Wandering continua for rational maps *

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Abstract

We prove that a Lattès map admits an always full wandering continuum if and only if it is flexible. The full wandering continuum is a line segment in a bi-infinite geodesic under the flat metric.

1 Introduction

Let f be a rational map of the Riemann sphere $\widehat{\mathbb{C}}$ with $\deg f \geq 2$. Denote by J_f and F_f the Julia set and the Fatou set of f respectively. One may refer to [12] for their definitions and basic properties. By a **continuum** we mean a connected compact set consisting of more than one point. A continuum $K \subset \widehat{\mathbb{C}}$ is called a **wandering continuum** for f if $K \subset J_f$ and $f^n(K) \cap f^m(K) = \emptyset$ for any $n > m \geq 0$.

The existence of wandering continua for polynomials has been studied by many authors. It was proved that all wandering components of the Julia set of a polynomial with disconnected Julia set are points [1, 8, 15]. For polynomials with connected Julia sets, it was proved that a polynomial without irrational indifferent periodic cycles has no wandering continuum if and only if the Julia set is locally connected [2, 5, 6, 9, 16].

The situation for non-polynomial rational maps is different. There are hyperbolic rational maps which have non-degenerate wandering components of their Julia sets. The first example was given by McMullen, where the wandering Julia components are Jordan curves [11]. In fact, it was proved that for a geometrically finite rational map, a wandering component of its Julia set is either a Jordan curve or a single point [14].

In this work we study wandering continua for rational maps with connected Julia sets. A continuum $K \subset \widehat{\mathbb{C}}$ is called **full** if $\widehat{\mathbb{C}} \setminus K$ is connected. A wandering continuum K for a rational map f is **always full** if $f^n(K)$ is full for all $n \geq 0$. Refer to [3] for the the following theorem and the definition of Cantor multicurves.

Theorem A. *Let f be a post-critically finite rational map and $K \subset J_f$ be a wandering continuum. Then either K is always full or there exists an integer $N \geq 0$ such that $f^n(K)$ is a Jordan curve for $n \geq N$. The latter case happens if and only if f has a Cantor multicurve.*

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Problem: Under what condition does a post-critically finite rational map f admit an always full wandering continuum?

In this paper, we solve this problem for Lattès maps (refer to §2 for its definition). Here is the main theorem:

Theorem 1.1. *A Lattès map f admits an always full wandering continuum if and only if it is flexible. In this case the wandering continuum is a line segment in an infinite geodesic under the flat metric.*

2 Lattès maps

This section is a review about Lattès maps. Refer to [10, 12, 13] for details. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with $\deg f \geq 2$. Denote by $\deg_z f$ the local degree of f at a point $z \in \widehat{\mathbb{C}}$,

$$\Omega_f = \{z : \deg_z f > 1\}.$$

the critical set and

$$P_f = \overline{\bigcup_{n>0} f^n(\Omega_f)}$$

the post-critical set of f. The rational map f is called **post-critically finite** if $\#P_f < \infty$.

Let f be a post-critically finite rational map. Define $\nu_f(z)$ for each point $z \in \widehat{\mathbb{C}}$ to be the least common multiple of the local degrees $\deg_y f^n$ for all $n > 0$ and $y \in \widehat{\mathbb{C}}$ with $f^n(y) = z$. By convention $\nu_f(z) = \infty$ if the point z is contained in a super-attracting cycle. The **orbiford** of f is defined by $\mathcal{O}_f = (\widehat{\mathbb{C}}, \nu_f)$. Note that $\nu_f(z) > 1$ if and only if $z \in P_f$. The **signature** of the orbifold $\mathcal{O}_f$ is the list of the values of ν_f restricted to P_f. The Euler Characteristic of $\mathcal{O}_f$ is given by

$$\chi(\mathcal{O}_f) = 2 - \sum_{z \in \widehat{\mathbb{C}}} \left(1 - \frac{1}{\nu_f(z)}\right).$$

It turns out in [10] that $\chi(\mathcal{O}_f) \leq 0$. The orbifold $\mathcal{O}_f$ is **hyperbolic** if $\chi(\mathcal{O}_f) < 0$, and **parabolic** if $\chi(\mathcal{O}_f) = 0$. It is easy to check that the signature of a parabolic orbifold $\mathcal{O}_f$ can only be (∞, ∞), $(2, 2, \infty)$, $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$.

