T}(\Gamma)$ such that $\tau_f \circ \Theta = \Theta \circ f$. Moreover, for each point $t \in \mathcal{T}(\Gamma)$, the fiber $\Theta^{-1}(t)$ is a component of either $\mathcal{J}(\Gamma)$ or $\mathcal{K}(\Gamma)$.*

Proof. Let $\{T_n, \iota_n, \tau_n\}_{n \geq 0}$ be the induced tower of tree maps of f with respect to Γ. We may identify T_n with $i_n(T_n) \subset \mathcal{T}(\Gamma)$ by Theorem 6.1. Then $\tau_n = \tau_f$. Let $\mathcal{I} \subset T_1$ be the union of open edges of T_1 contained in T_0. Let σ be the restriction of τ_f on $\mathcal{I}$. Then σ

is an expanding linear system. So $\mathcal{J}_\sigma$ is dense in $\mathcal{I}$. Let $g = f : \mathcal{A}^1 \to \mathcal{A}$ be the exact annular system obtained in Theorem 1.1. There exists a bijection from the components of $\mathcal{A}^1$ to the components of $\mathcal{I}$ according to the correspondence from Γ_1 to the edges of T_1 and the homotopy rel $f^{-1}(\mathcal{P}_f)$.

Define a map $\Theta_0 : \mathcal{J}_g \to \mathcal{J}_\sigma$ by the itinerary as in Proposition 3.6. It is order-preserving. Since its image $\mathcal{J}_\sigma$ is dense in $\mathcal{I}$, it can be extended to a continuous onto map $\Theta_0 : \widehat{\mathbb{C}} \to T_0$ such that each component of $\widehat{\mathbb{C}}\backslash\mathcal{A}$ maps to a vertex of T_0. It is easy to check that $\tau_f \circ \Theta_0 = \Theta_0 \circ f$ on $\mathcal{A}^1$.

Pullback the map Θ_0 by the above equation, we get a continuous onto map $\Theta_1 : \widehat{\mathbb{C}} \to T_1$ such that each component of $\widehat{\mathbb{C}}\backslash f^{-1}(\mathcal{A})$ maps to a vertex of T_1 and $\tau_f \circ \Theta_1 = \Theta_1 \circ f$ on $f^{-1}(\mathcal{A})$.

Inductively, we get a sequence of continuous maps $\Theta_n : \widehat{\mathbb{C}} \to T_n$ such that each component of $\widehat{\mathbb{C}}\backslash f^{-n}(\mathcal{A})$ maps to a vertex of T_n and $\tau_f \circ \Theta_n = \Theta_n \circ f$ on $f^{-n}(\mathcal{A})$. It is easy to check that Θ_n converges uniformly to a continuous onto map $\Theta : \widehat{\mathbb{C}} \to \mathcal{T}(\Gamma)$ as $n \to \infty$ and the map Θ satisfies all the conditions. $\square$

7 Wandering continua

We will prove Theorem 1.2 here.

Definition 6. *Let f be a rational map. By a* **wandering continuum** *we mean a non-degenerate continuum $K \subset \mathcal{J}_f$ (i.e. K is a connected compact set consisting of more than one point) such that $f^n(K) \cap f^m(K) = \emptyset$ for any $n > m \geq 0$.*

A continuum $E \subset \widehat{\mathbb{C}}\backslash\mathcal{P}_f$ is called **essential** if there are exactly two components of $\widehat{\mathbb{C}}\backslash E$ containing points of $\mathcal{P}_f$ and each of them contains at least two points of $\mathcal{P}_f$.

Lemma 7.1. *Let f be a post-critically finite rational map. Suppose that $K \subset \mathcal{J}_f$ is a wandering continuum. Then either $f^n(K)$ is 1-connected for all $n \geq 0$; or there exists an integer $N \geq 0$ such that $f^n(K)$ is essential for $n \geq N$.*

Proof. Set $K_n = f^n(K)$ for $n \geq 0$. Since $\#\mathcal{P}_f < \infty$ and K is wandering, we have $K_n \cap \mathcal{P}_f = \emptyset$ for all $n \geq 0$. Thus if K_n is 1-connected, then K_m is also 1-connected for $m \leq n$.

Suppose that there is an integer $n_0 \geq 1$ such that K_{n_0} is not 1-connected, then K_n is not 1-connected for all $n \geq n_0$. Let $p(K_n) \geq 1$ be the number of components of $\widehat{\mathbb{C}}\backslash K_n$ containing points of $\mathcal{P}_f$. Since K_n are pairwise disjoint, there are at most $(\#\mathcal{P}_f - 2)$ continua K_n such that $p(K_n) \geq 3$. Thus there is an integer $n_1 \geq n_0$ such that $p(K_n) \leq 2$ for all $n \geq n_1$.

If $p(K_n) \equiv 1$ for all $n \geq n_1$, let $\widehat{K}_n$ be the union of K_n together with the components of $\widehat{\mathbb{C}}\backslash K_n$ disjoint from $\mathcal{P}_f$, then $f : \widehat{K}_n \to \widehat{K}_{n+1}$ is a homeomorphism for $n \geq n_1$. Since K_{n_1} is not 1-connected, $\widehat{K}_{n_1}\backslash K_{n_1}$ is non-empty. Let U be a component of $\widehat{K}_{n_1}\backslash K_{n_1}$. Then $U \cap \mathcal{P}_f = \emptyset$. If $U \cap \mathcal{J}_f \neq \emptyset$, then $f^m(U) \supset \mathcal{J}_f$ for some $m \geq 1$. But $f^m(U)$ is a component of $\widehat{\mathbb{C}}\backslash K_{n_1+m}$. It is a contradiction. So $U \cap \mathcal{J}_f = \emptyset$. Noticing that $\partial U \subset K_{n_1} \subset \mathcal{J}_f$, the simply-connected domain U is exactly a Fatou domain. But ∂U is wandering. Thus it is a contradiction since there is no wandering Fatou domain by Sullivan's theorem (refer to [24]). Therefore there is an integer $n_2 \geq n_1$ such that $p(K_{n_2}) = 2$.

We claim that $p(K_n) \equiv 2$ for all $n \geq n_2$. Otherwise, assume that there is an integer $m > n_2$ such that $p(K_m) = 1$, then there is a disk D containing K_m such that $D \cap \mathcal{P}_f = \emptyset$. Let D_n be the component of $f^{n-m}(D)$ containing K_n for $n_2 \leq n \leq m$. Then D_n is disjoint from $\mathcal{P}_f$. So $p(K_n) = 1$ for $n_2 \leq n \leq m$. This contradicts $p(K_{n_2}) = 2$.

We may assume $\#\mathcal{P}_f \geq 3$ (otherwise f is conjugate to the map $z \to z^{\pm d}$ and hence has no wandering continuum), then f has at most one exceptional point. If there is an integer $m \geq n_2$ such that $\widehat{\mathbb{C}} \backslash K_m$ has a component containing exactly one $\mathcal{P}_f$ point, then there is a disk $D \supset K_m$ such that D contains exactly one $\mathcal{P}_f$ point. Let D_n be the component of $f^{n-m}(D)$ containing K_n for $n_2 \leq n \leq m$. Then D_n is simply-connected and contains at most one point of $\mathcal{P}_f$. Thus $\widehat{\mathbb{C}} \backslash K_n$ has a component containing exactly one $\mathcal{P}_f$ point for $n_2 \leq n \leq m$. Therefore either there exists an integer $N \geq n_2$ such that for $n \geq N$, $f^n(K)$ is essential, or $\widehat{\mathbb{C}} \backslash f^n(K)$ has a component containing exactly one $\mathcal{P}_f$ point for all $n \geq n_2$.

