

A Priori Estimates for Free Boundary Problem of Incompressible Inviscid Magnetohydrodynamic Flows

CHENGCHUN HAO & TAO LUO

Communicated by L. SAINT-RAYMOND

Abstract

In the present paper, we prove the a priori estimates of Sobolev norms for a free boundary problem of the incompressible inviscid magnetohydrodynamics equations in all physical spatial dimensions $n = 2$ and 3 by adopting a geometrical point of view used in Christodoulou and Lindblad (Commun Pure Appl Math 53:1536–1602, 2000), and estimating quantities such as the second fundamental form and the velocity of the free surface. We identify the well-posedness condition that the outer normal derivative of the total pressure including the fluid and magnetic pressures is negative on the free boundary, which is similar to the physical condition (Taylor sign condition) for the incompressible Euler equations of fluids.

Contents

1. Introduction	806
1.1. Formulation of the Problem and Main Results	806
1.2. Motivation of the Construction of Higher Order Energy Functional and Outline of the Proofs	812
2. Reformulation in Lagrangian Coordinates	814
3. The Energy Conservation and Some Conserved Quantities	818
4. The First Order Energy Estimates	820
5. The General r th Order Energy Estimates	824
6. Justification of a Priori Assumptions	837
Acknowledgments	841
Appendix A. Preliminaries and Some Estimates	841
References	845

Notations

$x = (x^i)$	Eulerian coordinates
$y = (y^a)$	Lagrangian coordinates
∂	Spatial derivative in x
∇	Covariant derivative in y
v	The velocity field in Eulerian coordinates
u	The velocity field in Lagrangian coordinates
B	The magnetic field in Eulerian coordinates
β	The magnetic field in Lagrangian coordinates
p	Fluid pressure
$P = p + \frac{1}{8\pi} B ^2$	Total pressure
D_t	$= \frac{\partial}{\partial t} \Big _{y=\text{const}} = \frac{\partial}{\partial t} \Big _{x=\text{const}} + v^k \frac{\partial}{\partial x^k}$
g	The Riemannian metric defined by $g_{ab} = \sum_{i,j} \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$
γ	The induced metric on the tangent space of the boundary which can be extended to be 0 on the orthogonal complement of the tangent space of the boundary. Also, it can be extended to be a pseudo-Riemannian metric in the whole domain
Π	Orthogonal projection to the tangent space of the boundary
θ	The second fundamental form of the boundary
ι_0	The injectivity radius of the normal exponential map

1. Introduction

1.1. Formulation of the Problem and Main Results

In the present paper, we consider the following incompressible inviscid magnetohydrodynamics (MHD) equations

$$v_t + v \cdot \partial v + \partial p = \frac{1}{4\pi} \left(B \cdot \partial B - \frac{1}{2} \partial |B|^2 \right), \quad \text{in } \mathcal{D}, \quad (1.1a)$$

$$B_t + v \cdot \partial B = B \cdot \partial v, \quad \text{in } \mathcal{D}, \quad (1.1b)$$

$$\operatorname{div} v = 0, \quad \operatorname{div} B = 0, \quad \text{in } \mathcal{D}, \quad (1.1c)$$

describing the motion of conducting fluids in an electromagnetic field, where the velocity field of the fluids $v = (v_1, \dots, v_n)$, the magnetic field $B = (B_1, \dots, B_n)$, the fluid pressure p and the domain $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ are the unknowns to be determined. Here $n \in \{2, 3\}$ is the spatial dimension, $\frac{1}{4\pi} B \cdot \partial B$ is the magnetic tension, $\frac{1}{8\pi} |B|^2$ is the magnetic pressure, $p + \frac{1}{8\pi} |B|^2$ is so called total pressure which will be denoted by P in this paper, and $|B| = (\sum_{j=1}^n B_j^2)^{1/2}$ is the modulus of B . $\partial = (\partial_1, \dots, \partial_n)$ and div are the usual gradient operator and spatial divergence under Eulerian coordinates.

Given a simply connected bounded domain $\mathcal{D}_0 \subset \mathbb{R}^n$ and the initial data v_0 and B_0 satisfying the constraints $\operatorname{div} v_0 = 0$ and $\operatorname{div} B_0 = 0$, we want to find a set

$\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ and the vector fields v and B solving (1.1) and satisfying the initial conditions:

$$\{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \quad (v, B) = (v_0, B_0) \quad \text{on } \{0\} \times \mathcal{D}_0. \tag{1.2}$$

Throughout the paper, we use the Einstein summation convention, that is, when an index variable appears twice in both the subscript and the superscript of a single term it implies summation of that term over all the values of the index.

Let $\mathcal{D}_t = \{x \in \mathbb{R}^n : (t, x) \in \mathcal{D}\}$, we also require the following boundary conditions on the free boundary $\partial\mathcal{D}_t$:

$$v_{\mathcal{N}} = \kappa \quad \text{on } \partial\mathcal{D}_t, \tag{1.3a}$$

$$p = 0 \quad \text{on } \partial\mathcal{D}_t, \tag{1.3b}$$

$$|B| = \zeta \text{ and } B \cdot \mathcal{N} = 0 \quad \text{on } \partial\mathcal{D}_t, \tag{1.3c}$$

for each $t \in [0, T]$ with $T > 0$, where \mathcal{N} is the exterior unit normal to $\partial\mathcal{D}_t$, $v_{\mathcal{N}} = \mathcal{N}^i v_i$, and κ is the normal velocity of $\partial\mathcal{D}_t$, ζ is a non-negative constant. Condition (1.3c) should be understood as the constraints on the initial data. Indeed, we will verify that the condition $B \cdot \mathcal{N} = 0$ on $\partial\mathcal{D}_t$ holds for all $t \in [0, T]$ if it holds initially. We remark here on the physical meaning of the boundary conditions. Condition (1.3a) means that the boundary of \mathcal{D}_t moves with the fluids, (1.3b) means that outside the fluid region \mathcal{D}_t is the vacuum, the condition $B \cdot \mathcal{N} = 0$ on $\partial\mathcal{D}_t$ comes from the assumption that the boundary $\partial\mathcal{D}$ is a perfect conductor. Indeed, if we use \mathbf{E} to denote the electric field induced by the magnetic field B , then the boundary condition $B \cdot \mathcal{N} = 0$ on $\partial\mathcal{D}_t$ gives rise to $\mathbf{E} \times \mathcal{N} = 0$ on $\partial\mathcal{D}_t$. The boundary condition $|B| = \text{const}$ on $\partial\mathcal{D}_t$ (the magnetic strength is constant on the boundary) is needed to guarantee that the total energy of the system is conserved, that is,

$$\frac{d}{dt} \int_{\mathcal{D}_t} \left(\frac{1}{2} |v|^2 + \frac{1}{8\pi} |B|^2 \right) (t, x) \, dx = 0.$$

Condition (1.3c) includes the widely used (e.g., [12]) zero magnetic field boundary condition as the special case, but it is much more general and physically reasonable.

