On Douady-Earle extension and its application

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June 13, 2013

Contents

Canonical extension

Let $h \in \mathrm{Hom}^{+}(S^1)$. Then h can be extended continuously to a homeomorphism of the unit disk $\mathbb{D}$.

Problem: Give a canonical extension $E : h \mapsto E(h) \in \mathrm{Hom}^{+}(\mathbb{D})$ such that $E(h)$ inherits certain particular properties of h. e.g.

(1) If h is the boundary value of a quasiconformal map of $\mathbb{D}$, then $E(h)$ is quasiconformal.

(2) E is conformal natural:

$$E(\gamma \circ h \circ \beta) = \gamma \circ E(h) \circ \beta, \text{ for } \gamma, \beta \in \text{Möb}(\mathbb{D}).$$

Beurling-Ahlfors extension

Beurling-Ahlfors (1956):

$h \in \mathrm{Hom}^+(S^1)$ is **quasi-symmetric** if there exists a constant $\rho \geq 1$ such that for any two adjacent intervals $I_1, I_2 \subset S^1$ with same length,

$$\frac{1}{\rho} \leq \frac{|h(I_1)|}{|h(I_2)|} \leq \rho.$$

Denote by $\mathrm{QS}(S^1)$ the set of quasisymmetric maps of S^1.

Boundary value of quasiconformal maps $\Rightarrow$ **quasisymmetricity** by the geometric definition of quasiconformal maps.

Beurling-Ahlfors extension

Define $f(x+iy) = u(x+iy) + iv(x+iy)$ by

$$\begin{cases} u(x+iy) &= \dfrac{1}{2y}\displaystyle\int_{-y}^{y} h(x+t)\,dt, \\ v(x+iy) &= \dfrac{1}{2y}\displaystyle\int_{0}^{y} (h(x+t) - h(x-t))\,dt. \end{cases}$$

Theorem (Beurling-Ahlfors)

$f : \mathbb{H} \to \mathbb{H}$ is quasiconformal if h is quasisymmetric.

Beurling-Ahlfors extension is "linear natural", but not conformal natural.

Conformal barycenter

For $z \in \mathbb{D}$, denote by $\gamma_z(\zeta) = \dfrac{\zeta - z}{1 - \bar{z}\zeta}$.

Theorem (Douady-Earle, 1986)

Let $h \in \mathrm{Hom}^+(S^1)$. For any $z \in \mathbb{D}$, there exits a unique point $w \in \mathbb{D}$ such that:

$$\frac{1}{2\pi} \int_{S^1} \gamma_w \circ h \circ \gamma_z(\zeta)|d\zeta| = 0.$$

The map $E(h) : z \mapsto w$ is a homeomorphism. Moreover, $E(h)$ is quasiconformal if $h \in \mathrm{QS}(S^1)$.

w: the **conformal barycenter** of the measure $h \circ \gamma_z(\zeta)|d\zeta|$.
$E(h)$ is called the **barycenter extension** or **Douady-Earle extension**.

Douady-Earle extesion

Properties:

(1) $E(h)$ is conformally natural.

(2) $E(h)$ is real analytic.

(3) $h \in \mathbf{T} \mapsto$ complex dilatation of $E(h)$ is real analytic.

$$\mathbf{T} = \{h \in \mathrm{QS}(S^1) : \ h \text{ fixes} \pm 1, i\}$$

is the universal Teichmüller space.