Suppose that the signature of $\mathcal{O}_f$ is (∞, ∞). Then f is Möbius conjugate to a power map $z \mapsto z^d$ with $|d| \geq 2$. Suppose that the signature of $\mathcal{O}_f$ is $(2, 2, \infty)$. Then f is Möbius conjugate to $\pm\Psi_d$, where Ψ_d is the **Chebyshev polynomial** of degree d defined by the equation

$$\Psi_d(z + \frac{1}{z}) = z^d + \frac{1}{z^d}.$$

Note that the Julia set of the map $\pm\Psi_d$ is the interval $[-2, 2]$. Thus in both cases, there exist no wandering continuum for f.

A post-critically finite rational map f with parabolic orbifold is called a **Lattès map** if $\nu_f(z) \neq \infty$ for any point $z \in \widehat{\mathbb{C}}$. Let $\nu(\mathcal{O}_f) = \max\{\nu_f(z) : z \in \widehat{\mathbb{C}}\}$. Refer to [13, Theorem 3.1] for the following theorem.

Theorem 2.1. *Let f be a Lattès map. Then there exist a lattice $\Lambda = \{n+m\omega,\ n,m \in \mathbb{Z}\}$ (Im $\omega > 0$), a finite holomorphic cover $\Theta : \mathbb{C}/\Lambda \to \mathcal{O}_f$, a finite cyclic group G of order $\nu(\mathcal{O}_f)$ generated by a conformal self-map ρ of $\mathbb{C}/\Lambda$ with fixed points, and an affine map $A(z) = az + b\,(mod\ \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, such that*

$$\Theta(z_1) = \Theta(z_2) \Leftrightarrow z_1 = \rho^n(z_2) \text{ for } n \in \mathbb{Z},$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{A} & \mathbb{C}/\Lambda \\ \Theta\big\downarrow & & \big\downarrow\Theta \\ \mathcal{O}_f & \xrightarrow{f} & \mathcal{O}_f. \end{array}$$

A Lattès map f is called **flexible** if $\mathcal{O}_f$ has siganature $(2,2,2,2)$ and the affine map $A(z) = az + b\,(\text{mod }\Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ defined in Theorem 2.1 has integer derivative $A' = a \in \mathbb{Z}$. A Lattès map admits a non-trivial quasiconformal deformation if and only if it is flexible by the following discussion.

Let f be a Lattès map. If $\#P_f = 3$ and f is topologically conjugate to another rational map g, then f and g are Möbius conjugate.

Now we assume that $\#P_f = 4$. Then the signature of $\mathcal{O}_f$ is $(2,2,2,2)$ and $\nu(\mathcal{O}_f) = 2$. Let $\widetilde{\rho} : \mathbb{C} \to \mathbb{C}$ be a lift of the generator ρ of G under the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$. Let $z_0 \in \mathbb{C}$ be the unique fixed point of $\widetilde{\rho}$. Then $\widetilde{\rho}(z) = 2z_0 - z$. Denote by $Q \subset \mathbb{C}/\Lambda$ the set of fixed points of ρ. Then $\#Q = 4$ and $\Theta(Q) = P_f$. Therefore

$$\pi^{-1}(Q) = \{n/2 + m\omega/2 + z_0,\ n,m \in \mathbb{Z}\}. \tag{1}$$

Let $A(z) = az + b\,(\text{mod }\Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be the affine map defined in Theorem 2.1. Write $\alpha(z) = az + b$. Since $f(P_f) \subset P_f$, we have $\alpha(\pi^{-1}(Q)) \subset \pi^{-1}(Q)$. Equivalently, there exist integers (p,q,r,s) such that

$$a = p + q\omega, \text{ and } a\omega = r + s\omega. \tag{2}$$

It follows that

$$q\omega^2 + (p-s)\omega - r = 0. \tag{3}$$

If a is a real number, then $q = r = 0$ and $a = p = s$. Thus the real number a must be an integer and equations (2) hold for any complex number ω. This shows that one can make a quasiconformal deformation for the map f to get another rational map such that they are not Möbius conjugate.