In the classical plasma–vacuum interface problem (cf. [10,23]), suppose that the interface between the plasma region $\Omega_p(t)$ and the vacuum region $\Omega_v(t)$ is $\Gamma(t)$ which moves with the plasma, then it requires that (1.1) holds in the plasma region $\Omega_p(t)$, while in the vacuum region $\Omega_v(t)$, the vacuum magnetic field \mathcal{B} satisfies

$$\nabla \times \mathcal{B} = 0, \quad \nabla \cdot \mathcal{B} = 0. \tag{1.4}$$

On the interface $\Gamma(t)$, it holds that

$$p = 0, \quad |B| = |\mathcal{B}|, \quad B \cdot \mathcal{N} = \mathcal{B} \cdot \mathcal{N} = 0, \tag{1.5}$$

where \mathcal{N} is the unit normal to $\Gamma(t)$. Therefore, the boundary conditions in (1.3) also model the plasma–vacuum problem for the case when $|\mathcal{B}|$ is constant.

We will prove a priori bounds for the free boundary problem (1.1)–(1.3) in Sobolev spaces under the following condition

$$\nabla_{\mathcal{N}} \left(p + \frac{1}{8\pi} |B|^2 \right) \leq -\varepsilon < 0 \quad \text{on } \partial\mathcal{D}_t, \tag{1.6}$$

where $\nabla_{\mathcal{N}} = \mathcal{N}^i \partial_i$. We assume that this condition holds initially, and will verify that it holds true within a period. For the free boundary problem of the motion of incompressible fluids in vacuum, without magnetic fields, the natural physical condition (cf. [2,4,5,8,15–17,21,24,25,27]) reads that

$$\nabla_{\mathcal{N}} p \leq -\varepsilon < 0 \quad \text{on } \partial\mathcal{D}_t, \tag{1.7}$$

which excludes the possibility of the Rayleigh–Taylor type instability (see [8]). In this paper, we find that the natural physical condition is (1.6) when the equations of magnetic field couple with the fluids equation. In fact, the quantity $p + \frac{1}{8\pi} |B|^2$, the total pressure of the system, will play an important role in our analysis. Roughly speaking, the velocity tells the boundary where to move, and the boundary is the level set of the total pressure that determines the acceleration.

The free surface problem of the incompressible Euler equations of fluids has attracted much attention in the recent decades. Important progress has been made for flows with or without vorticity, and with or without surface tension. We refer readers to [1,4,5,8,15–17,21,24,25,27].

On the other hand, there have been only few results on the interface problems for the MHD equations. This is due to the difficulties caused by the strong coupling between the velocity fields and magnetic fields. In this direction, the well-posedness of a linearized compressible plasma–vacuum interface problem was investigated in [23], and a stationary problem was studied in [9]. The current-vortex sheets problem was studied in [3] and [22]. For the incompressible viscous MHD equations, a free boundary problem in a simply connected domain of \mathbb{R}^3 was studied by a linearization technique and the construction of a sequence of successive approximations in [18] with an irrotational condition for magnetic fields in a part of the domain.

In this paper, we prove the a priori estimates for the free boundary problem (1.1)–(1.3) in all physical spatial dimensions $n = 2, 3$ by adopting a geometrical point of view used in [4], and estimating quantities such as the second fundamental form and the velocity of the free surface. Denote the material derivative $D_t = \partial_t + v \cdot \partial$ and the total pressure $P = p + \frac{1}{8\pi} |B|^2$, we can write the free boundary problem as

$$D_t v_j + \partial_j P = \frac{1}{4\pi} B^k \partial_k B_j \quad \text{in } \mathcal{D}, \tag{1.8a}$$

$$D_t B_j = B^k \partial_k v_j \quad \text{in } \mathcal{D}, \tag{1.8b}$$

$$\partial_j v^j = 0 \quad \text{in } \mathcal{D}; \quad \partial_j B^j = 0 \quad \text{on } \{t = 0\} \times \mathcal{D}_0, \tag{1.8c}$$

$$v_{\mathcal{N}} = \kappa \quad \text{on } [0, T] \times \partial\mathcal{D}_t, \tag{1.8d}$$

$$|B| = \zeta \quad \text{on } \partial\mathcal{D}, \quad B_j \mathcal{N}^j = 0 \quad \text{on } \{t = 0\} \times \partial\mathcal{D}_0, \tag{1.8e}$$

$$p = 0 \quad \text{on } \partial\mathcal{D}, \tag{1.8f}$$

$$\nabla_{\mathcal{N}} P \leq -\varepsilon < 0 \quad \text{on } \{t = 0\} \times \partial\mathcal{D}_0. \tag{1.8g}$$

We will derive the energy estimates from which the Sobolev norms of $H^s(\mathcal{D}_t)$ ($\mathbb{N} \ni s \leq n + 1$) of solutions will be derived. For this purpose, we define the energy norms as follows: the zeroth-order energy, $E_0(t)$, is defined as the total energy of the system, that is,

$$E_0(t) = \int_{\mathcal{D}_t} \delta^{ij} \left(\frac{1}{2} v_i v_j + \frac{1}{8\pi} B_i B_j \right) dx, \tag{1.9}$$

which is conserved, that is,

$$E_0(t) = E_0(0), \quad \text{for } 0 \leq t \leq T. \tag{1.10}$$

The higher order energy norm has a boundary part and an interior part. The boundary part controls the norms of the second fundamental form of the free surface, the interior part controls the norms of the velocity, magnetic fields and hence the pressure. We will prove that the time derivatives of the energy norms are controlled by themselves. A crucial point in the construction of the higher order energy norms is that the time derivatives of the interior parts will, after integrating by parts, contribute some boundary terms that cancel the leading-order terms in the corresponding time derivatives of the boundary integrals. To this end, we need to project the equations for the total pressure $P = p + \frac{1}{8\pi} |B|^2$ to the tangent space of the boundary. The orthogonal projection Π to the tangent space of the boundary of a $(0, r)$ tensor α is defined to be the projection of each component along the normal:

$$(\Pi\alpha)_{i_1 \dots i_r} = \Pi_{i_1}^{j_1} \dots \Pi_{i_r}^{j_r} \alpha_{j_1 \dots j_r}, \quad \text{where } \Pi_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j, \tag{1.11}$$

with $\mathcal{N}^j = \delta^{ij} \mathcal{N}_i = \mathcal{N}_j$.