Complex dilatation of the DE-extension

The Douady-Earle extension $w = E(h)(z)$ is defined by the equation

$$F(z,w) = \frac{1}{2\pi}\int_{S^1} \frac{h(\zeta)-w}{1-\bar{w}h(\zeta)}\frac{1-|z|^2}{|z-\zeta|^2}|d\zeta| = 0.$$

The complex dilatation μ of $E(h)$ is:

$$\frac{|\mu(z)|^2}{1-|\mu(z)|^2} = \frac{|\bar{F}_{\bar{z}}F_{\bar{w}} - F_{\bar{z}}\bar{F}_{\bar{w}}|^2}{(|F'_z|^2 - |F'_{\bar{z}}|^2)(|F'_w|^2 - |F'_{\bar{w}}|^2)}.$$

Complex dilatation of the DE-extension

Suppose that $\int_{S^1} h(\zeta)|d\zeta| = 0$. Then $E(h)(0) = 0$.

$$F_z(0,0) = \frac{1}{2\pi}\int_0^{2\pi} e^{-it}h(e^{it})dt,$$
$$F_{\bar{z}}(0,0) = \frac{1}{2\pi}\int_0^{2\pi} e^{it}h(e^{it})dt,$$
$$F_w(0,0) = -1, \quad \text{and}$$
$$F_{\bar{w}}(0,0) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})^2 dt.$$

Complex dilatation of the DE-extension

Denote

$$\mathrm{QS}_K(S^1) = \{h \in \mathrm{QS}(S^1) :\ h \text{ has a } K\text{-qc extension}\}.$$

There exists a universal constant $C > 1$ such that:

$$(|F_z|^2 - |F_{\bar{z}}|^2)(|F_w|^2 - |F_{\bar{w}}|^2) \geq C^{-K},$$

if $h \in \mathrm{QS}_K(S^1)$ [DE].

Let H be the harmonic function with boundary value h. Then $H(0) = 0$. We proved the following two inequalities:

$$|F_{\bar{z}}| \leq \frac{1}{\pi} \iint_{\mathbb{D}} |\overline{\partial} H|^2 dxdy.$$

$$|F_{\bar{w}}| \leq \frac{64}{\pi} \iint_{\mathbb{D}} |\overline{\partial} H|^2 dxdy.$$

Complex dilatation of the DE-extension

Lemma

Let $h \in \mathrm{QS}_K(S^1)$ with $\int_{S^1} h(\zeta)|d\zeta| = 0$. Let μ be the complex dilatation of $E(h)$. Then there exists a universal constant $C > 1$ such that:

$$\frac{|\mu(0)|^2}{1 - |\mu(0)|^2} \leq C^K \iint_{\mathbb{D}} |\bar{\partial} H|^2 dxdy,$$

where H is the harmonic function with boundary value h.

Complex dilatation of the DE-extension

Let $h \in \mathrm{QS}(S^1)$ and $\mu(w)$ be the complex dilatation of $E(h)^{-1}$.
Set $w = E(h)(z)$, i.e., $\int \gamma_w \circ h \circ \gamma_z(\zeta)\, d\zeta = 0$.
Since E is conformal natural, we have

$$E(\gamma_w \circ h \circ \gamma_z) = \gamma_w \circ E(h) \circ \gamma_z.$$

Let $\nu(\zeta)$ be the complex dilatation of $E(\gamma_w \circ h \circ \gamma_z)$. Then

$$\begin{aligned} \frac{|\mu(w)|^2}{1-|\mu(w)|^2} &= \frac{|\nu(0)|^2}{1-|\nu(0)|^2} \le C^K \iint_{\mathbb{D}} |\bar\partial(\gamma_w \circ H \circ \gamma_z)|^2 d\xi d\eta \\ &\le C^K \iint_{\mathbb{D}} |\bar\partial(\gamma_w \circ H)|^2 d\xi d\eta. \end{aligned}$$

Complex dilatation of the DE-extension

Theorem 1 [C, 2000]

Let $h \in \mathrm{QS}_K(S^1)$ and $\mu(w)$ be the complex dilatation of $E(h)^{-1}$. Then there exists a universal constant $C > 1$ such that

$$\frac{|\mu(w)|^2}{1-|\mu(w)|^2} \leq C^K \iint_{\mathbb{D}} |\bar{\partial}(\gamma_w \circ H)|^2 dxdy,$$

where H is the harmonic function with boundary value h.