If a is not real, then $q \neq 0$ and thus the complex number ω with Im $\omega > 0$ is uniquely determined by the integers (p,q,r,s) by equation (3). This shows that if the map f is topologically conjugate to another rational map g, then f and g are Möbius conjugate.

Remark. A Lattès map is flexible if and only if it has a Cantor multicurve. Therefore a Lattès map admits a wandering Jordan curve if and only if it is flexible by Theorem A.

3 Wandering continua for torus coverings

Let $\Lambda = \{n+m\omega : n,m \in \mathbb{Z}\}$ (Im $\omega > 0$) be a lattice. Then $\mathbb{C}/\Lambda$ is a torus. A continuum $E \subset \mathbb{C}/\Lambda$ is **full** if there exists a simply connected domain $U \subset \mathbb{C}/\Lambda$ such that $E \subset U$

and $U \backslash E$ is connected. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the natural projection. If $E \subset \mathbb{C}/\Lambda$ is a full continuum, then so is each component of $\pi^{-1}(E)$. In this section, we will prove the following theorem.

Theorem 3.1. *Let $A(z) = az + b \,(mod\ \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be a covering of the torus with* $\deg A \geq 2$. *Then the map A admits an always full wandering continuum E if and only if its derivative a is an integer. In this case, the wandering continuum E is a line segment in an infinite geodesic under the flat metric of $\mathbb{C}/\Lambda$.*

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.2. *Let $E \subset \mathbb{C}/\Lambda$ be a full continuum. For any line $L \subset \mathbb{C}$ and any connected component B of $\pi^{-1}(E)$, if I is a bounded component of $L \backslash B$, then π is injective on I.*

Proof. Let I be a bounded component of $L \backslash B$. Then there are exactly two components U, V of $\mathbb{C} \backslash (L \cup B)$ such that their boundaries contain the interval I. We claim that at least one of them, denoted it by U, is bounded. Otherwise one may find a Jordan curve γ in $U \cup V \cup I$ such that γ separates the two endpoints x_1 and y_1 of I. Since γ is disjoint from B, and both x_1 and y_1 are contained in B, this contradicts that fact that B is connected.

Assume by contradiction that π is not injective on I, i.e. there exist two distinct points $x, y \in I$ such that $\pi(x) = \pi(y)$. For each connected component G of $B \cap \partial U$, the set $G \cap L$ is non-empty. Denote by $H(G)$ the closed convex hull of $G \cap L$, i.e. $H(G)$ is the minimal closed interval in L with $H(G) \supset G$. Then for any two components G_1, G_2 of $B \cap \partial U$, $H(G_1)$ and $H(G_2)$ are either disjoint or one contains another. In particular, there exists a component G_0 of $B \cap \partial U$ such that $H(G_0) \supset H(G)$ for any component G of $B \cap \partial U$. Moreover, $H(G_0) \supset I$.

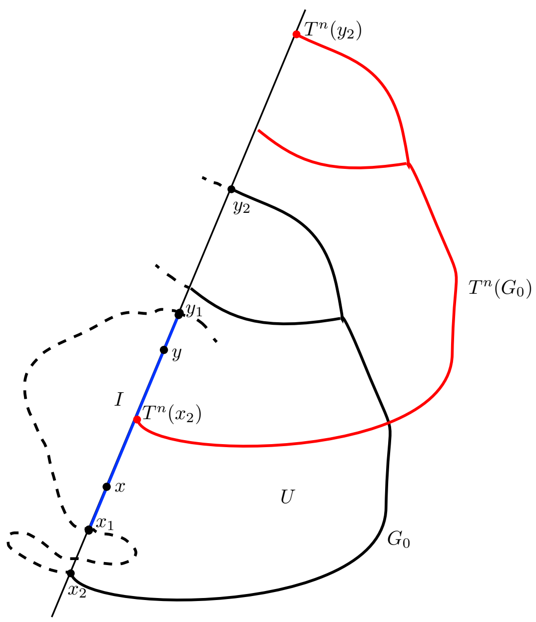

Figure 1. Lifting of a full continuum.

