Multicurves and Shishikura trees associated with disconnected Julia sets

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June 21, 2012

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§1. Introduction

Theorem(C. McMullen (1988))

Let f be a rational map with disconnected Julia set J_f. Let K be a non-trivial periodic component of J_f with period $p \geq 1$. Then there exists a rational map g with connected Julia set J_g such that (f^p, K) is quasi-conformally conjugated to (g, J_g).

Example.

$$f(z) = \frac{z^5 + \lambda}{z^3}, \quad \lambda \text{ is small.}$$

Then J_f is homeomorphic to the product of the Cantor set with a Jordan curve.

Theorem(K. Pilgrim and TAN Lei (2000))

Let f be a geometrically finite rational map with disconnected Julia set.
(a) There are uncountably many wandering components of J_f, and each of them is either a single point or a Jordan curve.
(b) There are at most countably many periodic components of J_f which are either points or Jordan curves.
(c) There are at most finitely many periodic components of J_f which are neither points nor Jordan curves.

For a polynomial f, it is known that:

- Every wandering Julia component is a single point.

Branner and Hubbard (1992) for $\deg f \leq 3$,
Kozlovski and van Strein (2009) for $\deg f > 3$,
Qiu and Yin (2009) for $\deg f > 3$.

- It is easy to see that:

$$\#\{\text{cycles of non-trivial Julia components}\} \leq \deg f - 2.$$

Problems:

1. Are there "complex type" wandering Julia components?

2. $\#\{\text{cycles of complex type Julia components}\} \le C(d)$?, where $d \ge 3$ is the degree of the rational map.

Theorem(counterexamples)

Given $d \ge 3$ and $n \ge 1$. There exists a sub-hyperbolic rational map f with $\deg f = d$ such that

$$\#\{\text{cycles of complex type Julia components}\} \ge n.$$

§2. Canonical multicurves

Let f be a sub-hyperbolic rational map with disconnected Julia set J_f. Denote by P_f the post-critical set of f.

We assume that for each periodic Fatou domain U, if each component of $f^{-n}(U)$ is simply-connected for all $n \geq 0$, then $\#(U \cap P_f) = 1$.

Since P_f has only finitely many accumulation points, we have

$$p(f) = \#\{\text{Fatou and Julia components that contains } P_f \text{ points }\} < \infty.$$

Definition. Let $E \subset \mathbb{C}$ be a connected subset. We call E is
simple type: if there is a disk $D \supset E$ such that $\#(D \cap P_f) \leq 1$;
annular type: if there is an annulus $A \supset E$ such that $A \cap P_f = \emptyset$;
complex type: otherwise.

Since $f(P_f) \subset P_f$, we have

- E is complex type $\Longrightarrow$ $f(E)$ is complex type.
- E is annular type $\Longrightarrow$ $f(E)$ is not simple type.

Lemma

$$\#\{\text{complex type Fatou and Julia components}\} \leq 2p(f) - 2.$$

Lemma(Pilgrim and Tan)

Let K be a Julia component.
(a) If $f^n(K)$ are simple type for all $n \geq 0$, then K is a single point.
(b) If there is an integer $N > 0$ such that $f^n(K)$ are annular type for $n > N$, then K is a Jordan curve.

§2. Canonical multicurves

For each complex type Fatou component U and each complex type component C of $\mathbb{C} \smallsetminus U$, take a Jordan curve $\gamma \subset U$ such that

$$(D_\gamma \smallsetminus C) \cap P_f = \emptyset,$$

where D_γ is the component of $\mathbb{C} \smallsetminus \gamma$ such that $D_\gamma \supset C$.

For each annular type Fatou component U, take an essential Jordan curve γ in U.

Denote by Γ_f the multicurve representive all these curves.

Lemma

(a) $\#\Gamma_f = \#\{\text{complex type Fatou and Julia components}\} - 1 \le 2p(f) - 3$.
(b) Γ_f is stable. Moreover, for each $\gamma \in \Gamma_f$, there is a curve $\beta \in \Gamma_f$ such that γ is homotopic to a component of $f^{-1}(\beta)$ rel P_f.

Definition. We call Γ_f the *canonical multicurve* of the rational map f.

§3. Shishikura trees

§3.1. Set (T_f, V_f) the dual tree of the canonical multicurve Γ_f (V_f denotes the set of vertices of T_f).

vertices of $T_f \longleftrightarrow$ components of $\mathbb{C} \setminus \Gamma_f$.

edges of $T_f \longleftrightarrow$ curves in Γ_f.