Let $\bar{\partial}_i = \Pi_i^j \partial_j$ be a tangential derivative. If $q = \text{const}$ on $\partial\mathcal{D}_t$, it follows that $\bar{\partial}_i q = 0$ there and

$$(\Pi \partial^2 q)_{ij} = \theta_{ij} \nabla \mathcal{N} q, \tag{1.12}$$

where $\theta_{ij} = \bar{\partial}_i \mathcal{N}_j$ is the second fundamental form of $\partial\mathcal{D}_t$.

The higher order energies are defined as: for $r \geq 1$

$$\begin{aligned} E_r(t) &= \int_{\mathcal{D}_t} \delta^{ij} \left(Q(\partial^r v_i, \partial^r v_j) + \frac{1}{4\pi} Q(\partial^r B_i, \partial^r B_j) \right) dx \\ &+ \int_{\mathcal{D}_t} \left(\left| \partial^{r-1} \text{curl } v \right|^2 + \frac{1}{4\pi} \left| \partial^{r-1} \text{curl } B \right|^2 \right) dx \\ &+ I(r) \int_{\partial\mathcal{D}_t} Q(\partial^r P, \partial^r P) \vartheta \, dS, \end{aligned} \tag{1.13}$$

where $I(r) = 0$ if $r = 1$ and $I(r) = 1$ if $r > 1$, so we do not need the boundary integral for $r = 1$, and

$$\vartheta = (-\nabla_{\mathcal{N}} P)^{-1}.$$

Here Q is a positive definite quadratic form which, when restricted to the boundary, is the inner product of the tangential components, that is, $Q(\alpha, \beta) = \langle \Pi\alpha, \Pi\beta \rangle$, and in the interior $Q(\alpha, \alpha)$ increases to the norm $|\alpha|^2$. To be more specific, let

$$Q(\alpha, \beta) = q^{i_1 j_1} \dots q^{i_r j_r} \alpha_{i_1 \dots i_r} \beta_{j_1 \dots j_r} \tag{1.14}$$

where

$$q^{ij} = \delta^{ij} - \eta(d)^2 \mathcal{N}^i \mathcal{N}^j, \quad d(x) = \text{dist}(x, \partial \mathcal{D}_t), \quad \mathcal{N}^i = -\delta^{ij} \partial_j d. \quad (1.15)$$

Here η is a smooth cutoff function satisfying $0 \leq \eta(d) \leq 1$, $\eta(d) = 1$ when $d < d_0/4$ and $\eta(d) = 0$ when $d > d_0/2$. d_0 is a fixed number that is smaller than the injectivity radius of the normal exponential map ι_0 , defined to be the largest number ι_0 such that the map

$$\partial \mathcal{D}_t \times (-\iota_0, \iota_0) \rightarrow \{x \in \mathbb{R}^n : \text{dist}(x, \partial \mathcal{D}_t) < \iota_0\} \quad (1.16)$$

given by

$$(\bar{x}, \iota) \rightarrow x = \bar{x} + \iota \mathcal{N}(\bar{x})$$

is an injection.

The main theorems in this paper are as follows:

Theorem 1.1. *For any smooth solution of the free boundary problem (1.8) for $0 \leq t \leq T$ satisfying*

$$|\partial P| \leq M, \quad |\partial v| \leq M, \quad \text{in } \mathcal{D}_t, \quad (1.17)$$

$$|\theta| + |\partial v| + \frac{1}{\iota_0} \leq K, \quad \text{on } \partial \mathcal{D}_t, \quad (1.18)$$

we have for $t \in [0, T]$

$$E_1(t) \leq 2e^{CMt} E_1(0) + CK^2 (\text{Vol } \mathcal{D}_t + E_0(0)) (e^{CMt} - 1), \quad (1.19)$$

for some positive constants C and M .

Theorem 1.2. *Let $r \in \{2, \dots, n + 1\}$, then there exists a $T > 0$ such that the following holds: For any smooth solution of the free boundary problem (1.8) for $0 \leq t \leq T$ satisfying*

$$|B| \leq M_1 \quad \text{for } r = 2, \quad \text{in } \mathcal{D}_t, \quad (1.20)$$

$$|\partial P| \leq M, \quad |\partial v| \leq M, \quad |\partial B| \leq M, \quad \text{in } \mathcal{D}_t, \quad (1.21)$$

$$|\theta| + 1/\iota_0 \leq K, \quad \text{on } \partial \mathcal{D}_t, \quad (1.22)$$

$$-\nabla_{\mathcal{N}} P \geq \varepsilon > 0, \quad \text{on } \partial \mathcal{D}_t, \quad (1.23)$$

$$|\partial^2 P| + |\nabla_{\mathcal{N}} D_t P| \leq L, \quad \text{on } \partial \mathcal{D}_t, \quad (1.24)$$

we have, for $t \in [0, T]$,

$$E_r(t) \leq e^{C_1 t} E_r(0) + C_2 (e^{C_1 t} - 1), \quad (1.25)$$

where the positive constants C_1 and C_2 depend on $K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \mathcal{D}_t, E_0(0), E_1(0), \dots$, and $E_{r-1}(0)$.

Most of the a priori bounds (1.20)–(1.24) can be obtained from the energy norms by the elliptic estimates which are used to control all components of $\partial^r v$, $\partial^r B$ and $\partial^r p$ from the tangential components $\Pi \partial^r P$ in the energy norms, and a bound for the second fundamental form of the free boundary

$$\left\| \bar{\partial}^{r-2} \theta \right\|_{L^2(\partial \mathcal{D}_t)} \leq C \left(K, L, M, \frac{1}{\varepsilon}, E_{r-1}, \text{Vol } \mathcal{D}_t \right) E_r$$

for $r \geq 2$, which controls the regularity of the free boundary.

Since $E_0(t) = E_0(0)$ and $\text{Vol } \mathcal{D}_t = \text{Vol } \mathcal{D}_0$, recursively we can prove the following main theorem from Theorems 1.1, 1.2.

Theorem 1.3. *Let*

$$\mathcal{H}(0) = \max \left(\|\theta(0, \cdot)\|_{L^\infty(\partial \mathcal{D}_0)}, 1/t_0(0) \right), \tag{1.26}$$

$$\mathcal{E}(0) = \|1/(\nabla_N P(0, \cdot))\|_{L^\infty(\partial \mathcal{D}_0)} = 1/\varepsilon(0) > 0. \tag{1.27}$$

There exists a continuous function $\mathcal{T} > 0$ such that if

$$T \leq \mathcal{T}(\mathcal{H}(0), \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \mathcal{D}_0), \tag{1.28}$$

then any smooth solution of the free boundary problem for MHD equations (1.8) for $0 \leq t \leq T$ satisfies

$$\sum_{s=0}^{n+1} E_s(t) \leq 2 \sum_{s=0}^{n+1} E_s(0), \quad 0 \leq t \leq T. \tag{1.29}$$

In order to prove the above theorems, we need to use the elliptic estimates of the pressure p . However, the time derivative of Δp involves a third-order term of the velocity which needs to be controlled by higher order energies. In order to overcome this difficulty, we work on the equations for the total pressure $P = p + \frac{1}{8\pi}|B|^2$, instead of those for the fluid pressure p .