Set $T(z) = z + (y - x)$. Let x_2 and y_2 be the two endpoints of $H(G_0)$. Then there exists an integer n such that $T^n(x_2) \in [x, y]$ and hence $T^n(y_2) \notin I$. Let $\mathbb{H}$ be the component of

$\mathbb{C}\backslash L$ that contains U, then $T^n(G_0)$ is a continuum in $\mathbb{H} \cup L$ joining $T^n(x_2)$ with $T^n(y_2)$, whereas G_0 is a continuum in $\mathbb{H} \cup L$ joining x_2 with y_2. Thus G_0 must intersect $T^n(G_0)$.

On the other hand, since $\pi(x) = \pi(y)$, we have $(x - y) \in \Lambda$. Thus $T^n(z) = z \mod \Lambda$ and $T^n(B)$ is another component of $\pi^{-1}(E)$ and hence is disjoint from B. This contradicts the facts that $G_0 \subset B$ and G_0 intersects $T^n(G_0)$. $\square$

Lemma 3.3. *Let $A(z) = az + b\,(mod\ \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be a covering with* $\deg A \geq 2$ *and $E \subset \mathbb{C}/\Lambda$ be an always full wandering continuum. Then E must be a line segment.*

Proof. Let B be a component of $\pi^{-1}(E)$. Assume by contradiction that B is not a line segment. We claim that there exists a line $L \subset \mathbb{C}$ such that $L\backslash B$ has a bounded component I. Otherwise, each line segment joining two points in B must be contained in B. Thus B is convex and hence has positive measure since it is not a line segment. This is impossible since A is expanding and E is wandering.

As in the proof of Lemma 3.2, there exists a bounded component U of $\mathbb{C}\backslash(L \cup B)$ such that $I \subset \partial U$. Since $\deg A \geq 2$, we have $a \neq 1$. Thus the map $\alpha(z) = az + b : \mathbb{C} \to \mathbb{C}$ has a unique fixed point $z_0 \in \mathbb{C}$. Denote by $\Gamma_0 = \{n + m\omega + z_0 : n, m \in \mathbb{Z}\}$ and $\Gamma_n = \alpha^{-n}(\Gamma_0)$. Then there exists two distinct points $x_n, y_n \in U \cap \Gamma_n$ for some integer $n \geq 0$ such that for the line L_n that passes through the points x_n, y_n, the set $L_n \cap U$ has a component I_n which contains both x_n and y_n, and the two endpoints of I_n are contained in B.

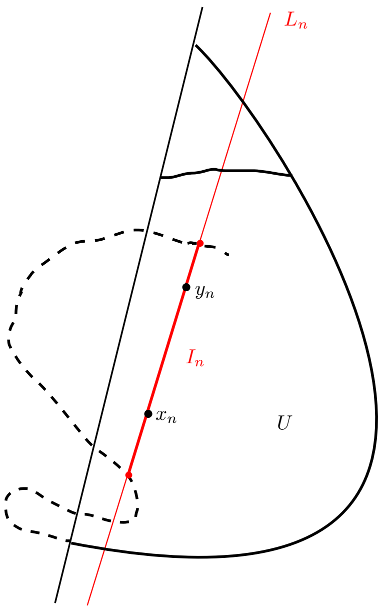

Figure 2. A wandering continuum is a line segment.