Let Γ_* to be the collection of non-peripheral curves in $f^{-1}(\Gamma_f)$. Then each component of $\mathbb{C} \setminus \Gamma_*$ is either an annulus disjoint from P_f, or homotopic to exactly a component of $\mathbb{C} \setminus \Gamma_f$. Therefore T_f is also the dual tree of Γ_* but with (possibly) more vertices (since there may be two curves in Γ_* which are homotopic rel P_f). Denote by V_* the set of its vertices.

§3. Shishikura trees

The rational map f induces a continuous map

$$\tau_f : (T_f, V_*) \to (T_f, V_f)$$

such that:

(1) $V_* = \tau_f^{-1}(V_f) \supset V_f$.

(2) By adopting a linear metric on T_f, we may define τ_f to be linear on every edges of (T_f, V_*).

We call (T_f, V_f) the *Shishikura tree* of f and $\tau_f : (T_f, V_*) \to (T_f, V_f)$ the *Shishikura tree map*.

M. Shishikura, Ergodic Th. & Dynam. Sys. (1989).
M. Shishikura, Proceeding of RIMS (2002).

§3.2. Weight on edges of (T_f, V_*). For each edge a of (T_f, V_*), define

$$w(a) = \deg f|_\gamma,$$

where $\gamma \in \Gamma_*$ is the curve corresponding to the edge a.

Marking on vertices V_f. Marking each vertex $v \in V_f$ by " + " or " − " according to its corresponding pieces of $\mathbb{C} \setminus \Gamma_f$ is contained in a Fatou domain or not. Set $V_f = V_f^+ \cup V_f^-$ according to the marking. Then

$$\tau_f(V_f^+) \subset V_f^+, \text{ and } \tau_f(V_f^-) \subset V_f^-.$$

Further properties of the tree map τ_f:

(3) For each (closed) edge e of (T_f, V_f), $\exists$ an integer $n \geq 0$ such that

$$\tau^{-n}(V_f^+) \cap e \neq \emptyset.$$

(4) (Γ_f *is not a Thurston obstruction*) There exists a linear metric ρ on T_f such that τ_f is expanding with respect to the weight:

$$L_\rho(e) > \sum_i \frac{L_\rho(\tau_f(a_i))}{w(a_i)}$$

for each edge e of (T_f, V_f), where a_i are all the edges of (T_f, V_*) in e.

(5) (*characterization of cycles in* V_f^+) Each cycle in V_f^+ can be realized as a cycle of multiply-connected Fatou domains.

§3.3. The degree of the tree map τ_f:

For each vertex $v \in V_f$ (or V_*), denote by $E(v)$ (or $E_*(v)$) the collection of edges of (T_f, V_f) (or (T_f, V_*)) such that the vertex v is an endpoint of each of them.

The *mapping degree* of τ_f at $v \in V_*$ is defined as:

$$\mathrm{Deg}(\tau_f, v) = \max_{e \in E(\tau_f(v))} \{\sum w(a_i),\ a_i \in E_*(v) \text{ and } \tau_f(a) = e\}.$$

The *critical degree* of τ_f at $v \in V_*$ is defined as:

$$\deg(\tau_f, v) = 2\mathrm{Deg}(\tau_f, v) - 2 - \sum_{a \in E_*(v)} [w(a) - 1] + 1.$$

The degree of τ_f is defined by:

$$2\deg\tau_f - 2 = \sum_{v\in V_*}[\deg(\tau_f, v) - 1].$$

It turns out that $\deg\tau_f \leq \deg f$.

§3.4. Reduced tree map. Note that the Shishikura tree map $\tau_f : (T_f, V_*) \to (T_f, V_f)$ need not to be surjective. However, there exists an integer $n \geq 0$ such that $\tau_f^n(T_f) = \tau_f^{n+1}(T_f)$. Set $T = \tau_f^n(T_f)$, $V_0 = V_f \cap T$ and $V_1 = V_* \cap T$, then $\tau_f(T) = T$ and $\tau_f^{-1}(V_0) \cap T = V_1$. Moreover, the tree map $\tau_f : (T, V_1) \to (T, V_0)$ still satisfies the conditions (1)-(5), and

$$\deg(\tau_f|_T) \leq \deg\tau_f.$$

We call the tree map $\tau_f : (T, V_1) \to (T, V_0)$ the *reduced Shishikura tree map* of the rational map f.

Theorem (realization of the tree maps)

Any surjective tree map $\tau : (T, V_1) \to (T, V_0)$ which satisfies the conditions (1)-(5) can be realized as the Shishikura tree map of a sub-hyperbolic rational map g such that $\deg g = \deg \tau$.