Before we close this introduction, we mention here some studies on viscous or inviscid MHD equations, including the Cauchy problem or initial boundary value problems for the fixed boundaries [6, 7, 11–14, 18–20, 26] and the references therein.

The rest of this paper is organized as follows. In Section 1.2 we give more remarks on the motivations of the construction of the higher order energy functional $E_r(t)$ in (1.13) and outline of the proof of our theorems. In Section 2, we use the Lagrangian coordinates to transform the free boundary problem to a fixed initial boundary problem. The Lagrangian transformation induces a Riemannian metric on \mathcal{D}_0 , for which we recall the time evolution properties derived in [4] and prove some new identities which will be used later. We also write the equations in Lagrangian coordinates, by using the covariant spatial derivatives with respect to the Riemannian metric induced by the Lagrangian transformation, instead of using the ordinary derivatives. In Section 3, we prove the conservation of the zeroth order energy $E_0(t)$, from which one can see that the boundary conditions on the magnetic fields B is necessary for this energy conservation. We also prove in Section 3 that the condition $B \cdot \mathcal{N} = 0$ on the boundary propagates along the boundary.

Section 4 is devoted to the first order energy estimates. In Section 5, we prove the higher order energy estimates by using the identities derived in Section 2, the time evolution property of the metric on the boundary induced by the above mentioned Riemannian metric induced by the Lagrangian transformation, the projection properties and the elliptic estimates. In the derivation of the higher order energy estimates in Section 5 some a priori assumptions are made which will be justified in Section 6. We also give an appendix on some estimates used in the previous sections, which are basically proved in [4].

1.2. Motivation of the Construction of Higher Order Energy Functional and Outline of the Proofs

We give more remarks on the motivations of the construction of the higher order energy functional $E_r(t)$ in (1.13) and outline of the proof of our theorems here. First, for divergence free vector fields, the L^2 norms of curl and tangential (or normal) derivatives control the L^2 norms of the derivatives of the vector fields [cf. (A.15)]. Therefore, the interior integral part in (1.13) controls the L^2 norms of $\partial^r v$ and $\partial^r B$. The reason for using $\partial^{r-1} \text{curl } v$ and $\partial^{r-1} \text{curl } B$ in the interior integral part of (1.13) is that it is relatively easy to obtain the estimates of the Sobolev norms of $\text{curl } v$ and $\text{curl } B$ by using the equations for them. The time derivative of the interior integral part of $E_r(t)$ produces a boundary integral term after the integration by parts, which cannot be bounded by the interior integral part of $E_r(t)$ directly. We need a boundary integral to cancel the leading term of it. The time derivative of the boundary integral $\int_{\partial \mathcal{D}_t} Q(\partial^r P, \partial^r P) \vartheta \, dS$ ($r \geq 2$) in (1.13) which involves the projection of the r -th derivatives of the total pressure $P = p + \frac{1}{8\pi} |B|^2$ to the tangent space of the boundary is constructed for this purpose, for example, when $r = 2$, we make use of the following second-order equations for the velocity and the total pressure

$$D_t^2 v_i - \partial_i v^k \partial_k P = -\partial_i D_t P + \frac{1}{4\pi} B^k \partial_k D_t B_i, \tag{1.30}$$

$$D_t \partial_i \partial_j P + (\partial_k P) \partial_i \partial_j v^k = \partial_i \partial_j D_t P - (\partial_i v^k) \partial_k \partial_j P - (\partial_j v^k) \partial_k \partial_i P, \tag{1.31}$$

restricted to the boundary together with the boundary condition

$$D_t P = 0 \quad \text{on } \partial \mathcal{D}_t, \tag{1.32}$$

since P is constant on $\partial \mathcal{D}_t$. Equations (1.30) and (1.31) can be derived from (1.8a) and (1.8b), with the help of the following commutator formula:

$$[D_t, \partial_i] = -(\partial_i v^k) \partial_k. \tag{1.33}$$

One can use elliptic estimates to control all components of $\partial^r v$, $\partial^r B$ and $\partial^r P$ from the energy functional $E_r(t)$, by the Dirichlet problems of the elliptic equations for the total pressure P and its Lagrangian time derivative $D_t P$ for which one has the boundary conditions $P = \text{const}$ and $D_t P = 0$ on $\partial \mathcal{D}_t$ [the elliptic equation for P can be obtained by taking the divergence of (1.8a), and the elliptic equation for

$D_t P$ can be obtained by taking D_t of the elliptic equation for P and using (1.33)]. It should be noted that a bound of the higher order energy functional $E_r(t)$ also gives the bound of $\|\bar{\partial}^{r-2}\theta\|_{L^2(\partial\mathcal{D}_t)}$ by using a higher-order version of the projection formula (1.12) and the physical condition (1.6). Once we have the bounds for the second fundamental form $\|\bar{\partial}^{s-2}\theta\|_{L^2(\partial\mathcal{D}_t)}$ ($2 \leq s \leq r$), we can get estimates for solution of the Dirichlet problems for the elliptic equations of P and $D_t P$. We outline the proof of the main theorem as follows. For $r \geq 2$, integration by parts gives:

$$\begin{aligned} \frac{d}{dt} E_r(t) &\leq \text{lower order terms} \\ &\quad - \frac{1}{2\pi} \int_{\mathcal{D}_t} \partial_c \left(q^{af} q^{AF} \right) \partial_F^{r-1} \partial_f v_d B^c \partial_{Aa}^r B^d \, dx \end{aligned} \tag{1.34}$$

$$+ 2 \int_{\mathcal{D}_t} \partial_b \left(q^{af} q^{AF} \right) \partial_F^{r-1} \partial_f v^b \partial_A^{r-1} \partial_a P \, dx \tag{1.35}$$

$$+ 2 \int_{\partial\mathcal{D}_t} q^{af} q^{AF} \partial_{Aa}^r P \left(D_t \partial_{Ff}^r P - \frac{1}{\vartheta} N_b \partial_{Ff}^r v^b \right) \vartheta \, dS, \tag{1.36}$$

where $A = (a_1, \dots, a_{r-1})$ and $F = (f_1, \dots, f_{r-1})$, $q^{AF} = q^{a_1 f_1} \dots q^{a_{r-1} f_{r-1}}$, $\partial_{Ff}^r = \partial_{f_1} \dots \partial_{f_{r-1}} \partial_f$, the definitions for others such as ∂_A^{r-1} are similar.

From the Sobolev lemma (cf. [4]), the Hölder inequality and the assumption of Theorem 1.1, one can estimate the term (1.34) for $r = 2$ and $r \geq 3$ separately.