In the latter case, denote by U the component of $\widehat{\mathbb{C}} \backslash K_{n_2}$ containing exactly one $\mathcal{P}_f$ point. If $U \cap \mathcal{J}_f \neq \emptyset$, then there is an integer $k > 0$ such that $\widehat{\mathbb{C}} \backslash f^k(U)$ contains at most one point (an exceptional point). On the other hand, there is a disk $D \supset K_{n_2+k}$ such that D contains exactly one $\mathcal{P}_f$ point. Let D_{n_2} be the component of $f^{-k}(D)$ containing K_{n_2}. Then D_n is simply-connected and contains at most one point of $\mathcal{P}_f$. Thus $U \subset D_{n_2}$. Therefore $f^k(U) \subset D$ and hence $\widehat{\mathbb{C}} \backslash D \subset \widehat{\mathbb{C}} \backslash f^k(U)$ contains at most one point. This contradicts $\#\mathcal{P}_f \geq 3$. So U is disjoint from $\mathcal{J}_f$ and hence is a simply-connected Fatou domain. This again contradicts Sullivan's no wandering Fatou domain theorem. □

Lemma 7.2. *Suppose that $K \subset \mathcal{J}_f$ is a wandering continuum and is not* 1*-connected. There is a multicurve Γ_K such that:*

(1) for each curve γ in Γ_K, there are infinitely many continua $f^n(K)$ homotopic to γ rel $\mathcal{P}_f$, and

(2) there is an integer $N_1 \geq 0$ such that for $n \geq N_1$, $f^n(K)$ is essential and homotopic rel $\mathcal{P}_f$ to a curve in Γ_K.

Proof. By Lemma 7.1, there is an integer $N \geq 0$ such that $f^n(K)$ is essential for $n \geq N$. Since the $f^n(K)$ are pairwise disjoint, for any integer $m \geq N$, we may choose an essential Jordan curve β_n in $\widehat{\mathbb{C}} \backslash \mathcal{P}_f$ for $N \leq n \leq m$ such that they are pairwise disjoint and $f^n(K)$ is homotopic to β_n rel $\mathcal{P}_f$. Let Γ_m be the collection of these curves. Let $\tilde{\Gamma}_m \subset \Gamma_m$ be a multicurve such that each curve in Γ_m is homotopic to a curve in $\tilde{\Gamma}_m$. Then each curve in $\tilde{\Gamma}_m$ is homotopic to a curve in $\tilde{\Gamma}_{m+1}$. This implies that $\#\tilde{\Gamma}_m$ is increasing and hence there is an integer $m_0 \geq N$ such that $\#\tilde{\Gamma}_m$ is a constant for $m \geq m_0$ since any multicurve contains at most $\#\mathcal{P}_f - 3$ curves. Therefore each curve in $\tilde{\Gamma}_{m+1}$ is homotopic to a curve in $\tilde{\Gamma}_m$ for $m \geq m_0$. This shows that the multicurves $\tilde{\Gamma}_m$ are homotopic to each other for all $m \geq m_0$.

Let $\Gamma_K \subset \tilde{\Gamma}_{m_0}$ be the sub-collection consisting of curves $\gamma \in \tilde{\Gamma}_{m_0}$ such that there are infinitely many $f^n(K)$ homotopic to γ rel $\mathcal{P}_f$. Then it is non-empty and hence is a multicurve. Obviously, Γ_K is uniquely determined by K and there is an integer $N_1 \geq 0$ such that for $n \geq N_1$, $f^n(K)$ is essential and homotopic rel $\mathcal{P}_f$ to a curve in Γ_K. □

Lemma 7.3. *Γ_K is an irreducible Cantor multicurve.*

Proof. By Lemma 7.2, there exists an integer $N_1 \geq 0$ such that $f^n(K)$ for $n \geq N_1$ is homotopic to a curve in Γ_K rel $\mathcal{P}_f$. Thus Γ_K is pre-stable. For any pair $(\gamma, \alpha) \in \Gamma_K \times \Gamma_K$, there are integers $k_2 > k_1 \geq N_1$ such that $f^{k_1}(K)$ is homotopic to γ and $f^{k_2}(K)$ is homotopic to α rel $\mathcal{P}_f$. Thus $f^{k_1-k_2}(\alpha)$ has a component δ homotopic to γ rel $\mathcal{P}_f$. So for $1 < i < k_2 - k_1$ the curve $f^i(\delta)$ is homotopic rel $\mathcal{P}_f$ to $f^{k_1+i}(K)$ and hence to a curve in Γ_K rel $\mathcal{P}_f$. This shows that Γ_K is irreducible.

Now we want to prove that Γ_K is a Cantor multicurve. We may apply Lemma 2.3 and assume by contradiction that $f^{-1}(\gamma)$ for each $\gamma \in \Gamma_K$ has exactly one component homotopic rel $\mathcal{P}_f$ to a curve in Γ_K.

Assume $N_1 = 0$ for simplicity. Denote by $\Gamma_K = \{\gamma_0, \cdots, \gamma_{p-1}\}$ such that γ_0 is homotopic to K and γ_n is homotopic to a component of $f^{-1}(\gamma_{n+1})$ for $0 \le n < p$ (set $\gamma_p = \gamma_0$). It makes sense since for each $\gamma \in \Gamma_K$, $f^{-1}(\gamma)$ has exactly one component homotopic rel $\mathcal{P}_f$ to a curve in Γ_K. Then $f^n(K)$ is homotopic to γ_k if $n \equiv k (\mathrm{mod} p)$.

For $n \ge 0$ and $k \ge 1$ denote by $A(n, n+kp)$ the unique annular component of $\widehat{\mathbb{C}} \backslash (f^n(K) \cup f^{n+kp}(K))$. Then $f^m : A(n, n+kp) \to A(n+m, n+kp+m)$ is proper for any $m \ge 1$. This is because that $A(n+m, n+kp+m)$ is disjoint from $\mathcal{P}_f$ and homotopic to $f^{n+m}(K)$, so $f^{-m}(A(n+m, n+kp+m))$ has a unique component homotopic to $f^n(K)$, which must be $A(n, n+kp)$.

One may choose (n, k) such that $A(n, n+kp)$ contains points of $\mathcal{J}_f$. On the other hand, f^m is proper on $A(n, n+kp)$ for all $m \ge 1$, whose image contains no points of $\mathcal{P}_f$. It is a contradiction. $\square$

Proof of Theorem 1.2. Suppose that $K \subset \mathcal{J}_f$ is a wandering continuum and is not 1-connected. Then Γ_K is an irreducible Cantor multicurve by Lemma 7.3. By Lemma 7.2, there exists an integer $N_1 \ge 0$ such that $f^n(K)$ for $n \ge N_1$ is homotopic to a curve in Γ_K rel $\mathcal{P}_f$. We assume $N_1 = 0$ for simplicity.

Let $\mathcal{E}$ be the collection of the essential components E of $f^{-m}(f^n(K))$ for $n, m \ge 0$ such that $f^i(E)$ is homotopic to a curve in Γ_K for $0 \le i < m$. Then $f(E) \in \mathcal{E}$ for any element $E \in \mathcal{E}$, and any two elements in $\mathcal{E}$ are either disjoint or one contains another as subsets of $\widehat{\mathbb{C}}$.

For each $\gamma \in \Gamma_K$, let $\mathcal{E}(\gamma)$ be the sub-collection of continua in $\mathcal{E}$ homotopic to γ rel $\mathcal{P}_f$. We claim that for any continuum $E \in \mathcal{E}(\gamma)$, there are two disjoint continua $E_1, E_2 \in \mathcal{E}(\gamma)$ such that $E \subset A(E_1, E_2)$, where $A(E_1, E_2)$ denotes the unique annular component of $\widehat{\mathbb{C}} \backslash (E_1 \cup E_2)$.

Consider $\{f^n(E)\}$ for $0 \le n \le 2 \cdot \#\Gamma_K + 1$. There is a curve $\beta \in \Gamma_K$ such that at least three of them are contained in $\mathcal{E}(\beta)$. Numerate them by $f^{n_i}(E)$ $(i = 1, 2, 3)$ such that $f^{n_3}(E) \subset A(f^{n_1}(E), f^{n_2}(E))$. Let A be the component of $f^{-n_3}(A(f^{n_1}(E), f^{n_2}(E)))$ that contains E. Then $A = A(E_1, E_2)$ where E_i $(i = 1, 2)$ is a component of $f^{-n_3}(f^{n_i}(E))$. Now the claim is proved.

Denote $A(\gamma) = \cup A(E, E')$ for all disjoint pairs $E, E' \in \mathcal{E}(\gamma)$. Then $A(\gamma)$ is an annulus in $\widehat{\mathbb{C}} \backslash \mathcal{P}_f$ homotopic to γ rel $\mathcal{P}_f$, and $A(\gamma) \cap A(\beta) = \emptyset$ for distinct curves $\beta, \gamma \in \Gamma_K$.