Complex dilatation of the DE-extension

Let $g : \mathbb{D} \to \mathbb{D}$ be a qc map with boundary value h^{-1} and ν be the complex dilatation of g.

Theorem 2 [C, 2000]

$$\frac{|\mu(w)|^2}{1-|\mu(w)|^2} \le C^K \iint_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\bar{w}\zeta|^4} \frac{|\nu(\zeta)|^2}{1-|\nu(\zeta)|^2} d\xi d\eta.$$

Corollary

Let K_1 be the maximal dilatation of the DE extension of $h \in \mathrm{QS}_K(S^1)$. Then

$$(K_1 - 1) \le C^K (K-1),$$

where $C > 1$ is a universal constant.

Boundary behavior

Let $h \in \mathrm{QS}_K(S^1)$ and $\mu(w)$ be the complex dilatation of $E(h)^{-1}$.
Let $g : \mathbb{D} \to \mathbb{D}$ be a quasiconformal map with boundary value h^{-1}.
Let ν be the complex dilatation of g.

Corollary

Suppose that there exist constants $\alpha > 0$ and $C > 0$ such that

$$|\nu(w)| \leq C(1 - |w|^2)^{\alpha}.$$

Then there is a constant $C_1 > 0$ such that

$$|\mu(w)| \leq C_1(1 - |w|^2)^{\frac{\alpha}{1+\alpha}}.$$

If h has boundary dilatation 1, then $\mu(w) \to 0$ as $|w| \to 1$ [H, EMS].
The above corollary provide a control for the speed of the convergence.

WP-homeomorphisms

The universal Weil-Petersson Teichmüller space is a subspace of the universal Teichmüller space defined by the following class.

Definition. $h \in \mathrm{WP}(S^1)$ if it has a qc extension to $\mathbb{D}$ such that its complex dilatation satisfies that:

$$\iint_{\mathbb{D}} \frac{|\mu|^2}{1-|\mu|^2}\rho(z)^2 dxdy < \infty,$$

where $\rho(z) = 2/(1-|z|^2)$ is the Poincaré density.

We denote by $\mathrm{WP}(\mathbb{D})$ the space of qc maps such that their complex dilatations satisfies the above condition.

WP-homeomorphisms

Let $g \in \mathrm{WP}(\mathbb{D})$. Let $\nu(z)$ be the complex dilatation of g.
Let f be the Douady-Earle extension with boundary value g^{-1}.
Let $\mu(w)$ be the complex dilatation of f^{-1}.

Theorem 3 [C, 2000]

There exists a universal constant $C > 1$ such that

$$\iint_{\mathbb{D}} \frac{|\mu|^2}{1-|\mu|^2}\rho(z)^2\, dxdy \le C^{K(g)} \iint_{\mathbb{D}} \frac{|\nu|^2}{1-|\nu|^2}\rho(z)^2\, dxdy.$$

Since DE-extension is bi-Lipschitz under Poincaré metric, we obtain:

Corollary

$\mathrm{WP}(S^1)$ is a group under composition as maps.

The universal WP Teichmüller space

Let f be a univalent function on $\mathbb{D}$ such that it admits a quasiconformal extension to $\Delta = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with complex dilatation μ.

Theorem 4 [C, 2000]

The following statements are equivalent:

(a) $\displaystyle\iint_{\Delta} |\mu(z)|^2 |\rho^2(z)\, dxdy < \infty.$

(b) $\displaystyle\iint_{\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right|^2 dxdy < \infty.$

(c) $\displaystyle\iint_{\mathbb{D}} |S_f(z)|^2 \rho^{-2}(z)\, dxdy < \infty.$

The universal WP Teichmüller space

Theorem 5 [C, 2000]

The universal WP-Teichmüller space is complete.

This result was also proved in by L. Takhtajan and Lee-Peng Teo (Mem. Amer. Math. Soc. 183, 2006).