The integral (1.35) can be bounded $CK E_r^{1/2}(t) \|\partial^r P\|_{L^2(\mathcal{D}_t)}$ by the Hölder inequality. The estimate for $\|\partial^r P\|_{L^2(\mathcal{D}_t)}$ can be obtained by the elliptic estimates, a higher-order version of the projection formula (1.12) and the physical condition (1.6). For the estimate of the boundary integral (1.36), we notice that the tangential derivative P on $\partial\mathcal{D}_t$ vanishes to infer that

$$\int_{\partial\mathcal{D}_t} q^{af} q^{AF} \partial_{Aa}^r P \left(D_t \partial_{Ff}^r P - \frac{1}{\vartheta} N_b \partial_{Ff}^r v^b \right) \vartheta \, dS$$

can be bounded by $C \|\vartheta\|_{L^\infty(\partial\mathcal{D}_t)}^{1/2} E_r^{1/2}(t) \|\Pi(D_t(\partial^r P) + \partial^r v \cdot \partial P)\|_{L^2(\partial\mathcal{D}_t)}$. On the other hand, we can have

$$D_t \partial^r P + \partial^r v \cdot \partial P = \partial^r D_t P + \text{lower order terms.} \tag{1.37}$$

Since $D_t P = 0$ on $\partial\mathcal{D}_t$, one can use the elliptic estimates to bound $\|\partial^r D_t P\|_{L^2(\mathcal{D}_t)}$ in terms of $\|\Pi \partial^r D_t P\|_{L^2(\partial\mathcal{D}_t)}$ and $\sum_{s \leq r-2} \|\partial^s \Delta D_t P\|_{L^2(\mathcal{D}_t)}$ under the a priori assumptions in Theorem 1.2. The estimate for $\|\Pi \partial^r D_t P\|_{L^2(\partial\mathcal{D}_t)}$ can be obtained from the higher-order version of the projection formula (1.12), in terms of $\|\theta\|_{L^\infty(\partial\mathcal{D}_t)}$, $\sum_{k \leq r-3} \|\bar{\partial}^k \theta\|_{L^2(\partial\mathcal{D}_t)}$ and $\sum_{k \leq r} \|\partial^k D_t P\|_{L^2(\mathcal{D}_t)}$. The estimate for $\|\partial^s \Delta D_t P\|_{L^2(\mathcal{D}_t)}$ can be obtained by the elliptic equation for $D_t P$ together with the boundary condition $D_t P = 0$ on $\partial\mathcal{D}_t$.

Once we have the above estimates, the justification of a priori assumptions in Theorems 1.1 and 1.2 can mainly follow the argument in [4].

It is clear from the above discussion to see the role played by the total pressure $P = p + \frac{1}{8\pi}|B|^2$, where the magnetic field B comes in for which additional estimates are needed.

We will prove the above estimates by using the Lagrangian coordinates. One of the advantages of doing so is that we can work on a fixed domain.

2. Reformulation in Lagrangian Coordinates

Assume that we are given a velocity vector field $v(t, x)$ defined in a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ such that the boundary of $\mathcal{D}_t = \{x : (t, x) \in \mathcal{D}\}$ moves with the velocity, that is, $(1, v) \in T(\partial\mathcal{D})$ which denotes the tangent space of $\partial\mathcal{D}$. We will now introduce Lagrangian or co-moving coordinates, that is, coordinates that are constant along the integral curves of the velocity vector field so that the boundary becomes fixed in these coordinates (cf. [4]). Let $x = x(t, y) = f_t(y)$ be the trajectory of the fluid given by

$$\begin{cases} \frac{dx}{dt} = v(t, x(t, y)), & (t, y) \in [0, T] \times \Omega, \\ x(0, y) = f_0(y), & y \in \Omega \end{cases} \tag{2.1}$$

where, when $t = 0$, we can start with either the Euclidean coordinates in $\Omega = \mathcal{D}_0$ or some other coordinates $f_0 : \Omega \rightarrow \mathcal{D}_0$ where f_0 is a diffeomorphism in which the domain Ω becomes simple. For each t , we will then have a change of coordinates $f_t : \Omega \rightarrow \mathcal{D}_t$, taking $y \rightarrow x(t, y)$. The Euclidean metric δ_{ij} in \mathcal{D}_t then induces a metric

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \tag{2.2}$$

and its inverse

$$g^{cd}(t, y) = \delta^{kl} \frac{\partial y^c}{\partial x^k} \frac{\partial y^d}{\partial x^l} \tag{2.3}$$

in Ω for each fixed t .

We will use covariant differentiation in Ω with respect to the metric $g_{ab}(t, y)$, since it corresponds to differentiation in \mathcal{D}_t under the change of coordinates $\Omega \ni y \rightarrow x(t, y) \in \mathcal{D}_t$, and we will work in both coordinate systems. This also avoids possible singularities in the change of coordinates. We will denote covariant differentiation in the y_a -coordinates by $\nabla_a, a = 0, \dots, n$, and differentiation in the x_i -coordinates by $\partial_i, i = 1, \dots, n$. The covariant differentiation of a $(0, r)$ tensor $k(t, y)$ is the $(0, r + 1)$ tensor given by

$$\nabla_a k_{a_1 \dots a_r} = \frac{\partial k_{a_1 \dots a_r}}{\partial y^a} - \Gamma_{aa_1}^d k_{d \dots a_r} - \dots - \Gamma_{aa_r}^d k_{a_1 \dots d}, \tag{2.4}$$

where the Christoffel symbols Γ_{ab}^d are given by

$$\Gamma_{ab}^c = \frac{g^{cd}}{2} \left(\frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d} \right) = \frac{\partial y^c}{\partial x^i} \frac{\partial^2 x^i}{\partial y^a \partial y^b}. \tag{2.5}$$

If $w(t, x)$ is the $(0, r)$ tensor expressed in the x -coordinates, then the same tensor $k(t, y)$ expressed in the y -coordinates is given by

$$k_{a_1 \dots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} w_{i_1 \dots i_r}(t, x), \quad x = x(t, y), \tag{2.6}$$

and by the transformation properties for tensors,

$$\nabla_a k_{a_1 \dots a_r} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial w_{i_1 \dots i_r}}{\partial x^i}. \tag{2.7}$$

Covariant differentiation is constructed so the norms of tensors are invariant under changes of coordinates,

$$g^{a_1 b_1} \dots g^{a_r b_r} k_{a_1 \dots a_r} k_{b_1 \dots b_r} = \delta^{i_1 j_1} \dots \delta^{i_r j_r} w_{i_1 \dots i_r} w_{j_1 \dots j_r}. \tag{2.8}$$