Denote by $\mathcal{A} = \cup_{\gamma \in \Gamma_K} A(\gamma)$ and $\mathcal{A}^1$ the union of components of $f^{-1}(\mathcal{A})$ homotopic to curves in Γ_K. Then $\mathcal{A}^1 \subset \mathcal{A}$ and $\partial \mathcal{A} \subset \partial \mathcal{A}^1$ by the claim and the definition of $\mathcal{E}$. So $g = f|_{\mathcal{A}^1} : \mathcal{A}^1 \to \mathcal{A}$ is an exact annular system. In particular, $f^n(K) \subset \mathcal{A}$ and hence $f^n(K) \subset \mathcal{A}^1$ for all $n \ge 0$. So $K \subset \mathcal{J}_g$. Since K is connected, it must be contained in a component of $\mathcal{J}_g$ which is a Jordan curve by Theorem 3.10. But K is essential. Therefore K coincides with the Jordan curve. $\square$

8 Foldings of polynomials

In this section, we introduce a topological surgery to produce branched coverings with Cantor multicurves from polynomials. We give two criteria for these maps to be equivalent to rational maps. This section is self-contained and can be read independently to the previous sections.

8.1 Folding maps

Let F be a post-critically finite branched covering of $\widehat{\mathbb{C}}$ and β be an essential Jordan curve in $\widehat{\mathbb{C}}\backslash\mathcal{P}_F$. The pair (F,β) is called a **folding map** if $F^{-1}(\beta)$ contains at least two curves and each of them is homotopic to β rel $\mathcal{P}_F$. The curve β is called the **folding curve** and

$$m(F,\beta) := \#\{\text{components of } F^{-1}(\beta)\}$$

is called the **folding times**.

Two folding maps (F,β) and (G,α) are called **Thurston equivalent** if F is Thurston equivalent to G through a pair of homeomorphisms (ϕ,ψ) such that $\phi(\beta)$ is homotopic to α rel $\mathcal{P}_G$.

Let (F,β) be a folding map. Denote by U,V the two components of $\widehat{\mathbb{C}}\backslash\beta$. Denote by U_1,V_1 the two disc components of $\widehat{\mathbb{C}}\backslash F^{-1}(\beta)$ such that U_1 is homotopic to U (i.e. there is an isotopy θ of $\widehat{\mathbb{C}}$ rel $\mathcal{P}_F$ such that $U_1=\theta(U)$). Then V_1 is homotopic to V. There are three possibilities:

Type A: $F(U_1)=U$ and $F(V_1)=U$.

Type B: $F(U_1)=U$ and $F(V_1)=V$.

Type C: $F(U_1)=V$ and $F(V_1)=U$.

All our examples in §8.5 are of Type A. Following the recipe of Example 3 one can easily construct folding maps of the other two types.

Define

$$d(F,\beta)=\begin{cases}\deg(F|_{U_1}), & \text{in type A},\\ \min\{\deg(F|_{U_1}),\deg(F|_{V_1})\}, & \text{in type B},\\ \sqrt{\deg(F|_{U_1})\deg(F|_{V_1})}, & \text{in type C}.\end{cases}$$

The following facts are easy to check:

• (F,β) is of type A if and only if $m(F,\beta)$ is even.

• (F^n,β) is also a folding map with $m(F^n,\beta)=m(F,\beta)^n$ and $d(F^n,\beta)=d(F,\beta)^n$ for $n\geq 1$.

• If (F,β) is of type A (resp. type B), then (F^n,β) is also of type A (resp. type B) for $n\geq 1$; If (F,β) is of type C, then (F^{2k-1},β) is of type C and (F^{2k},β) is of type B for $k\geq 1$.

We will prove the following theorems in this section.

Theorem 8.1. *Let (F,β) be a folding map. Suppose that*

(a) the multicurve $\{\beta\}$ is not a Thurston obstruction;

(b) any stable multicurve disjoint from β is not a Thurston obstruction; and

(c) $d(F,\beta)<m(F,\beta)$.

Then F has no Thurston obstructions.

Theorem 8.2. *Let (F, β) be a folding map. Suppose that*

(a) the multicurve $\{\beta\}$ is not a Thurston obstruction;

(b) any stable multicurve disjoint from β is not a Thurston obstruction; and

(c') there exist an integer $p \geq 1$ and a finite tree $T \subset \widehat{\mathbb{C}} \setminus \beta$ whose vertices are contained in $\mathcal{P}_F$ such that $F^p(T)$ is contained in T homotopically (i.e., there exists a homeomorphism θ of $\widehat{\mathbb{C}}$ isotopic to the identity rel $\mathcal{P}_F$ such that $F^p(T) \subset \theta(T)$), F^p is injective on T and $F^{-p}(T)$ has a component homotopic to β rel $\mathcal{P}_F$.

Then F has no Thurston obstructions.

Remark. (1) Obviously the conditions (a) and (b) in the theorems are necessary. We will give examples in §8.5 to show that these two conditions are not sufficient, and the conditions (c) and (c') are not necessary. In fact, The map F_2 in Example 2 satisfies the conditions (a) and (b) and (c') but not (c), whereas the map F_3 in Example 3 satisfies the conditions (a) and (b) and (c) but not (c').

(2) Denote $m = m(F, \beta)$ and d_i ($1 \leq i \leq m$) the degree of F on the components of $F^{-1}(\beta)$. Then the multicurve $\{\beta\}$ is not a Thurston obstruction if and only if

$$\lambda(\{\beta\}) = \frac{1}{d_1} + \cdots + \frac{1}{d_m} < 1.$$

8.2 Foldings of polynomials

Let (F, β) be a folding map of type A and g be a polynomial with connected Julia set. We say (F, β) is the **folding of** g if it is Thurston equivalent to another folding map (G, α) such that $G^{-1}(U)$ has a disc component $U_1 \Subset U$, where U is one of two Jordan domains enclosed by α, and $G|_{U_1} = g$.

Let (F, β) be a folding map of type B and (g_1, g_2) be a pair of polynomials with connected Julia sets. We say (F, β) is the **folding of** (g_1, g_2) if it is Thurston equivalent to another folding map (G, α) such that there are disjoint Jordan domains U and V in $\widehat{\mathbb{C}}$ with both ∂U and ∂V homotopic to α rel $\mathcal{P}_G$, both $G^{-1}(U)$ and $G^{-1}(V)$ have a disc component $U_1 \Subset U$ and $V_1 \Subset V$, $G|_{U_1} = g_1$ and $G|_{V_1} = g_2$.

The following result relates a folding map to a folding of polynomials, without taking into account whether the latter map is Thurston equivalent to a rational map or not.

Theorem 8.3. *Let (F, β) be a folding map of type A (or type B). Suppose that $\{\beta\}$ is not a Thurston obstruction. Then (F, β) is the folding of a polynomial g (or a pair of polynomials (g_1, g_2)) if and only if any stable multicurve disjoint from β is not a Thurston obstruction. Moreover, the polynomial g (or the pair of polynomials (g_1, g_2)) is unique up to holomorphic conjugation.*

Proof. Suppose that (F, β) is the folding of a polynomial g whose Julia set is connected. By the definition, there is a folding map (G, α) which is Thurston equivalent to (F, β), such that $G^{-1}(U)$ has a disc component $U_1 \Subset U$, where U is one of two Jordan domains enclosed by α, and $G|_{U_1} = g$.

Let Γ be an irreducible multicurve of G which is disjoint from α. If there is one curve $\gamma \in \Gamma$ homotopic to α rel $\mathcal{P}_G$, then $\Gamma = \{\gamma\}$ and hence $\lambda(\Gamma) < 1$. Now we assume that for any $\gamma \in \Gamma$, γ is not homotopic to α rel $\mathcal{P}_G$. Then γ is homotopic rel $\mathcal{P}_G$ to a curve in U.

Set $\mathcal{P} = (\mathcal{P}_G \cap U) \cup \{\infty\}$. Then $\mathcal{P}_g \subset \mathcal{P}$, $g(\mathcal{P}) \subset \mathcal{P}$ and Γ is a multicurve of the marked polynomial $(g, \mathcal{P})$. By Theorem 3.3 in [8], $\lambda(\Gamma) < 1$. Applying Lemma 2.2, we see that $\lambda(\Gamma) < 1$ for any stable multicurve Γ.

