Rational maps with constant Thurston map

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Theorem 1

There exists 3 roots of the equation $f(z) = b$ such that as b varies in $\mathbb{C} \setminus \{0, 1\}$, these roots always lie on the vertices of an equilateral triangle.

$$f(z) = P_3 \circ g \circ P_2, \quad P_d(z) = z^d,$$
$$g(z) = -\frac{(z-1)(z+3)}{4z}.$$

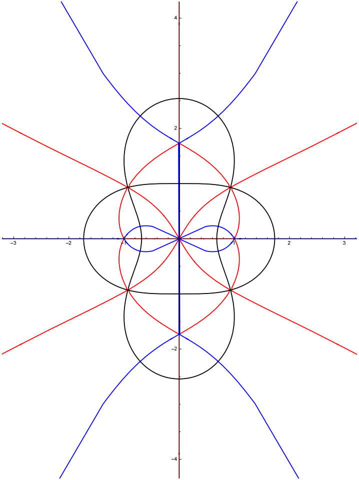
$$f: \quad 0, \infty \mapsto \infty, \quad \deg = 6$$
$$f: \quad \pm 1, \pm i\sqrt{3} \mapsto 0, \quad \deg = 3$$
$$f: \quad \text{4 simple critical points and 4 regular points to 1.}$$

Let $V \subset \mathbb{C}$ be a simply-connected domain with $0 \in V$ and $1 \notin V$. Then $f^{-1}(V)$ has 4 components:

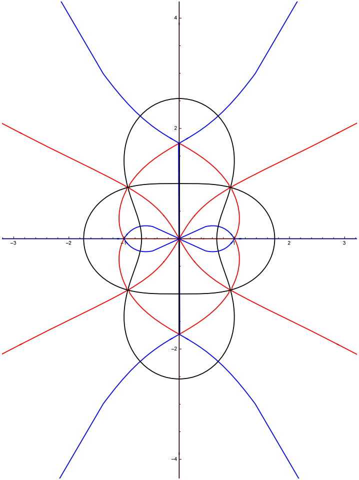
$$U_1 \ni 1,\ U_{-1} \ni -1,\ U_2 \ni i\sqrt{3} \text{ and } U_{-2} \ni -i\sqrt{3},$$

which map to V with $\deg = 3$. Pick a point $b \in V \setminus \{0\}$. The 3 roots is taken to be

$$E = f^{-1}(b) \cap U_2.$$



Topological configuration of $f(z)$.



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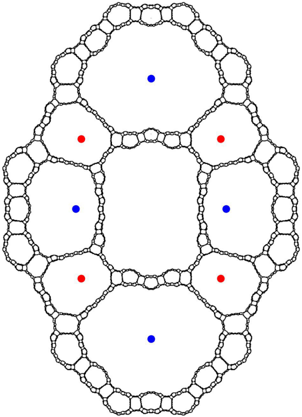

The Julia set of $f(z)$.

The Thurston map

A **marked rational map** $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ is a rational map f with $\deg f \geq 2$ and two finite set $A, B \subset \widehat{\mathbb{C}}$ such that $\#A > 3$, $\#B > 3$ and $f(A) \cup V_f \subset B$.

Let $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ be a marked rational map. For any $\phi \in \mathrm{Hom}^+(\widehat{\mathbb{C}})$, by the uniformalization theorem, there exists a $\psi \in \mathrm{Hom}^+(\widehat{\mathbb{C}})$ and a rational map R such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\psi} & \widehat{\mathbb{C}} \\ f\downarrow & & \downarrow R \\ \widehat{\mathbb{C}} & \xrightarrow{\phi} & \widehat{\mathbb{C}} \end{array}$$

Moreover, ψ is unique up to Mübius transformation.