Now consider the full continuum $\alpha^n(B)$ and the line $\alpha^n(L_n)$. The set $\alpha^n(L_n)\backslash\alpha^n(B)$ has a component $\alpha^n(I_n)$, which contains $x = \alpha^n(x_n)$ and $y = \alpha^n(y_n)$. Since $x_n, y_n \in \Gamma_n$, we have $x, y \in \Gamma_0$ and hence $\pi(x) = \pi(y)$. This contradicts Lemma 3.2. $\square$

Lemma 3.4. *Let $A(z) = az+b\,(mod\ \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be a torus covering with* $\deg A \geq 2$*. If a is not real, then any line segment in $\mathbb{C}/\Lambda$ is not wandering.*

Proof. Let $E \subset \mathbb{C}/\Lambda$ be a line segment. We want to show that there exists an integer $n > 0$ such that $A^n(E)$ intersects $A^{n+1}(E)$.

Let R be the full parallelogram with vertices $0, 1, \omega$ and $1+\omega$. Then R is a fundamental domain of the group Λ. Thus for any $n \geq 0$, the set $\pi^{-1}(A^n(E))$ has a component B_n such that the midpoint $m(B_n)$ of the line segment B_n is contained in the closure of R. Since the diameter of R is less than $1 + |\omega|$, for any $n \geq 0$, the Euclidean distance

$$|m(B_n) - m(B_{n+1})| \leq 1 + |\omega|. \tag{4}$$

Denote by $a = |a| \exp(i\theta)$. Then $0 < |\theta| < \pi$ since a is not real. Let L_n be the line containing B_n for $n \geq 0$. Then L_n and L_{n+1} must intersect at a point O_n and the angle formed by these two lines is $|\theta|$. If B_n is disjoint from B_{n+1}, then $O_n \notin B_n$ or $O_n \notin B_{n+1}$. In the former case, we have

$$|O_n - m(B_n| \geq \frac{|B_n|}{2},$$

where $|B_n|$ is the length of B_n. Therefore the Euclidean distance from $m(B_n)$ to L_{n+1} satisfies

$$d(m(B_n), L_{n+1}) \geq \frac{|B_n|}{2} \sin |\theta|.$$

It follows that

$$1 + |\omega| \geq |m(B_n) - m(B_{n+1})| \geq d(m(B_n), L_{n+1}) \geq \frac{|B_n|}{2} \sin |\theta|.$$

So

$$|B_n| \leq \frac{2(1 + |\omega|)}{|\sin \theta|}. \tag{5}$$

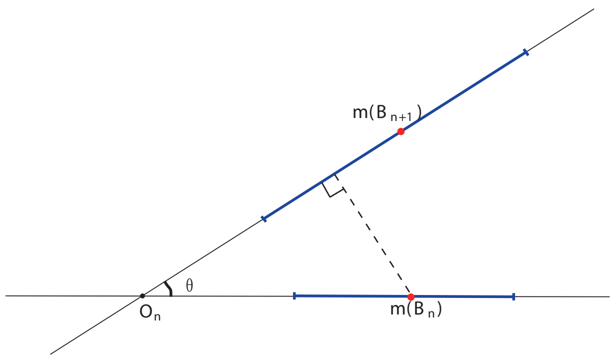

Figure 3. The upper bound of the length.

In the latter case, we have:

$$|B_{n+1}| \leq \frac{2(1 + |\omega|)}{|\sin \theta|}. \tag{6}$$

Noticing that $\deg A = |a|^2 \geq 2$, we have $|B_n| = |a|^n |B_0| \to \infty$ as $n \to \infty$. Thus both cases are impossible. $\square$

Now suppose that $A(z) = az + b \,(\mathrm{mod}\ \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is a covering with $\deg A \geq 2$ and a is an integer. Let $L \subset \mathbb{C}$ be a line. Then either $\pi(L)$ is a Jordan curve on $\mathbb{C}/\Lambda$ or π is injective on L. Write $\alpha(z) = az + b$. Then for any $n, m \geq 0$, $\alpha^n(L)$ and $\alpha^m(L)$ either coincide or are parallel. Thus if π is injective on L, then $\pi(L)$ is either eventually periodic or a **wandering line**, i.e. $A^n(\pi(L)) \cap A^m(\pi(L)) = \emptyset$ for any $n > m \geq 0$.