Conversely, suppose that any stable multicurve of F disjoint from β is not a Thurston obstruction. Let W be the component of $\widehat{\mathbb{C}}\backslash\beta$ such that $F(W_1) = W$, where W_1 is the component of $\widehat{\mathbb{C}}\backslash F^{-1}(\beta)$ homotopic to W. There is an isotopy θ of $\widehat{\mathbb{C}}$ rel $\mathcal{P}_F$ such that $\theta(W_1) \Subset W$. Set $G_1 = F \circ \theta^{-1}$. Then (G_1, β) is Thurston equivalent to (F, β).

Denote $\mathcal{P} = \mathcal{P}_F \cap W$. Then $G_1(\mathcal{P}) \subset \mathcal{P}$ and $G_1 : (\overline{\theta(W_1)}, \mathcal{P}) \to (\overline{W}, \mathcal{P})$ is a marked repelling system (ref to [8]). Applying Lemma 2.1 and Theorem 3.5 in [8], there exist a polynomial-like map $g_1 : V_1 \to V$ with both V and V_1 Jordan domains and a pair of homeomorphisms (ϕ, ψ) from $\overline{W}$ to $\overline{V}$ such that ψ is isotopic to ϕ rel $\mathcal{P} \cup \partial W$, $\psi(\theta(W_1)) = V_1$ and $\phi \circ G_1 \circ \psi^{-1} = g_1$ on $\overline{V_1}$. Extend (ϕ, ψ) to homeomorphisms of $\widehat{\mathbb{C}}$ such that they coincide with each other outside of W. Let $G_2 = \phi \circ G_1 \circ \psi^{-1}$. Then $(G_2, \phi(\beta))$ is Thurston equivalent to (F, β) and $G_2|_{V_1} = g_1$ is a polynomial-like map. By Straightening Theorem (refer to [11] Theorem 1), there is a quasiconformal map h of $\widehat{\mathbb{C}}$ such that for $G := h \circ G_2 \circ h^{-1}$, $G|_{h(V_1)}$ is a polynomial g. Therefore (F, β) is the folding of the polynomial g. The uniqueness of g comes from Thurston Theorem.

This argument also works for type B. We omit its proof. $\square$

Combining Theorems 8.1, 8.2 and 8.3, we obtain

Corollary 8.4. *Let (F, β) be a folding of a polynomial g (or a pair of polynomials (g_1, g_2)). Suppose that $\{\beta\}$ is not a Thurston obstruction.*

(a) If $d(F, \beta) < m(F, \beta)$, then F has no Thurston obstruction.

(b) Suppose that there exist an integer $p \geq 1$ and a finite tree $T \subset \widehat{\mathbb{C}}\backslash\beta$ whose vertices are contained in $\mathcal{P}_F$ such that $F^p(T)$ is contained in T homotopically, F^p is injective on T and $F^{-p}(T)$ has a component homotopic to β rel $\mathcal{P}_F$. Then F has no Thurston obstruction.

8.3 Proof of Theorems 8.1 and 8.2

Let (F, β) be a folding map. For any two essential Jordan curves γ and α in $\widehat{\mathbb{C}}\backslash\mathcal{P}_F$, set $k(\gamma, \alpha)$ to be their geometric intersection number. It is defined by

$$k(\gamma, \alpha) = \min\{\#(\delta \cap \alpha)\},$$

where the minimum is taken over all the choices of δ in the homotopy class of γ. By definition $k(\gamma, \alpha) = 0$ if γ is homotopic to α rel $\mathcal{P}_F$.

Lemma 8.5. *Let Γ be an irreducible multicurve of F. Then either $k(\gamma, \beta) \neq 0$ for all $\gamma \in \Gamma$ or $k(\gamma, \beta) = 0$ for all $\gamma \in \Gamma$.*

Proof. Suppose that $k(\gamma, \beta) = 0$ for some $\gamma \in \Gamma$. For any curve $\alpha \in \Gamma$, since Γ is irreducible, α is homotopic to a component of $F^{-n}(\gamma)$ rel $\mathcal{P}_F$ for some $n \geq 0$. Let δ be a Jordan curve in $\widehat{\mathbb{C}}\backslash\mathcal{P}_F$ homotopic to γ rel $\mathcal{P}_F$ such that it is disjoint from β, then α is homotopic to a component of $F^{-n}(\delta)$ rel $\mathcal{P}_F$, which is disjoint from $F^{-n}(\beta)$. Thus $k(\alpha, \beta) = 0$ since $F^{-n}(\beta)$ contains a curve homotopic to β rel $\mathcal{P}_F$. $\square$

Lemma 8.6. *Let γ and α be essential Jordan curves in $\widehat{\mathbb{C}}\setminus\mathcal{P}_F$ such that $k(\gamma,\beta)\neq 0$. Suppose that $F^{-1}(\gamma)$ has a component δ homotopic to α rel $\mathcal{P}_F$. Then*

$$\deg(F:\delta\to\gamma)\geq\frac{m\cdot k(\alpha,\beta)}{k(\gamma,\beta)},$$

where $m=m(F,\beta)$.

Proof. Denote by $d(\delta)=\deg(F:\delta\to\gamma)$. We may assume that $\#(\gamma\cap\beta)=k(\gamma,\beta)$. Denote by $\alpha_1,\cdots,\alpha_m$ the components of $F^{-1}(\beta)$. Then

$$\#\Big(\delta\cap\bigcup_{p=1}^{m}\alpha_p\Big)=d(\delta)\cdot\#(\gamma\cap\beta)=d(\delta)\cdot k(\gamma,\beta)\ .$$

On the other hand,

$$\#\Big(\delta\cap\bigcup_{p=1}^{m}\alpha_p\Big)=\sum_{p=1}^{m}\#(\delta\cap\alpha_p)\geq\sum_{p=1}^{m}k(\alpha,\beta)=m\cdot k(\alpha,\beta)\ .$$

Combining the above two inequalities we get the lemma. □

Lemma 8.7. *Suppose that $\Gamma=\{\gamma_1,\cdots,\gamma_n\}$ is an irreducible multicurve of F such that $k_i=k(\gamma_i,\beta)\neq 0$. Let $M_\Gamma=(a_{ij})$ be the transition matrix of Γ. Then*

$$a_{ij}\leq\frac{d_0k_j^2}{mk_i^2}\ ,$$

where $d_0=d(F,\beta)$ and $m=m(F,\beta)$.

Proof. We may assume that $\#(\gamma_i\cap\beta)=k(\gamma_i,\beta)=k_i$ for any $\gamma_i\in\Gamma$. Fix a pair (i,j). If $F^{-1}(\gamma_j)$ has no component homotopic to γ_i, then $a_{ij}=0$. Now suppose that $F^{-1}(\gamma_j)$ has $n>0$ components homotopic to γ_i. Denote them by $\{\delta_s,s=1,\cdots,n\}$. We claim that $n\leq d_0k_j/k_i$.

Denote by U,V the two components of $\widehat{\mathbb{C}}\setminus\beta$. Denote by U_1, V_1 the two disc components of $\widehat{\mathbb{C}}\setminus F^{-1}(\beta)$ such that U_1,V_1 are homotopic to U and V, respectively. Denote $d_1=\deg(F|_{U_1})$ and $d_2=\deg(F|_{V_1})$. Then both $U\cap\gamma_j$ and $V\cap\gamma_j$ have exactly $k_j/2$ components (notice that $k_j=\#(\gamma_j\cap\beta)$ is an even number). It follows that $U_1\cap F^{-1}(\gamma_j)$ has exactly $d_1k_j/2$ components and $V_1\cap F^{-1}(\gamma_j)$ has exactly $d_2k_j/2$ components.

On the other hand, since both ∂U_1 and ∂V_1 are homotopic to β rel $\mathcal{P}_F$, both $U_1\cap\delta_s$ and $V_1\cap\delta_s$ have at least $k_i/2$ components for $s=1,\cdots,n$. It follows that

$$\begin{aligned}\frac{nk_i}{2}&\leq\#\Big\{\text{components of }U_1\cap(\cup\delta_s)\Big\}\\&\leq\#\Big\{\text{components of }U_1\cap F^{-1}(\gamma_j)\Big\}=\frac{d_1k_j}{2}\ .\end{aligned}$$

So $n\leq d_1k_j/k_i$. We also have $n\leq d_2k_j/k_i$ if we replace U_1 by V_1 in the above inequality. Hence $n\leq\min\{d_1,d_2\}k_j/k_i\leq d_0k_j/k_i$.