The Thurston map

Let $\phi_0 \in \mathrm{Hom}^+(\widehat{\mathbb{C}})$ such that $\phi_0 \sim_B \phi$, i.e. $\phi_0 \circ \phi^{-1}$ is isotopic to a conformal map rel $\phi(B)$. Let ψ_0 be the lift of ϕ_0. Then $\psi_0 \sim_{f^{-1}(B)} \psi$ and hence $\psi_0 \sim_A \psi$.

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\psi_0 \sim_A \psi} & \widehat{\mathbb{C}} \\ f \downarrow & & \downarrow R \\ \widehat{\mathbb{C}} & \xrightarrow{\phi_0 \sim_B \phi} & \widehat{\mathbb{C}} \end{array}$$

Recall that the Teichmüller space $T(\widehat{\mathbb{C}}, B)$ is the quotient space

$$T(\widehat{\mathbb{C}}, B) = \mathrm{Hom}^+(\widehat{\mathbb{C}})/\sim_B = \{[\phi]_B, \phi \in \mathrm{Hom}^+(\widehat{\mathbb{C}})\}.$$

Define a map $\sigma_{f,A,B} : T(\widehat{\mathbb{C}}, B) \to T(\widehat{\mathbb{C}}, A)$ by $\sigma_{f,A,B}([\phi]_B) = [\psi]_A$. It is called the **Thurston map** induced by f.

Constant Thurston map

Theorem [BEKP,2009]

The Thurston map $\sigma_{f,A,B}$ is a constant if and only if for any Jordan curve γ in $\widehat{\mathbb{C}} \smallsetminus B$, each component of $f^{-1}(\gamma)$ is either trivial or peripheral in $\widehat{\mathbb{C}} \smallsetminus A$.

A Jordan curve $\gamma \subset \widehat{\mathbb{C}} \smallsetminus A$ is **trivial** if one component of $\widehat{\mathbb{C}} \smallsetminus \gamma$ is disjoint from A, and **peripheral** if one component of $\widehat{\mathbb{C}} \smallsetminus \gamma$ contains exactly one point of A.

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Problem: Classify the marked rational maps with constant Thurston map.

Belyi maps

Let $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ be a marked rational map. Assume that

$$\#(f(A) \cup V_f) = 3.$$

Then $\sigma_{f,A,B}$ is a constant. In particular, $f(z)$ is a **Belyi map**, i.e. $\#V_f \leq 3$.

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Example 1 (McMullen). Let $s(z)$ be a Belyi map and $f = g \circ s$, where g is an arbitrary rational map. Let $A \subset \widehat{\mathbb{C}}$ be a finite set with $\#A > 3$ s.t.

$$\#(s(A) \cup V_s) = 3.$$

Then $\sigma_{f,A,B}$ is a constant for any possible choice of the set B. Note that $\#(f(A) \cup V_f) > 3$ if $\#V_g > 3$.

We will call a marked rational map with the above form **having a Belyi factor**, or in particular, **having a power factor** if $\#V_s = 2$.

Question [BEKP]: Does any marked rational map with constant Thurston map have a Belyi factor?

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Theorem 2

Let $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ be a marked rational map such that $\sigma_{f,A,B}$ is a constant. Then there exists a Belyi map s with $\deg s \leq \deg f$ such that $\#(s(A) \cup V_s) = 3$.

This theorem may support an affirmative answer of the above question. However, we will see later that it is not true.

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Definition. A marked rational map $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ will be called **regular** if $f(A)$ is disjoint from V_f; **branched** if $f(A) \subset V_f$ or **mixing** otherwise.

Regular and mixing cases may happen for maps with power factor.

Power factoring

Let $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ be a marked rational map. Denote by

$$A_1 = A \smallsetminus f^{-1}(V_f) \quad \text{and} \quad A_2 = A \cap f^{-1}(V_f).$$

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Lemma 1

If $\sigma_{f,A,B}$ is a constant, then $f(A_1)$ contains at most one point.

Theorem 3

If $\sigma_{f,A,B}$ is a constant and $A_1 \neq \emptyset$, then $\#A_2 \leq 2$. Moreover if $\#A_2 = 2$, then f has a power factor.