Furthermore, expressed in the y -coordinates,

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}. \tag{2.9}$$

Since the curvature vanishes in the x -coordinates, it must do so in the y -coordinates, and hence

$$[\nabla_a, \nabla_b] = 0. \tag{2.10}$$

Let us introduce the notation $k_{a \dots b \dots c} = g^{bd} k_{a \dots d \dots c}$, and recall that covariant differentiation commutes with lowering and rising indices: $g^{ce} \nabla_a k_{b \dots e \dots d} = \nabla_a g^{ce} k_{b \dots e \dots d}$. Let us also introduce a notation for the material derivative

$$D_t = \frac{\partial}{\partial t} \Big|_{y=\text{const}} = \frac{\partial}{\partial t} \Big|_{x=\text{const}} + v^k \frac{\partial}{\partial x^k}. \tag{2.11}$$

Then we have, from [4, Lemma 2.2], that

$$D_t k_{a_1 \dots a_r} = \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left(D_t w_{i_1 \dots i_r} + \frac{\partial v^\ell}{\partial x^{i_1}} w_{\ell \dots i_r} + \dots + \frac{\partial v^\ell}{\partial x^{i_r}} w_{i_1 \dots \ell} \right). \tag{2.12}$$

Now we recall a result concerning time derivatives of the change of coordinates and commutators between time derivatives and space derivatives (cf. [4, Lemma 2.1]).

Lemma 2.1. *Let $x = f_t(y)$ be the change of variables given by (2.1), and let g_{ab} be the metric given by (2.2). Let $v_i = \delta_{ij} v^j = v^i$, and set*

$$u_a(t, y) = v_i(t, x) \frac{\partial x^i}{\partial y^a}, \quad u^a = g^{ab} u_b, \tag{2.13}$$

$$h_{ab} = \frac{1}{2} D_t g_{ab}, \quad h^{ab} = g^{ac} h_{cd} g^{db}. \tag{2.14}$$

Then

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a} \frac{\partial v^i}{\partial x^k}, \quad D_t \frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^k} \frac{\partial v^k}{\partial x^i}, \tag{2.15}$$

$$D_t g_{ab} = \nabla_a u_b + \nabla_b u_a, \quad D_t g^{ab} = -2h^{ab}, \quad D_t d\mu_g = g^{ab} h_{ab} d\mu_g, \tag{2.16}$$

$$D_t \Gamma_{ab}^c = \nabla_a \nabla_b u^c, \tag{2.17}$$

where $d\mu_g$ is the Riemannian volume element on Ω in the metric g .

Proof. The proof is the same as that of [4, Lemma 2.1] except that we need to make some modification due to the difference of the definition of h_{ab} . Indeed, the proof of (2.15), (2.17) and the first part of (2.16) is the same as the mentioned. The second part of (2.16) follows from (2.14) since $0 = D_t(g^{ad}g_{dc}) = (D_t g^{ad})g_{dc} + g^{ad}D_t g_{dc} = (D_t g^{ad})g_{dc} + 2g^{ad}h_{dc}$ and then $D_t g^{ab} = (D_t g^{ad})\delta_d^b = (D_t g^{ad})g_{dc}g^{cb} = -2g^{ad}h_{dc}g^{cb} = -2h^{ab}$. The last part of (2.16) follows since in local coordinates $d\mu_g = \sqrt{\det g}dy$ and $D_t(\det g) = (\det g)g^{ab}D_t g_{ab}$. \square

We now recall the estimates of commutators between the material derivative D_t and space derivatives ∂_i and covariant derivatives ∇_a .

Lemma 2.2. ([4, Lemma 2.3]) *Let ∂_i be given by (2.9). Then*

$$[D_t, \partial_i] = -(\partial_i v^k) \partial_k. \tag{2.18}$$

Furthermore,

$$[D_t, \partial^r] = -\sum_{s=0}^{r-1} \binom{r}{s+1} (\partial^{1+s} v) \cdot \partial^{r-s}, \tag{2.19}$$

where the symmetric dot product is defined to be in components

$$\left((\partial^{1+s} v) \cdot \partial^{r-s} \right)_{i_1 \dots i_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \left(\partial_{i_{\sigma_1} \dots i_{\sigma_{1+s}}}^{1+s} v^k \right) \partial_{k i_{\sigma_{s+2}} \dots i_{\sigma_r}}^{r-s}, \tag{2.20}$$

and \sum_r denotes the collection of all permutations of $\{1, 2, \dots, r\}$.

Lemma 2.3. (cf. [4, Lemma 2.4]) *Let $T_{a_1 \dots a_r}$ be a $(0, r)$ tensor. We have*

$$[D_t, \nabla_a] T_{a_1 \dots a_r} = -(\nabla_{a_1} \nabla_a u^d) T_{da_2 \dots a_r} - \dots - (\nabla_{a_r} \nabla_a u^d) T_{a_1 \dots a_{r-1} d}. \tag{2.21}$$

If $\Delta = g^{cd} \nabla_c \nabla_d$ and q is a function, we have

$$\left[D_t, g^{ab} \nabla_a \right] T_b = -2h^{ab} \nabla_a T_b - (\Delta u^e) T_e, \tag{2.22}$$

$$[D_t, \nabla] q = 0, \tag{2.23}$$

$$[D_t, \Delta] q = -2h^{ab} \nabla_a \nabla_b q - (\Delta u^e) \nabla_e q. \tag{2.24}$$

Furthermore,

$$[D_t, \nabla^r]q = \sum_{s=1}^{r-1} - \binom{r}{s+1} (\nabla^{s+1}u) \cdot \nabla^{r-s}q, \tag{2.25}$$

where the symmetric dot product is defined to be in components

$$\left((\nabla^{s+1}u) \cdot \nabla^{r-s}q \right)_{a_1 \dots a_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \left(\nabla_{a_{\sigma_1} \dots a_{\sigma_{s+1}}} u^d \right) \nabla_{da_{\sigma_{s+2}} \dots a_{\sigma_r}} q. \tag{2.26}$$

Proof. The proof is similar to that of [4, Lemma 2.4]. We only need to verify (2.22) and (2.24) since they involve the term $D_t g^{ab}$. Now from (2.16) and (2.21), it follows that

$$\begin{aligned} [D_t, g^{ab} \nabla_a] T_b &= D_t(g^{ab} \nabla_a T_b) - g^{ab} \nabla_a D_t T_b \\ &= (D_t g^{ab}) \nabla_a T_b + g^{ab} D_t \nabla_a T_b - g^{ab} \nabla_a D_t T_b \\ &= -2h^{ab} \nabla_a T_b + g^{ab} [D_t, \nabla_a] T_b \\ &= -2h^{ab} \nabla_a T_b - g^{ab} \nabla_b \nabla_a u^e T_e \\ &= -2h^{ab} \nabla_a T_b - (\Delta u^e) T_e. \end{aligned}$$