Now applying Lemma 8.6, we have

$$a_{ij}=\sum_{s=1}^{n}\frac{1}{\deg(F:\delta_s\to\gamma_j)}\leq\sum_{s=1}^{n}\frac{k_j}{mk_i}=n\frac{k_j}{mk_i}\leq d_0\frac{k_j}{k_i}\frac{k_j}{mk_i}=\frac{d_0k_j^2}{mk_i^2}\ .$$

This proves the lemma. □

Proof of Theorem 8.1. Denote $m = m(F,\beta)$, $d_0 = d(F,\beta)$ and $p = \#\mathcal{P}_F$. By the condition $d_0 < m$, there is an integer $N \geq 1$ such that $(p-3)d_0^N < m^N$.

Now we consider the folding map (F^N, β). Note that $\#\mathcal{P}_{F^N} = p$, $m(F^N,\beta) = m^N$ and $d(F^N,\beta) = d_0^N$. Let $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ be an irreducible multicurve of F^N. If $k(\gamma_i,\beta) = 0$ for some $\gamma_i \in \Gamma$, then $k(\gamma_j,\beta) = 0$ for all $\gamma_j \in \Gamma$ by Lemma 8.5. So Γ is homotopic rel $\mathcal{P}_F$ to a multicurve disjoint from β. Thus it is not a Thurston obstruction by the conditions (a) and (b).

Now we assume that $k_i = k(\gamma_i,\beta) \neq 0$ for all $\gamma_i \in \Gamma$. Set the vector $\mathbf{v} = (v_i)$ with $v_i = 1/k_i^2$, then by Lemma 8.7

$$(M_\Gamma \mathbf{v})_i = \sum_{j=1}^{n} a_{ij} v_j \leq \sum_{j=1}^{n} \frac{d_0^N k_j^2}{m^N k_i^2} \cdot \frac{1}{k_j^2} = \frac{n\, d_0^N}{m^N k_i^2}.$$

Notice that $n = |\Gamma| \leq \#\mathcal{P}_{F^N} - 3 = p - 3$. Therefore

$$(M_\Gamma \mathbf{v})_i \leq \frac{(p-3)d_0^N}{m^N k_i^2} < \frac{1}{k_i^2} = v_i\ .$$

It follows that $M_\Gamma \mathbf{v} < \mathbf{v}$ and hence $\lambda(\Gamma) < 1$ (refer to Lemma A.1 in [8]). Thus F^N has no Thurston obstruction by Lemma 2.2. This implies that F has no Thurston obstruction. $\square$

Proof of Theorem 8.2. Denote $m = m(F,\beta)$. Let $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ be an irreducible multicurve of F. If $k(\gamma_i,\beta) = 0$ for some $\gamma_i \in \Gamma$, then $k(\gamma_j,\beta) = 0$ for all $\gamma_j \in \Gamma$ by Lemma 8.5. So Γ is homotopic rel $\mathcal{P}_F$ to a multicurve disjoint from β. Thus it is not a Thurston obstruction by the conditions (a) and (b).

Now we assume that $k(\gamma_i,\beta) \neq 0$ for all $\gamma_i \in \Gamma$. Denote by $k(\gamma_i, T)$ the geometric intersection number.

Case 1. $k(\gamma_i, T) = 0$ *for some curve* $\gamma_i \in \Gamma$. Assume that γ_i is disjoint from T. By condition (c'), $F^{-p}(T)$ has a component homotopic to β rel $\mathcal{P}_F$. Thus $k(\delta,\beta) = 0$ for all the components δ of $F^{-p}(\gamma_i)$. But Γ is irreducible, so $k(\gamma_j,\beta) = 0$ for some $\gamma_j \in \Gamma$. It is a contradiction.

Case 2. $k(\gamma_i, T) \neq 0$ *for all curves* $\gamma_i \in \Gamma$. We may assume that $p = 1$ and $F(T) \subset T$. We also assume that $\#(\gamma_i \cap T) = k(\gamma_i, T)$ for $\gamma_i \in \Gamma$. Let δ_s $(s = 1, \cdots, n)$ be all the components of $F^{-1}(\gamma_j)$ homotopic to a curve in Γ, then $F : (\cup \delta_s) \cap T \to \gamma_j \cap T$ is injective since $F(T) \subset T$ and $F|_T$ is injective. So

$$\sum_{s=1}^{n} \#(\delta_s \cap T) \leq \#(\gamma_j \cap T) = k(\gamma_j, T).$$

Therefore if γ_i is homotopic to a curve in $F^{-1}(\gamma_j)$, then $k(\gamma_i, T) \leq k(\gamma_j, T)$. Since Γ is irreducible, we have $k(\gamma_i, T)$ is a constant for all $\gamma_i \in \Gamma$ and $F^{-1}(\gamma_j)$ has exactly one component homotopic to a curve in Γ. Relabel the index such that $F^{-1}(\gamma_{j+1})$ has a curve δ_j homotopic to γ_j for $j = 1, \cdots, n$ (set $\gamma_{n+1} = \gamma_1$). Let $M_\Gamma = (a_{ij})$ be the transition matrix of Γ. Then

$$a_{j,j+1} = \frac{1}{\deg(F : \delta_j \to \gamma_{j+1})}$$

for $j = 1, \cdots, n$ (set $a_{n,n+1} = a_{n,1}$), and $a_{ij} = 0$ otherwise. By Lemma 8.6, we have

$$\deg(F : \delta_j \to \gamma_{j+1}) \geq \frac{mk_j}{k_{j+1}}.$$

So $a_{j,j+1} \leq k_{j+1}/(mk_j)$. Set the vector $\mathbf{v} = (1/k_i)$, then $M_\Gamma \mathbf{v} \leq (1/m)\mathbf{v}$ and hence $\lambda(\Gamma) \leq 1/m < 1$ (refer to Lemma A.1 in [8]). Applying Lemma 2.2 again we conclude that F has no Thurston obstructions. □

8.4 Sierpinski maps

A connected compact subset $E \subset \widehat{\mathbb{C}}$ is called a **Sierpinski carpet** if it is locally connected, nowhere dense and its complementary components are Jordan domains with pairwise disjoint closures.

A rational map is called a **Sierpinski map** if its Julia set is a Sierpinski carpet. A polynomial is called of **Sierpinski type** if its Julia set is connected and locally connected, it has at least two bounded Fatou domains, and the bounded Fatou domains have pairwise disjoint closures.

Theorem 8.8. *Let f be a post-critically finite rational map. Suppose that (f, β) is a folding of a Sierpinski type polynomial g. Then f is a Sierpinski map.*

Proof. Note that $\{\beta\}$ is a Cantor multicurve of f. Applying Theorem 1.1, there exists an annulus $A \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ homotopic to β rel $\mathcal{P}_f$, such that $f : f^{-1}(A) \to A$ is an exact annular system.

Denote by B_1, B_2 the two components of $\widehat{\mathbb{C}} \backslash A$, then they are components of $f^{-1}(B_1 \cup B_2)$. Rearranging the indices if necessary, we may assume that $f(B_1) = f(B_2) = B_1$. There exists a Jordan curve β_0 essentially contained in A such that $U_1 \Subset U_0$, where U_0 is the domain enclosed by β_0 and containing B_1 and U_1 is the pre-image of $f^{-1}(U_0)$ containing B_1. Since β_0 is homotopic to β rel $\mathcal{P}_f$, the polynomial-like map $g_1 = f|_{U_1} : U_1 \to U_0$ is quasiconformally conjugate to a restriction of the polynomial g.

Each periodic Fatou domain of f is contained in B_1. Thus it is a periodic Fatou domain of the polynomial-like map g_1. Since the polynomial g is of Sierpinski type, every periodic Fatou domain of f is a Jordan domain. Since f is hyperbolic, every Fatou domain of f is a Jordan domain.

Any two Fatou domains of f are either contained in the same component of $f^{-n}(B_1)$ for some integer $n \geq 0$, or separated by a component of $f^{-m}(A)$ for some integer $m \geq 0$ and hence have disjoint closures. In the former case they have disjoint closures since g is of Sierpinski type. So f is a Sierpinski map. □

8.5 Examples

All our examples will be deformations of the airplane quadratic polynomial $Q_c(z) = z^2 + c$, i.e. the parameter c is chosen to be the unique real solution of the equation $(c^2+c)^2+c = 0$.