Lift of arcs

Lemma 2

Let $x_0 \in \widehat{\mathbb{C}} \smallsetminus f^{-1}(V_f)$ be a point. Then for any point $x_1 \in \widehat{\mathbb{C}}$ with $f(x_0) \neq f(x_1)$, there exists an open arc

$$\delta : (0,1) \to \widehat{\mathbb{C}} \smallsetminus V_f$$

joining $f(x_0)$ with $f(x_1)$ such that $f^{-1}(\delta)$ has a component $\widetilde{\delta}$ joining x_0 with x_1.

Remark. The result is not true if both x_0 and x_1 are contained in $f^{-1}(V_f)$.

The monodromy group

Pick $y \in \widehat{\mathbb{C}} \setminus V_f$ to be a base point. Then each $\gamma \in \pi_1(\widehat{\mathbb{C}} \setminus V_f, y)$ induces a permutation p_γ on $f^{-1}(y)$, which forms the **monodromy group** $\mathrm{Mon}(f)$. For each point $x \in X$, we denote by $\mathrm{Stab}(x) \subset \mathrm{Mon}(f)$ the stabilizer.

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Lemma 3

Assume that $\sigma_{f,A,B}$ is a constant and $A_1 \neq \emptyset$. For any $p \in \mathrm{Mon}(f)$, let $A' = p(A_1) \cup A_2$. Then $\sigma_{f,A',B}$ is also a constant.

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Lemma 4

Assume that $\sigma_{f,A,B}$ is a constant and $\#A_2 = 1$. Then f has a power factor if and only if $\mathrm{Stab}(a_i) = \mathrm{Stab}(a_j)$ for any two points $a_i, a_j \in A_1$.

The Arturo map

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Theorem (Arturo)

Let $B = V_f \cup \{b\}$ for some point $b \in \widehat{\mathbb{C}} \setminus V_f$ and $A = f^{-1}(b)$. Then $\sigma_{f,A,B}$ is a constant.

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Theorem (Arturo)

Let $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ be a marked rational map without power factor such that $\sigma_{f,A,B}$ is a constant. Assume that $A = f^{-1}(b)$ for some point $b \in B \smallsetminus V_f$. Then f is an Arturo map.

Mixing case

$$f(z) = -\left[\frac{(z^2-1)(z^2+3)}{4z^2}\right]^3 = P_3 \circ g \circ P_2.$$

Let E be the set defined above. The next theorem is equivalent to Theorem 1.

Theorem 4

Let $A = E \cup \{\infty\}$ and $B = \{0, 1, b, \infty\}$. Then $\sigma_{f,A,B}$ is a constant.

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Theorem 4

Let $A = E \cup \{\infty\}$ and $B = \{0, 1, b, \infty\}$. Then $\sigma_{f,A,B}$ is a constant.

Lemma 5

Denote by $S = \bigcap_{a_i \in E} \mathrm{Stab}(a_i)$. Then for any point $a_i \in E$, $\mathrm{Stab}(a_i) \setminus S \neq \emptyset$ and $p^2 \in S$ for any $p \in \mathrm{Stab}(a_i) \setminus S$.

Corollary

The marked rational map $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ has no power factor.

Proof of Theorem 4

We only need to prove that for any Jordan curve $\gamma \subset \mathbb{C} \setminus \{0, 1, b\}$, each component of $f^{-1}(\gamma)$ is trivial or peripheral by Theorem [BEKP]. It is a consequence of the following statements.

(1) For any arc $\delta \subset \mathbb{C} \setminus \{0, 1, b\}$ connecting b with a critical value 0 or 1, the 3 components of $f^{-1}(\delta)$ which connect points in E land either on the same point or 3 distinct points from another direction.

(2) For any arc $\delta \subset \mathbb{C} \setminus \{0, 1, b\}$ connecting b with the critical value ∞, consider the 3 components of $f^{-1}(\delta)$ which connect points in E. Either two of them or nor of them land on the infinity.