Theorem 6 [C, 2000]

Let $b_{n,m}$ be the n-th Fourier coefficient of the power h^m. Then $h \in \mathrm{WP}(S^1)$ iff

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n}{m} |b_{n,m}|^2 < \infty.$$

The universal WP Teichmüller space

Recently, Y. Shen gave a direct characterization for the space $\mathrm{WP}(S^1)$.

Theorem [Shen, 2013]

$h \in \mathrm{WP}(S^1)$ iff h is absolutely continuous and $\log h' \in \mathrm{H}^{\frac{1}{2}}$.

$u \in \mathrm{H}^{\frac{1}{2}}$ if

$$\|u\| = \left(\sum_{n=-\infty}^{+\infty} |n| |a_n|^2 \right)^{\frac{1}{2}} < \infty,$$

where a_n is the n-th Fourier coefficient of u.

Strong quasisymmetricity

$h \in QS(S^1)$ is **strongly quasisymmetric** if $\forall \epsilon > 0,\ \exists \delta > 0$ such that for any interval $I \subset S^1$ and any Borel set $E \subset I$,

$$|E| \leq \delta |I| \Rightarrow |h(E)| \leq \epsilon |h(I)|.$$

Denote by $\mathrm{SQS}(S^1)$ the set of strongly quasisymmetric maps. It is the group of h s.t.

$$V_h : b \mapsto b \circ h$$

is an isomorphism of the space $\mathrm{BMO}(S^1)$.

Carleson measures

A positive measure m on $\mathbb{D}$ is called a **Carleson measure** if

$$\sup_{I\subset S^1}\frac{m(C(I))}{|I|}<+\infty,$$

where $C(I)=\{rz:\ z\in I,\ (1-\frac{|I|}{2\pi})\leq r\leq 1\}$. Define

$$\mathrm{CM}(\mathbb{D})=\left\{\mu(z):\ \frac{|\mu|^2(z)}{1-|z|}dxdy \text{ is a Carleson measure}\right\}.$$

The universal BMO Teichmüller space

Let f be a univalent function on Δ such that it admits a qc extension to $\widehat{\mathbb{C}}$ with complex dilatation μ.

Theorem

The following are equivalent:
(1) $\mu \in \mathrm{CM}(\mathbb{D})$.
(2) $f \in \mathrm{SQS}(S^1)$.
(3) $|S_f|^2(|z|-1)^3 dxdy$ is a Carleson measure in Δ.

[Astala-Zinsmeister, 1990], [Fefferman-Kenig-Pipher, 1991]

BMO Teichmüller spaces

As an application of Theorem 1, we proved:

Theorem 7 [CZ, 2004]

Let $h \in \mathrm{SQS}(S^1)$ and μ be the complex dilatation of $E(h)$. Then $\mu \in \mathrm{CM}(\mathbb{D})$.

Since $E(h)$ is conformal natural, by this theorem, we can apply the universal BMO Teichmüler theory to Fuchsian groups.

Earthquakes

W. Thurston (1984):

A **geodesic lamination** λ on the Poincaré disk $\mathbb{D}$ is a closed subset $L \subset \mathbb{D}$ together with a foliation of the set by geodesics.
A **stratum** is a leaf or a component of $\mathbb{D} \setminus L$.

A λ-**left earthquake** E is a (possibly discontinuous) injective and surjective map of $\mathbb{D}$ such that:

(a) for each stratum A of λ, $E|_A \in \text{Möb}(\mathbb{D})$;

(b) for any two strata $A \neq B$ of λ, $(E|_A)^{-1} \circ (E|_B)$ is hyperbolic whose axis weakly separates A and B and which translates to the left, as viewed from A.

Earthquakes

Theorem (Thurston)

(1) A left earthquake is continuous on S^1.

(2) For any $h \in \mathrm{Hom}^+(S^1)$, there exists a "unique" left earthquake E with boundary value h.