The critical point $z=0$ forms a super attracting cycle with period 3. Denote by:

$$\begin{aligned}
&U=\{z:\ |z|<1\Big\} \text{ and } \beta=\partial U,\\
&V=\{z:\ |z|>2\}\cup\{\infty\} \text{ and } \beta_*=\partial V,\\
&A=\{z:\ 1<|z|<2\},\\
&\rho: U\to\mathbb{C},\ re^{i\theta}\mapsto \tfrac{re^{i\theta}}{1-r},\ \text{ an angle-preserving homeomorphism},\\
&\sigma: V\to U,\ \sigma(z)=2/z,\ \text{ a homeomorphism.}
\end{aligned}$$

Example 1. Set $g(z)=Q_c^{\circ 2}(z)$. We want to construct a branched covering F which is the folding of the polynomial g. We will define F piecewisely as following:

$$F_1=\begin{cases} G_1:\ \ \overline{U}\to\overline{U}\\ G_2:\ \ \overline{V}\to\overline{U}\\ G_3:\ \ A\to A\cup\overline{V}\end{cases}$$

• $G_1: U\to U,\ z\mapsto \rho^{-1}\circ g\circ\rho(z)$. Then G_1 can be extended continuously to the boundary with $G_1(z)=z^4$ on $\beta=\partial U$.
• $G_2:\overline{V}\to\overline{U}$ by $G_2(z)=G_1\circ\sigma(z)$. Then $G_2|_{\beta_*}(z)=(2/z)^4$.
• $G_3: A\to A\cup\overline{V}$ is a branched covering such that its boundary value coincides with G_1 on β, and G_2 on β_*, and its critical values are contained in V. The precise definition of G_3 requires some care in order to control the obstructions of F_1.

Let $x_0<0$ be the unique fixed point of Q_c on the negative real axis (it is called the α-fixed point). Then $g^{-1}(x_0)=Q_c^{-2}(x_0)$ has four points x_{-1},x_0,x_1,x_2, displaced in $\mathbb{R}$ relative to the super-attracting cycle as follows:

$$c<x_{-1}<x_0<0<x_1<c^2+c<x_2.$$

Denote by $R(\theta)$ the external ray of Q_c of angle $2\pi\theta$. It is also a ray of g. Both $R(1/3)$ and $R(2/3)$ land on the α-fixed point x_0. Denote by $L_0=R(1/3)\cup\{x_0\}\cup R(2/3)$. Then $Q_c(L_0)=L_0$ and Q_c switches the two rays. And $g(L_0)=L_0$ by fixing each ray. Now $g^{-1}(L_0)$ has 4 arcs L_{-1},L_0,L_1,L_2 with $x_i\in L_i$.

Pullback these four arcs by ρ we get $S_k:=\rho^{-1}(L_k)$ ($k=-1,0,1,2$) in U. The two ends of S_k are $(e^{2\pi i\theta_k},e^{2\pi i\varphi_k})$, with $\theta_{-1}=5/12$, $\theta_0=1/3$, $\theta_1=1/6$, $\theta_2=1/12$, and $\varphi_k=-\theta_k$. As $G_1=\rho^{-1}\circ g\circ\rho$, we have $G_1^{-1}(S_0)=\cup_{k=-1}^2 S_k$.

Pullback then by σ, we get $E_k:=\sigma^{-1}(S_k)$ in V. As $G_2=G_1\circ\sigma$, we have $G_2^{-1}(S_0)=\cup_{k=-1}^2 E_k$. As $\sigma(re^{2\pi i\eta})=\rho(2/(re^{2\pi i\eta}))$ and $\varphi_k=-\theta_k$ we know that the two ends of the arc E_k are $(2e^{2\pi i\theta_k},2e^{2\pi i\varphi_k})$.

Set I_k to be the union of two radial arcs in A for $k=-1,0,1,2$ by

$$I_k:=\Big\{re^{2\pi i\theta_k},r\in[1,2]\Big\}\cup\Big\{re^{2\pi i\varphi_k},r\in[1,2]\Big\}.$$

Then $\gamma_k:=S_k\cup E_k\cup I_k$ is a Jordan curve.

• Define G_3 on each of the 8 radial arcs in A such that it maps the arc homeomorphically onto $I_0\cup E_0$, and maps $(e^{2\pi it},2e^{2\pi it})$ to $(e^{2\pi i(4t)},e^{-2\pi i(4t)})$.

Extend G_3 continuously in each of the 8 quadrilaterals of $A\backslash(\cup_k I_k)$ as a orientation preserving branched covering of degree two. The image must be one of the two components of $(A\cup\overline{V})\backslash(I_0\cup E_0)$. We also assure that the unique critical point in each quadrilateral is mapped to either $y_c=\sigma^{-1}(\rho^{-1}(c))$ or $y_0=\sigma^{-1}(0)$.

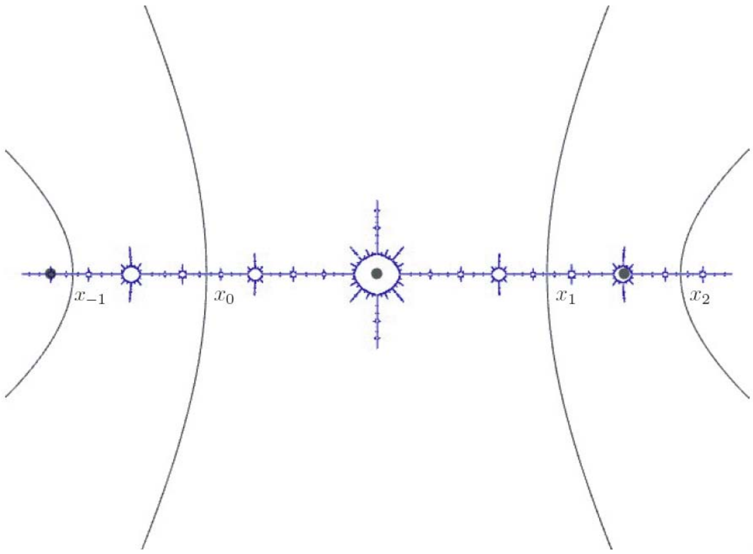

Figure 3. The four arcs of rays L_{-1}, L_0, L_1, L_2 (from left to right)

This ends the definition of F_1. It is a branched covering of $\widehat{\mathbb{C}}$ with post-critical set $\mathcal{P}_{F_1} = \rho^{-1}(\mathcal{P}_g) \cup \{y_0, y_c\}$.

Example 2. This example will be a little modification of the map F_1 above. Let D be the Fatou domain of Q_c containing the point c. Let $x_r \in (c, x_{-1})$ be the right intersection point of ∂D with real axis, which is also called the root of D. Set $y_r = \sigma^{-1}(\rho^{-1}(x_r))$. Choose a homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that h is the identity in $A \cup \overline{U}$, $h(y_c) = y_c$ and $h(y_0) = y_r$. Set $F_2 := h \circ F_1$. Notice that $\mathcal{P}_{F_2} = \rho^{-1}(\mathcal{P}_g) \cup \{y_r, y_c\}$.

Proposition 8.9. *The map F_1 has a Thurston obstruction, whereas the map F_2 has none.*

Proof. The curve $F_1^{-1}(\gamma_0)$ has 4 components $\gamma_k \ni x_k$ $(k = -1, 0, 1, 2)$ with $\deg(F_1 : \gamma_k \to \gamma_0) = 2$ for every k. The curve γ_2 is null-homotopic and γ_1 is peripheral. Both γ_0 and γ_{-1} are essential and homotopic to each other rel $\mathcal{P}_{F_1}$. Set $\Gamma = \{\gamma_0\}$. Then Γ is a stable multicurve with $\lambda(\Gamma) = 1/2 + 1/2 = 1$. So Γ is a Thurston obstruction.

Let $T_0 = [y_c, y_r]$. Then $F_2(\Omega_{F_2} \cap A) = \{y_c, y_r\} \subset T_0$. The annulus $V \backslash T_0$ contains no critical values of F_2. It follows that $F_2^{-1}(T_0)$ has a component K essentially contained in A. Thus K is homotopic to β rel $\mathcal{P}_{F_2}$. Set $T_1 := F_2(T_0) = \rho^{-1} \circ g([c, x_r])$. Then the line segment T_1 is a 3-periodic interval and F_2^3 is injective on T_1. This allows us to apply our criterion Theorem 8.2 to conclude that the map F_2 has no Thurston obstructions. $\square$

Example 3. Let β_i $(i = 1, \cdots, 6)$ be disjoint Jordan curves essentially contained in A labelled by the order from U to V. Denote by A_i the annulus bounded by β_{i-1} and β_i (set $\beta_0 = \beta$ and $\beta_7 = \beta_*$) for $i = 1, \cdots, 7$.