From (2.12) and (2.18), we have

$$\begin{aligned} D_t \nabla_a q &= D_t \left(\frac{\partial x^i}{\partial y^a} \partial_i q \right) = \frac{\partial x^i}{\partial y^a} \left(D_t \partial_i q + \partial_\ell \frac{\partial v^\ell}{\partial x^i} \right) \\ &= \frac{\partial x^i}{\partial y^a} \left([D_t, \partial_i] q + \partial_i D_t q + \partial_\ell q \frac{\partial v^\ell}{\partial x^i} \right) \\ &= \frac{\partial x^i}{\partial y^a} \left(-\partial_i v^k \partial_k q + \partial_i D_t q + \partial_i v^\ell \partial_\ell q \right) = \frac{\partial x^i}{\partial y^a} \partial_i D_t q = \nabla_a D_t q, \end{aligned}$$

namely, (2.23) follows. Then, (2.24) follows from (2.22) and

$$\begin{aligned} [D_t, \Delta]q &= D_t \Delta q - \Delta D_t q = D_t \left(g^{ab} \nabla_a \nabla_b q \right) - g^{ab} \nabla_a \nabla_b D_t q \\ &= [D_t, g^{ab} \nabla_a] \nabla_b q + g^{ab} \nabla_a [D_t, \nabla_b] q \\ &= [D_t, g^{ab} \nabla_a] \nabla_b q. \end{aligned}$$

Therefore, we complete the proof.

Denote

$$B_i = \delta_{ij} B^j = B^i, \quad \beta_a = B_j \frac{\partial x^j}{\partial y^a}, \quad \beta^a = g^{ab} \beta_b, \quad \text{and } |\beta|^2 = \beta_a \beta^a. \tag{2.27}$$

It follows, from (2.8), that

$$|\beta|^2 = |B|^2, \quad B_j = \frac{\partial y^a}{\partial x^j} \beta_a, \quad P = p + \frac{1}{8\pi} |\beta|^2. \tag{2.28}$$

Then $P = \frac{1}{8\pi} \zeta^2$ on the boundary $\partial\Omega$.

From (2.13), (1.8a), (2.28), (2.15), (2.7), we have

$$\begin{aligned}
 D_t u_a &= D_t \left(v_j \frac{\partial x^j}{\partial y^a} \right) = \frac{\partial x^j}{\partial y^a} D_t v_j + v_j D_t \frac{\partial x^j}{\partial y^a} \\
 &= \frac{\partial x^j}{\partial y^a} \left(-\partial_j P + \frac{1}{4\pi} B^k \partial_k B_j \right) + v_j \frac{\partial x^k}{\partial y^a} \frac{\partial v^j}{\partial x^k} \\
 &= -\nabla_a P + \frac{1}{4\pi} \frac{\partial x^j}{\partial y^a} \delta^{ki} \frac{\partial y^b}{\partial x^i} \beta_b \delta'_k \frac{\partial y^d}{\partial x^l} \frac{\partial y^c}{\partial x^j} \nabla_d \beta_c + \frac{\partial y^b}{\partial x^j} u_b \delta^{lj} \frac{\partial y^c}{\partial x^l} \nabla_a u_c \\
 &= -\nabla_a P + \frac{1}{4\pi} g^{bd} g_{ae} g^{ec} \beta_b \nabla_d \beta_c + g^{bc} u_b \nabla_a u_c \\
 &= -\nabla_a P + \frac{1}{4\pi} \beta^d \nabla_d \beta_a + u^c \nabla_a u_c.
 \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 D_t \beta_a &= \frac{\partial x^j}{\partial y^a} D_t B_j + B_j D_t \frac{\partial x^j}{\partial y^a} = \frac{\partial x^j}{\partial y^a} B^k \partial_k v_j + B_j \frac{\partial x^k}{\partial y^a} \frac{\partial v^j}{\partial x^k} \\
 &= \beta^d \nabla_d u_a + \beta^c \nabla_a u_c.
 \end{aligned}$$

Thus, the system (1.1) can be written in the Lagrangian coordinates as

$$D_t u_a + \nabla_a P = u^c \nabla_a u_c + \frac{1}{4\pi} \beta^d \nabla_d \beta_a, \quad (2.29a)$$

$$D_t \beta_a = \beta^d \nabla_d u_a + \beta^c \nabla_a u_c, \quad (2.29b)$$

$$\nabla_a u^a = 0 \quad \text{in } [0, T] \times \Omega; \quad \nabla_a \beta^a = 0 \quad \text{in } \{t = 0\} \times \Omega, \quad (2.29c)$$

$$|\beta| = \varrho \quad \text{and } \beta_a N^a = 0 \quad \text{on } [0, T] \times \partial\Omega, \quad (2.29d)$$

$$p = 0 \quad \text{on } [0, T] \times \partial\Omega. \quad (2.29e)$$

3. The Energy Conservation and Some Conserved Quantities

Firstly, the divergence free property of β , that is, $\text{div } \beta = 0$, is preserved for all times under the Lagrangian coordinates or in view of the material derivative, that is, $D_t \text{div } \beta = 0$. Indeed, from (2.22) and Lemma 2.1, the divergence of (2.29b) gives

$$\begin{aligned}
 D_t \left(g^{ab} \nabla_b \beta_a \right) &= \left[D_t, g^{ab} \nabla_b \right] \beta_a + g^{ab} \nabla_b D_t \beta_a \\
 &= -2h^{ab} \nabla_b \beta_a - (\Delta u^e) \beta_e + g^{ab} \nabla_b \left(\beta^d \nabla_d u_a + \beta^c \nabla_a u_c \right) \\
 &= -2h^{ab} \nabla_b \beta_a - (\Delta u^e) \beta_e + \nabla_b \beta^d \nabla_d u^b + \beta^d \nabla_d \nabla_b u^b \\
 &\quad + g^{ab} \nabla_b \beta^c \nabla_a u_c + \beta^c \Delta u_c \\
 &= -g^{ac} (\nabla_c u_d + \nabla_d u_c) g^{db} \nabla_b \beta_a + \nabla_b \beta^d \nabla_d u^b + g^{ab} \nabla_b \beta^c \nabla_a u_c \\
 &= 0.
 \end{aligned}$$