Lemma 3.5. *Let $L \subset \mathbb{C}$ be a line and $B \subset L$ be a line segment.*

(a) If $\pi(L)$ is a Jordan curve, then $A^n(\pi(B))$ is not full when n is large enough.

(b) If $\pi(L)$ is a wandering line, then $\pi(B)$ is a wandering continuum.

(c) If $\pi(L)$ is an eventually periodic line, then there exists a line segment $B_0 \subset B$ such that $\pi(B_0)$ is a wandering continuum.

Proof. (a) Since $\pi(L)$ is a Jordan curve, there exist two distinct points $x, y \in L$ such that $\pi(x) = \pi(y)$. Since $\deg A = |a|^2 \geq 2$, there exists an integer $n_0 > 0$ such that the Euclidean length $|\alpha^n(B)| \geq |x - y|$ when $n \geq n_0$. Thus $A^n(\pi(B)) = \pi(\alpha^n(B)) = \pi(L)$, which is a a Jordan curve, since a is real.

(b) This is obviously.

(c) Assume that $\pi(L)$ is periodic with period $p \geq 1$ for simplicity. Since $\deg A \geq 2$, there exists a unique point $x_0 \in L$ such that $A^p(\pi(x_0)) = \pi(x_0)$. Pick a point y_0 in the interior of B with $y_0 \neq x_0$. Then for any $n \geq 1$, there exists a unique point $y_n \in L$ such that $A^{np}(\pi(y_0)) = \pi(y_n)$. Moreover, $y_n \to \infty$ as $n \to \infty$.

Suppose that the integer a is positive. Then all points y_n are contained in the same component of $L \backslash \{x_0\}$. Since y_0 is contained in the interior of B, there exists a closed line segment $B_0 \subset B$ such that $B_0 \subset (y_0, y_1)$. Then $\pi(B_0)$ is a wandering continuum.

Now suppose that the integer a is negative. Then the points y_{2k} are contained in the same component of $L \backslash \{x_0\}$ for $k \geq 0$. Since y_0 is contained in the interior of B, there exists a closed line segment $B_0 \subset B$ such that $B_0 \subset (y_0, y_2)$. Then $\pi(B_0)$ is a wandering continuum for A. □

Proof of Theorem 3.1. The result follows directly from Lemmas 3.2, 3.3, 3.4 and 3.5. □

4 Proof of Theorem 1.1

Proof of Theorem 1.1. Let f be a Lattès map. By Theorem 2.1, there exist a lattice $\Lambda = \{n + m\omega,\, n, m \in \mathbb{Z}\}$ with $\operatorname{Im} \omega > 0$, a finite holomorphic cover $\Theta : \mathbb{C}/\Lambda \to \mathcal{O}_f$, a finite cyclic group G of order $\nu(\mathcal{O}_f)$ generated by a conformal self-map ρ of $\mathbb{C}/\Lambda$ with fixed points, and an affine map $A(z) = az + b \,(\mathrm{mod}\ \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, such that

$$\Theta(z_1) = \Theta(z_2) \Leftrightarrow z_1 = \rho^n(z_2) \text{ for } n \in \mathbb{Z},$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{A} & \mathbb{C}/\Lambda \\ \Theta \big\downarrow & & \big\downarrow \Theta \\ \mathcal{O}_f & \xrightarrow{f} & \mathcal{O}_f. \end{array}$$

Suppose that K is an always full wandering continuum of the map f. Then for each $n \geq 0$, every component of $\Theta^{-1}(f^n(K))$ is a full continuum in $\mathbb{C}/\Lambda$ since $f^n(K)$ is disjoint from P_f. Let E be a component of $\Theta^{-1}(K)$. It is an always full wandering continuum for the map A. Therefore the derivative a is an integer and K is a line segment in an infinite geodesic under the flat metric on $\mathbb{C}/\Lambda$ by Theorem 3.1.