Idea of the proof. For each cycle in V_0^-, we may choose a special construction to be realized as a cycle of Julia components (e.g. with star Hubbard tree, refer to Godillon's thesis), no Thurston obstruction there. Then apply a theorem of Cui and Tan to show that such a branched covering is Thurston equivalent to a sub-hyperbolic rational map.

Let Γ be a stable multicurve of the sub-hyperbolic rational map f. For each $\gamma \in \Gamma$ and any integer $m \geq 1$, define $k_{m,\Gamma}(\gamma) = k_m(\gamma)$ to be the number of components of $f^{-m}(\cup_{\beta\in\Gamma}\beta)$ homotopic to γ rel P_f.

Γ is called a *Cantor multicurve* if $k_m(\gamma) \to \infty$ as $m \to \infty$ for each $\gamma \in \Gamma$.

Theorem (Cantor multicurve)

The following conditions are equivalent;
(a) Γ_f contains a Cantor multicurve;
(b) The map f has infinitely many periodic Jordan curves;
(c) The map f has wandering Jordan curves;
(d) The tree map τ_f has infinitely many repelling cycles;
(e) The tree map τ_f has wandering points.

Theorem (Producing cycles of branched vertices)

Let $\tau : (T, V_1) \to (T, V_0)$ be a Shishikura tree map of a sub-hyperbolic rational map. Let $X = \{x_0, x_1, \cdots, x_{p-1}\}$ be a repelling τ-cycle disjoint from V_0. Then there is a new tree map $\tau' : (T', V_1') \to (T', V_0')$ such that:

(a) $(T, V_0) \subset (T', V_0')$;

(b) $T' \setminus T$ has exactly p components and each of them connects to T at some point x_i;

(c) the tree map τ' satisfies the conditions (1)-(5);

(d) $\deg \tau' = \deg \tau$;

(e) for each point $y \in T$, either $\tau'(y) = \tau(y)$ or $\tau'(y) = \tau^{p+1}(y)$;

(f) $\#\{\text{cycles in } V_0'\} = \#\{\text{cycles in } V_0\} + 1$.

Proof. $T \setminus X$ has $p+1$ components, the closure of one of them contains exactly one point in X. Denote it by B and assume that $x_0 \in \overline{B}$ by relabeling the index. Set T' to be the disjoint union of T with p copies B_i of B such that B_i connects to T at the point x_i.

Define $\tau'' : T' \to T'$ by $\tau'' = \tau$ on T and τ'' maps B_i to B_{i+1} by the identification. Define $R : T' \to T'$ to be a reflection such that $R(B_0) = B$, $R(B) = B_0$ and R is the identity otherwise. Let $\tau' = R \circ \tau''$. Then it satisfies all the conditions.

Example.

§7 First return map of τ_f

Denote by $T_c \subset T_f$ the smallest tree that contains all the critical values. Then T_c contains at most $2d-2$ end points and

$$\sum (B(v)-2) = \text{the number end points},$$

where the summation is taken over all the branched points and $B(v)$ is the branched number.

It turns out that every orbits in T_f must pass through T_c. Define $R : T_c \to T_c$ to be the first return map. As an application, we have:

Theorem (Exposed complex type Julia cycles)

There are at most $6d-10$ exposed complex type Julia cycles for degree d sub-hyperbolic rational maps.

A Julia component K is called *exposed* if it intersects with the boundary of ONE Fatou domain.

§8. Periodic points of τ_f

From §4, we know that τ_f has infinitely many repelling cycles iff the f has a Cantor multicurve. From that we can produce rational maps of degree d such that they have arbitrary many complex type Julia cycles.

However, this condition is not necessary. There are rational maps of degree d such that f has no Cantor multicurve (thus τ_f has only finitely many repelling cycles), but τ_f has arbitrary many repelling cycles. Such a tree map can be construct by the Hubbard tree of a quadratic polynomial.

Consider the Mandelbrot set M. Let H_0 be the central hyperbolic component (i.e. the hyperbolic component containing zero). Let $H_1, H_2, \cdots, H_n, \cdots$ be the hyperbolic components symmetric with respect to the real axis such that H_{k+1} touch H_k on the left. Denote by c_k the center of H_k and T_k be the Hubbard tree of the polynomial $P_k(z) = z^2 + c_k$. Then $P_k : T_k \to T_k$ has exact $k-1$ repelling cycles.

It is not hard to see that $P_k : T_k \to T_k$ can be realized as the Shishikura tree map of degree 3 rational maps.

Thank you !