Denote by $\mathcal{O}$ the grand orbit of the super-attracting cycle of Q_c. Define a branched covering F_3 of $\widehat{\mathbb{C}}$ piecewisely as follows:

- $F_3 : U \to U$, $z \mapsto \rho^{-1} \circ Q_c \circ \rho(z)$.
- $F_3 : V \to U$, $z \mapsto (2/z)^4$, or any degree 4 branched covering with critical values in $\rho^{-1}(\mathcal{O})$.

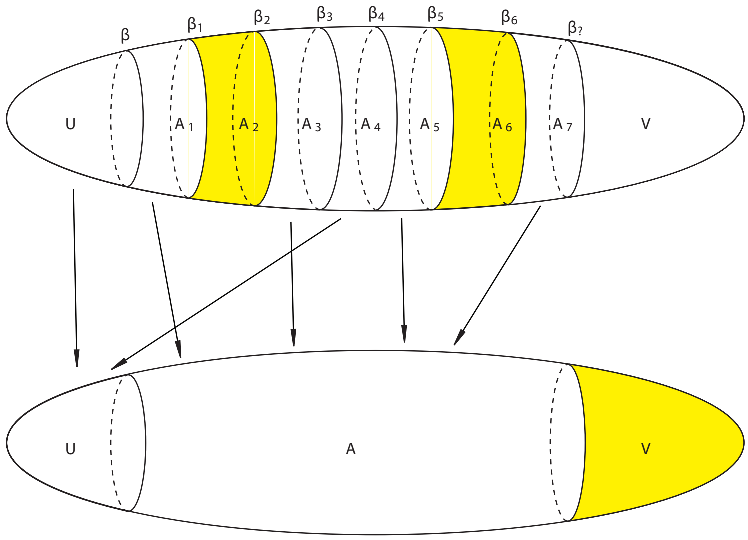

Figure 4. Construction of F_3

- $F_3 : A_1, A_3, A_5, A_7 \to A$ are coverings with degree 2,8,16,4 respectively such that they can be continuously extended to covering maps on the closures with β_3, β_4 being mapped to β and with the actions on $\beta \cup \beta_*$ coincide with previous defined boundary maps.
- $F_3 : A_4 \to U$ is a branched covering[2] such that its boundary map coincides with the previous defined boundary maps on $\overline{A}_3 \cup \overline{A}_5$, and its critical values are located in the set $\rho^{-1}(\mathcal{O})$. Its degree is the sum of the degrees on A_3, A_5, i.e. 24.
- Similarly $F_3 : A_2, A_6 \to V$ are branched coverings such that their boundary maps coincide with the previous defined boundary maps and their critical values are located in $(F_3|_V)^{-1} \circ \rho^{-1}(\mathcal{O})$. Their degrees are 10 and 20, respectively.

Every critical point of F_3 is eventually super-attracting. Note that $F_3^{-1}(\beta)$ has 4 connected components with the sum of the inverse of the degrees satisfying

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{4} = \frac{15}{16} < 1.$$

Each of the 4 pulled back curves is homotopic to β. Furthermore $2 = \deg(F_3|_U) < 4$. These properties allow us to apply our criterion Theorem 8.1 to F_3 to conclude that it has no Thurston obstructions. Thus F_3 is Thurston equivalent to a post-critically finite hyperbolic rational map f (note that $\#\mathcal{P}_{F_3} \geq 5$ and hence its orbifold is hyperbolic) by Thurston Theorem.

The bounded Fatou components of Q_c have pairwise disjoint closures. Applying Theorem 8.8, we may furthermore conclude that the Julia set of f is a Sierpinski carpet.

One may wonder if there is a finite tree $T \subset U$ whose vertices are contained in $\mathcal{P}_{F_3}$ such that $F_3^p(T)$ is contained in T homotopically for some integer $p \geq 1$ and F_3^p is injective on T. The existence of such a tree would allow us to apply Theorem 8.2 instead of Theorem 8.1 to discard any eventual obstructions.

[2] The existence of such a branched covering is known since Hurwitz. For a concrete construction see e.g. [27].

We shall see that such a tree T can not exist. Note that each point in $\mathcal{P}_{F_3} \cap U$ eventually maps to the unique critical cycle. Since $F_3|_U$ is topologically conjugated to the polynomial Q_c, we only need to prove that there is no finite tree $T \subset \widehat{\mathbb{C}}$ whose vertices are contained in the grand orbit of the super-attracting cycle of Q_c, such that $Q_c^p(T)$ is contained in T homotopically and Q_c^p is injective on T for some $p \geq 1$.

But this is true for any post-critically finite polynomial g whose bounded Fatou components have pairwise disjoint closures (which is the case of Q_c). Take the centers z_1, z_2 of any pair of distinct Fatou domains of g. There can not exist an arc δ connecting z_1, z_2 such that g^p is injective on δ for some integer $p \geq 1$ and $g^p(\delta)$ is homotopic to δ rel $\mathcal{P}_g \cup \{z_1, z_2\}$. Otherwise, by taking pre-images consecutively and apply Shrinking Lemma in [21], we can show that the Fatou domains of z_1, z_2 have common boundary points.

A Rees-Shishikura's semi-conjugacy

Let F be a formal mating of two post-critically finite polynomials. Suppose that F is Thurston equivalent to a rational map f. There is a semi-conjugacy from F to f. This result was obtained by M. Rees if both polynomials are hyperbolic [28], and proved by Shishikura in general [30]. The same result is still true for general post-critically finite branched coverings. Here we provide a statement with a general form.

Theorem A.1. *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a post-critically finite branched covering which is Thurston equivalent to a rational map f through a pair of homeomorphisms (ψ_0, ψ_1) of $\widehat{\mathbb{C}}$. Suppose that F is holomorphic in a neighborhood of the critical cycles of F. There exist a neighborhood U of the critical cycles of F and a sequence of homeomorphisms $\{\phi_n\}$ $(n \geq 0)$ of $\widehat{\mathbb{C}}$ homotopic to ψ_0 rel $\mathcal{P}_F$, such that $\phi_n|_U$ is holomorphic, $\phi_n|_U = \phi_0|_U$ and $f \circ \phi_{n+1} = \phi_n \circ F$. The sequence $\{\phi_n\}$ converges uniformly to a continuous map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Moreover, the following statements hold:*

(1) $h \circ F = f \circ h$.

(2) h is surjective.

(3) $h^{-1}(w)$ is a single point for $w \in \mathcal{F}_f$ and $h^{-1}(w)$ is either a single point or a full continuum for $w \in \mathcal{J}_f$.

(4) For points $x, y \in \widehat{\mathbb{C}}$ with $f(x) = y$, $h^{-1}(x)$ is a component of $F^{-1}(h^{-1}(y))$. Moreover, $\deg F|_{h^{-1}(x)} = \deg_x f$.

(5) $h^{-1}(E)$ is a continuum if $E \subset \widehat{\mathbb{C}}$ is a continuum.

(6) $h(F^{-1}(E)) = f^{-1}(h(E))$ for any $E \subset \widehat{\mathbb{C}}$.

(7) $F^{-1}(\widehat{E}) = \widehat{F^{-1}(E)}$ for any $E \subset \widehat{\mathbb{C}}$, where $\widehat{E} = h^{-1}(h(E))$.

One may also refer to [9] for a detailed account in a generalized form. The crucial part of theorem is the construction of the homotopy (ϕ_0, ϕ_1) rel a neighborhood U of critical cycles and the convergence of the sequence $\{\phi_n\}$. The other statements are directly deduced. The statements (5)-(7) is used in this paper, so we add them in the theorem and provide a proof here.

Proof of (5)-(7).