$P_3^{-1}(S^1) = S^1$. $g^{-1}(S^1)$ divides $\widehat{\mathbb{C}}$ into 4 domains, $W_0 \ni 0$, $W_\infty \ni \infty$, $W_1 \ni 1$ and $W_2 \ni -3$. Denote by

$$U_0 = P_2^{-1}(W_0), \quad U_\infty = P_2^{-1}(W_\infty),$$
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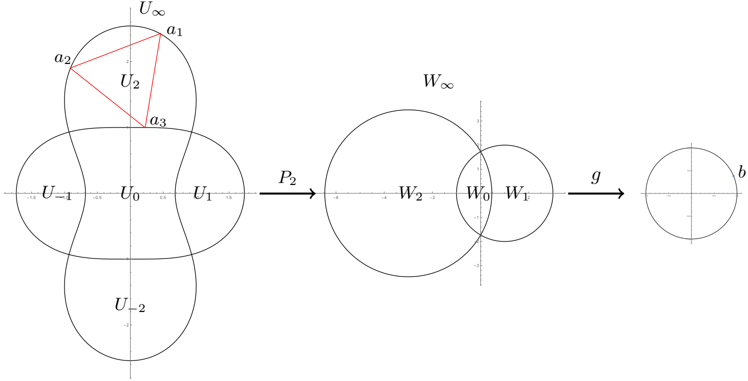

$E = \partial U_2 \cap f^{-1}(b) = \{a_1, a_2, a_3\}$, $a_1, a_2 \in \partial U_\infty$ and $a_3 \in \partial U_0$.

Let $p_\infty \in \mathrm{Mon}(f)$ be generated by a loop around the infinity. Then $p_\infty^6 = \mathrm{id}$. Set $E_k := p_\infty^k(E)$ for each $0 \le k < 6$.

Let $p_1 \in \mathrm{Mon}(f)$ be generated by a loop around the critical value 1. Then $p_1^2 = \mathrm{id}$, $p_1(E_0) = E_1$ and $p_1(E_3) = E_4$.

Set $E_6 = p_1(E_2)$ and $E_7 = p_1(E_5)$. Then $p_\infty(E_6) = E_7$ and $p_\infty(E_7) = E_6$.

Note that $\mathrm{Mon}(f)$ is generated by (p_∞, p_1). We prove that:

Proposition

For each $p \in \mathrm{Mon}(f)$ and each E_i, $p(E_i) = E_j$ for some $0 \le j \le 7$.
Moreover,
(a) $E_i \cap \partial U_\infty$ contains 0 or 2 points.
(b) $E_i \cap \partial U_0$ contains 1 or 3 points.
(c) $E_i \cap \partial U_j$ for $j = \pm 1, \pm 2$ contains 0,1 or 3 points.

Proof of Theorem 4

Now, let δ_0 be an arc defined in (1) or (2) such that it is disjoint from S^1. Then it satisfies the above conditions due to the location of E.

Any arc δ defined in (1) or (2) differs from δ_0 by an element of the fundamental group. Thus the landing points of $f^{-1}(\delta)$ differ from $f^{-1}(\delta_0)$ by a monodromy element. Therefore the above conditions are always true due to the location of E_i for $0 \leq i \leq 7$. □

Mixing case

Example 2. Let $s(z)$ be the map defined above. Let $f = g \circ s$, where g is an arbitrary rational map with $\deg g \geq 2$. Pick a point

$$b \in \widehat{\mathbb{C}} \smallsetminus (V_s \cup g^{-1}(V_g)).$$

Let A be the finite set defined above. Then $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ is a mixing marked rational map and $\sigma_{f,A,B}$ is a constant for any choice of B.

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Conjecture

Let $f : (\widehat{\mathbb{C}}, A) \to (\widehat{\mathbb{C}}, B)$ be a mixing marked rational map such that $\sigma_{f,A,B}$ is a constant and f has no power factor. Then it has the above form.

Thanks for your attention!