(3) $h \in \mathrm{QS}(S^1)$ iff E is uniformly bounded.

Earthquake is conformal natural.

Harmonic maps

$(S_1, \rho(z)|dz|)$, $(S_2, \sigma(w)|dw|)$: Riemann surfaces with conformal metrics.
A map $f : S_1 \to S_2$ is called **harmonic** if

$$f_{z\bar{z}} + \frac{2\sigma_w}{\sigma} f_z f_{\bar{z}} = 0.$$

Theorem (Schoen-Yau, 1978)

Let $f : S_1 \to S_2$ be an orientation preserving homeomorphism between hyperbolic and compact Riemann surfaces. Then there exists a unique quasiconformal harmonic map in the homotopy class of f.

Harmonic maps

Schoen Conjecture

Let $h \in \mathrm{QS}(S^1)$. Then h can be extended continuously to a unique harmonic quasiconformal map of $\mathbb{D}$.

If Schoen Conjecture is true, then it is a conformal natural extension.

Complex natural definition of quasisymmetricity

For $h \in \mathrm{Hom}^+(S^1)$, denote by

$$M(h) = \{\alpha \circ h \circ \beta : \ \alpha, \beta \in \text{Möb}(\mathbb{D}) \text{ and } \alpha \circ h \circ \beta \text{ fixes } \pm 1, i\}$$

Theorem

h is quasisymmetric iff $M(h)$ is equivariant continuous.

Theorem

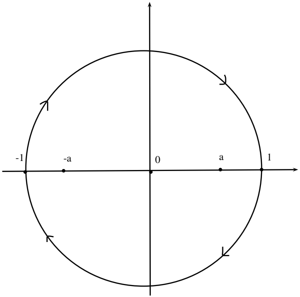
h is strong quasisymmetric iff $M(h)$ is uniformly absolutely continuous.

Examples

Example. $h \in \mathrm{QS}(S^1)$ is defined by the boundary value of the map:

$$E(z) = \begin{cases} \gamma_a(z) = \frac{z+a}{1+az}, & \text{if } \mathrm{Im} z \geq 0, \\ \gamma_a^{-1}(z) = \frac{z-a}{1-az}, & \text{if } \mathrm{Im} z < 0, \end{cases}$$

where $a \in (0,1)$. E is a simple earthquake.



Extremal qc map with boundary value h

There is an extremal qc map f with boundary h such that $|\mu_f|$ is a constant and

$$\frac{|\mu_f|^2}{1-|\mu_f|^2} = \frac{\beta^2}{\pi^2},$$

where $\beta = 2\log(\frac{1+a}{1-a})$.

DE extension of h

Let $\mu(z)$ be the complex dilatation of the DE extension of h. Then

$$\lim_{\beta\to 0}\left(\frac{|\mu(0)|^2}{1-|\mu(0)|^2}\right)\frac{\pi^2}{\beta^2}=1,$$

$$\lim_{\beta\to\infty}\frac{1}{\beta}\log\left(\frac{|\mu(0)|^2}{1-|\mu(0)|^2}\right)=1.$$

Recall that the maximal dilatation K_1 of the DE extension satisfies

$$(K_1-1)\leq C^K(K-1),$$

for a universal constant $C>1$. This example shows that the exponential term C^K is necessary.

The harmonic map with boundary value h

The map h has a harmonic extension. It can be expressed by an ODE. Let $\nu(z)$ be its complex dilatation. Then

$$\lim_{\beta\to 0}\left(\frac{|\nu(0)|^2}{1-|\nu(0)|^2}\right)\frac{\pi^2}{\beta^2}=1,$$

$$\lim_{\beta\to\infty}\frac{1}{\beta}\log\left(\frac{|\nu(0)|^2}{1-|\nu(0)|^2}\right)=\frac{1}{2}.$$

By comparing the two limits as $\beta\to\infty$. We get $\mu\not\equiv\nu$.