(5) Suppose that $E \subset \widehat{\mathbb{C}}$ is a connected closed subset. The closeness of $h^{-1}(E)$ is easy to see. Now suppose that $h^{-1}(E)$ is not connected, i.e., there are open sets U_1, U_2 in $\widehat{\mathbb{C}}$ such that $h^{-1}(E) \subset U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ and both U_1 and U_2 intersect $h^{-1}(E)$. Then

$K := h(\widehat{\mathbb{C}}\backslash(U_1 \cup U_2))$ is a compact set disjoint from E. Since E is connected, there is a connected neighborhood V of E such that $\overline{V} \cap K = \emptyset$. Since $\{\phi_n\}$ converges uniformly to h, there exists an integer $n > 0$ such that

$$\mathrm{d}(h, \phi_n) = \sup_{z \in \widehat{\mathbb{C}}} \mathrm{d}\Big(h(z), \phi_n(z)\Big) < \min \Big\{\mathrm{d}(E, \partial V), \mathrm{d}(\overline{V}, K)\Big\},$$

where $d(\cdot, \cdot)$ denotes the spherical distance. Then it follows that $\phi_n(\widehat{\mathbb{C}}\backslash(U_1 \cup U_2)) \cap \overline{V} = \emptyset$, hence $\phi_n^{-1}(V) \subset U_1 \cup U_2$. On the other hand, since $V \supset E$, both U_1 and U_2 intersect $\phi_n^{-1}(V)$. This contradicts the fact that $\phi_n^{-1}(V)$ is connected.

(6) From $f \circ h(F^{-1}(E)) = h \circ F(F^{-1}(E)) = h(E)$, we have $h(F^{-1}(E)) \subset f^{-1}(h(E))$. Conversely, for any point $w \in f^{-1}(h(E))$, $f(w) \in h(E)$. So there is a point $z_0 \in E$ such that $f(w) = h(z_0)$. By (5), the map

$$F : h^{-1}(w) \to h^{-1}\Big(f(w)\Big)$$

is surjective. Noticing that $z_0 \in h^{-1}(f(w))$, there is a point $z_1 \in h^{-1}(w)$ such that $F(z_1) = z_0$. So $w = h(z_1) \in h(F^{-1}(z_0)) \subset h(F^{-1}(E))$. Therefore, $f^{-1}(h(E)) \subset h(F^{-1}(E))$.

(7) $F^{-1}(\widehat{E}) = F^{-1}\Big(h^{-1}\big(h(E)\big)\Big) = h^{-1}\Big(f^{-1}\big(h(E)\big)\Big)$. From (6), we obtain

$$F^{-1}(\widehat{E}) = h^{-1}\Big(h\big(F^{-1}(E)\big)\Big) = \widehat{F^{-1}(E)}.$$

□

B Buff's example

Example. Denote by $U = \{z : 1 < |z| < r_0\}$ with $r_0 > 2$. Define a spiral in U by:

$$\delta = \Big\{\rho e^{i\theta} : \frac{r_0}{2} \leq \rho < r_0,\ \theta = \frac{1}{r_0 - \rho}\Big\}.$$

Set $A_1 = U\backslash\delta$. Then A_1 is an annulus with modulus $\mathrm{mod}(A_1) < \log r_0/(2\pi)$. Pick an integer $d > 2$ such that $d\,\mathrm{mod}(A_1) > \log r_0/\pi$. Set $r_1 > 1$ be the constant such that $\log r_1/(2\pi) = d\,\mathrm{mod}(A_1)$. Then $r_1 > r_0^2$. Set $A = \{z : 1 < |z| < r_1\}$ and $A_2 = h(A_1)$ with $h(z) = r_1/z$. Then A_2 is disjoint from A_1 and there is a holomorphic covering g of degree d from A_i $(i = 1, 2)$ to A such that g fixes the two components of ∂A. Then $g : A_1 \cup A_2 \to A$ is an exact annular system .

Theorem B.1. *Let $\mathscr{J}$ be the collection of the components of the Julia set of the exact annular system $g : A_1 \cup A_2 \to A$. With the topology induced by the corresponding linear system, $\mathscr{J}$ has a dense subset whose elements are not locally connected.*

Set $B = \{\zeta, 0 < \mathrm{Im}\ \zeta < \log r_1\}$. Then $\pi(\zeta) = \exp(\zeta) : B \to A$ is a universal covering. Denote by $E_0 = A\backslash(A_1 \cup A_2)$. For each component E_n of $g^{-n}(E_0)$ $(n \geq 0)$ and any point $z \in A\backslash E_n$, denote by

$$\text{h-dist}(z, \partial A; E_n) = \inf \Big\{\mathrm{diam}(\pi^{-1}(\gamma))\Big\},$$

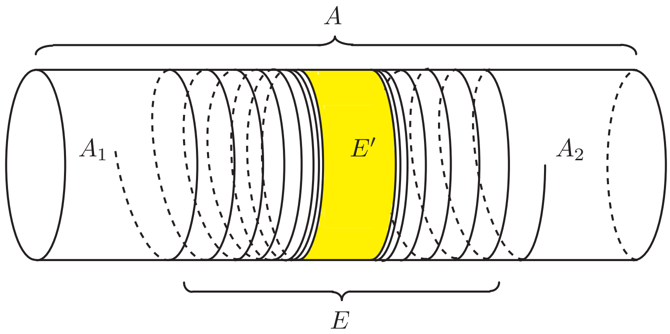

Figure 5. An annular system with spirals

where the infimum is taken over all the open arc in $A\backslash E_n$ connecting z with ∂A, and $\mathrm{diam}(\pi^{-1}(\gamma))$ is the Euclidean diameter of a component of $\pi^{-1}(\gamma)$.

Denote by $E_0' = A\backslash(U \cup h(U)) = E_0\backslash(\delta \cup h(\delta))$. It is easy to check that for any constant $M < \infty$, there exists a constant $\varepsilon > 0$ such that for any point $z \in A\backslash E_0$, if the Euclidean distance $\mathrm{dist}(z, E_0') < \varepsilon$, then $\text{h-dist}(z, \partial A; E_0) > M$.

For each component E_n of $g^{-n}(E_0)$ and any $k > n \geq 0$, denote by $V_k(E_n)$ the union of the two components of $g^{-k}(A)$ whose closures meet E_n. Then $\widetilde{V}_k(E_n) := V_k(E_n) \cup E_n$ is an annulus with $\widetilde{V}_{k+1}(E_n) \subset \widetilde{V}_k(E_n)$ and $\cap_{k\geq n}\widetilde{V}_k(E_n) = E_n$.

By the above argument, we see that for any constant $M < \infty$, there is an integer $k(M) \geq 1$ such that for any component K of $\mathcal{J}_g \cap V_{k(M)}(E_0)$, there exists a point $z \in K$ such that $\text{h-dist}(z, \partial A; E_0) > M$. By Koëbe distortion theorem, we may prove the next lemma.

Lemma B.2. *For any component E_n of $g^{-n}(E_0)$ $(n \geq 0)$ and any constant $M < \infty$, there is an integer $k(M, E_n) > n$ such that for any component K of $\mathcal{J}_g \cap V_{k(M,E_n)}(E_n)$, there exists a point $z \in K$ such that h-dist$(z, \partial A; E_n) > M$.*

Proof of Theorem B.1. For each integer $m > 0$ and any component E_n of $g^{-n}(E_0)$ $(n \geq 1)$, define $\mathscr{N}(m, E_n)$ to be the sub-collection of $\mathscr{J}$ such that $K \in \mathscr{N}(m, E_n)$ if $K \subset V_{k(m,E_n)}(E_n)$. Then $\mathscr{N}(m, E_n)$ is an open set in $\mathscr{J}$. Set $\mathscr{N}(m)$ to be the union of $\mathscr{N}(m, E_n)$ for all $n \geq 1$ and all the components E_n of $g^{-n}(E_0)$. Then it is an open dense subset of $\mathscr{J}$. Thus $\mathscr{N} = \bigcap_{m\geq 1} \mathscr{N}(m)$ is a dense subset of $\mathscr{J}$ in Baire's category.

For each $K \in \mathscr{N}$, we want to show that K is not locally connected. Otherwise K is a Jordan curve and hence there is an constant $M < \infty$ such that for any point $z \in K$, there are open arcs $\delta_+(z)$ and $\delta_-(z)$ in $A\backslash K$ connecting z with the two components of ∂A, respectively, such that $\mathrm{diam}(\pi^{-1}(\delta_\pm(z))) < M$.

Fix an integer $m > M$. Since $K \in \mathscr{N}(m)$, there exist an integer $n \geq 0$ and a component E_n of $g^{-n}(E_0)$ such that $K \in \mathscr{N}(m, E_n)$. Thus there is a point $z \in K$ such that $\text{h-diam}(z, \partial A; E_n) > m > M$, contradicting the fact that $\mathrm{diam}(\pi^{-1}(\delta_\pm(z))) < M$. $\square$

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