Let E_n be a component of $\Theta^{-1}(f^n(K))$. Then $\rho(E_n)$ is also a component of $\Theta^{-1}(f^n(K))$, where ρ is the generator of the group G. Let R be the full parallelogram with vertices

$0, 1, \omega$ and $1 + \omega$. Then R is a fundamental domain of the group Λ. Thus there are components I_n and J_n of $\pi^{-1}(E_n)$ and $\pi^{-1}(\rho(E_n))$ respectively, such that the midpoints of I_n and J_n are contained in R.

Assume that $\#P_f = 3$. Let $\widetilde{\rho}$ be a lift of the map ρ under the projection π. Then $\widetilde{\rho}$ is a rotation around its fixed point with angle $(2\pi)/\nu$, where $\nu = 3, 4$ or 6. Thus the angle formed by the two lines containing I_n and J_n respectively, is $(2\pi)/\nu$. As in the proof of Lemma 3.4, we have:

$$|I_n| \leq \frac{2(1 + |\omega|)}{\sin(\pi/3)},$$

where $|I_n|$ is the length of I_n. This leads to a contradiction since $|I_n| = |a|^n |I_0| \to \infty$ as $n \to \infty$. Therefore $\#P_f = 4$ and f is a flexible Lattès map.

Conversely, suppose that the map f is flexible. Denote by Q the set of fixed points of ρ. Then $A(Q) \subset Q$ since $f(P_f) \subset P_f$.

Let $L \subset \mathbb{C}$ be a line. If L passes through at least two points $z_1, z_2 \in \pi^{-1}(Q)$, then it passes through the point $(z_2 + (z_2 - z_1))$. Note that $z_2 - z_1 \equiv 0 \bmod (\Lambda/2)$ by (1). So we have $2(z_2 - z_1) \equiv 0 \bmod \Lambda$, i.e., $\pi(z_2 + (z_2 - z_1)) = \pi(z_1)$. So $\pi(L)$ is a Jordan curve on $\mathbb{C}/\Lambda$. If L passes through exactly one point $z_0 \in \pi^{-1}(Q)$, then π is injective on L, $\Theta(\pi(L))$ is a ray from a point in P_f, and $\Theta : \pi(L) \to \Theta(\pi(L))$ is a folding with exactly one fold point at $\pi(z_0) \in Q$. If L is disjoint from $\pi^{-1}(Q)$, then $\Theta \circ \pi$ is injective on L.

Suppose that $\pi(L) \subset \mathbb{C}/\Lambda$ is a wandering line. Then $A^n(\pi(L))$ is disjoint from Q for all $n \geq 0$. Thus Θ is injective on each line $A^n(\pi(L))$. On the other hand, if $\Theta(A^n(\pi(L)))$ intersects $\Theta(A^{n+p}(\pi(L)))$ for some integer $p > 0$, then they coincide since the map ρ in G preserves the slopes of the lines. Thus $A^{n+p}(\pi(L))) = \rho(A^n(\pi(L))$. Therefore

$$A^{n+2p}(\pi(L)) = \rho(A^n(\pi(L)))$$

since $A \circ \rho = \rho \circ A$. But ρ^2 is the identity. So $A^{n+2p}(\pi(L)) = A^{n+p}(\pi(L))$. This is a contradiction. Therefore $\Theta(A^n(\pi(L)))$ is pairwise disjoint. Thus for any line segment $E \subset \pi(L)$, the set $\Theta(E)$ is an always full wandering continuum for f.

Now suppose that $\pi(L) \subset \mathbb{C}/\Lambda$ is an eventually periodic line with period $p \geq 1$. Then $\Theta(A^n(\pi(L)))$ are either bi-infinite or one-side-infinite geodesics depending on whether $A^n(\pi(L))$ passes through a point in Q. Since ρ^2 is the identity, either they are disjoint and have the same period, or two of them coincide and the period is $p/2$. Let $E \subset \pi(L)$ be a wandering line segment. In the former case, $\Theta(E)$ is an always full wandering continuum of f. In the latter case, there exists a line segment $E_0 \subset E$ such that $\Theta(E_0)$ is an always full wandering continuum for f. $\square$

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