Secondly, we assume that

$$|\nabla u(t, y)| \leq C \quad \text{on } [0, T] \times \partial\Omega, \quad (3.1)$$

then that $\beta \cdot N = 0$ is preserved for all times t in the lifespan $[0, T]$, that is, we have $\beta \cdot N = 0$ on $[0, T] \times \partial\Omega$ if $\beta \cdot N = 0$ on $\{t = 0\} \times \partial\Omega$. Indeed, we have, from (2.29b) and Lemmas 2.1 and A.1, that

$$\begin{aligned} D_t(\beta_a N^a) &= D_t(g^{ab}\beta_a N_b) = N^a D_t \beta_a + \beta_a (D_t g^{ab}) N_b + \beta_a g^{ab} D_t N_b \\ &= N^a (\beta^d \nabla_d u_a + \beta^d \nabla_a u_d) - \nabla_c u^b \beta^c N_b - N^d \nabla_d u^a \beta_a + \beta_a g^{ab} h_{NN} N_b \\ &= h_{NN} \beta_a N^a, \end{aligned}$$

which implies, by the Gronwall inequality and the identity $|D_t|f|| = |D_t f|$, that

$$|(\beta_a N^a)(t, y)| \leq e^{Ct} |(\beta_a N^a)(0, y)| = 0. \quad (3.2)$$

Thus, in view of the above three preserved quantities, the system (2.29), or (1.1), can be written in the Lagrangian coordinates as

$$D_t u_a + \nabla_a P = u^c \nabla_a u_c + \frac{1}{4\pi} \beta^d \nabla_d \beta_a, \quad (3.3a)$$

$$D_t \beta_a = \beta^d \nabla_d u_a + \beta^c \nabla_a u_c, \quad (3.3b)$$

$$\nabla_a u^a = 0, \quad \nabla_a \beta^a = 0, \quad \text{in } [0, T] \times \Omega, \quad (3.3c)$$

$$P = \frac{1}{8\pi} \zeta^2, \quad |\beta| = \zeta, \quad \beta \cdot N = 0, \quad \text{on } [0, T] \times \partial\Omega. \quad (3.3d)$$

Finally, the energy defined by

$$E_0(t) = \int_{\Omega} \left(\frac{1}{2} |u|^2 + \frac{1}{8\pi} |\beta|^2 \right) d\mu_g \quad (3.4)$$

is conserved. In fact, by (2.16), (2.29), Gauss' formula and the fact $D_t d\mu_g = 0$ due to $\operatorname{div} u = 0$, it yields

$$\begin{aligned} \frac{d}{dt} E_0(t) &= \int_{\Omega} D_t \left(\frac{1}{2} g^{ab} u_a u_b + \frac{1}{8\pi} g^{ab} \beta_a \beta_b \right) d\mu_g \\ &= \int_{\Omega} \left(u^a D_t u_a + \frac{1}{4\pi} \beta^a D_t \beta_a \right) d\mu_g \\ &\quad + \int_{\Omega} \frac{1}{2} (D_t g^{ab}) \left(u_a u_b + \frac{1}{4\pi} \beta_a \beta_b \right) d\mu_g \\ &= \int_{\Omega} \left[-u^a \nabla_a P + u^a u^c \nabla_a u_c + \frac{1}{4\pi} u^a \beta^d \nabla_d \beta_a \right] d\mu_g \\ &\quad + \int_{\Omega} \left(\frac{1}{4\pi} \beta^a \beta^d \nabla_d u_a + \frac{1}{4\pi} \beta^a \beta^c \nabla_a u_c \right) d\mu_g \\ &\quad - \int_{\Omega} h^{ab} \left(u_a u_b + \frac{1}{4\pi} \beta_a \beta_b \right) d\mu_g \\ &= - \int_{\partial\Omega} N_a u^a p d\mu_{\gamma} + \int_{\Omega} u^a u^c \nabla_a u_c d\mu_g + \frac{1}{4\pi} \int_{\partial\Omega} N_d \beta^d u^a \beta_a d\mu_{\gamma} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4\pi} \int_{\Omega} \beta^a \beta^c \nabla_a u_c \, d\mu_g - \frac{1}{2} \int_{\Omega} g^{ac} \\
& \quad \times (\nabla_c u_d + \nabla_d u_c) g^{db} \left(u_a u_b + \frac{1}{4\pi} \beta_a \beta_b \right) \, d\mu_g \\
& = 0.
\end{aligned}$$

4. The First Order Energy Estimates

From (2.21) and (3.3a), we have

$$\begin{aligned}
D_t(\nabla_b u_a) + \nabla_b \nabla_a P &= [D_t, \nabla_b] u_a + \nabla_b D_t u_a + \nabla_b \nabla_a P \\
&= - \left(\nabla_a \nabla_b u^d \right) u_d + \frac{1}{4\pi} \nabla_b \left(\beta^d \nabla_d \beta_a \right) + \nabla_b (u^c \nabla_a u_c) \\
&= - \left(\nabla_a \nabla_b u^d \right) u_d + \frac{1}{4\pi} \left(\nabla_b \beta^d \nabla_d \beta_a + \beta^d \nabla_b \nabla_d \beta_a \right) \\
& \quad + \nabla_b u^c \nabla_a u_c + u^c \nabla_b \nabla_a u_c \\
&= \nabla_b u^c \nabla_a u_c + \frac{1}{4\pi} \left(\nabla_b \beta^d \nabla_d \beta_a + \beta^d \nabla_b \nabla_d \beta_a \right).
\end{aligned}$$

From (2.21) and (3.3b), we get

$$\begin{aligned}
D_t(\nabla_b \beta_a) &= [D_t, \nabla_b] \beta_a + \nabla_b D_t \beta_a \\
&= - \left(\nabla_a \nabla_b u^d \right) \beta_d + \nabla_b \left(\beta^d \nabla_d u_a + \beta^c \nabla_a u_c \right) \\
&= - \left(\nabla_a \nabla_b u^d \right) \beta_d + \nabla_b \beta^d \nabla_d u_a + \beta^d \nabla_b \nabla_d u_a + \nabla_b \beta^c \nabla_a u_c + \beta^c \nabla_b \nabla_a u_c \\
&= \nabla_b \beta^c (\nabla_c u_a + \nabla_a u_c) + \beta^d \nabla_d \nabla_b u_a.
\end{aligned}$$

Thus, we obtain

$$D_t(\nabla_b u_a) + \nabla_b \nabla_a P = \nabla_b u^c \nabla_a u_c + \frac{1}{4\pi} \left(\nabla_b \beta^d \nabla_d \beta_a + \beta^d \nabla_b \nabla_d \beta_a \right), \quad (4.1)$$

$$D_t(\nabla_b \beta_a) = \nabla_b \beta^c (\nabla_c u_a + \nabla_a u_c) + \beta^d \nabla_d \nabla_b u_a. \quad (4.2)$$

Now, we calculate the material derivative of $g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d$. From (2.16), (2.14), (A.13), we get

$$\begin{aligned}
D_t \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d \right) &= \left(D_t g^{bd} \right) \gamma^{ae} \nabla_a u_b \nabla_e u_d + g^{bd} (D_t \gamma^{ae}) \nabla_a u_b \nabla_e u_d \\
& \quad + 2g^{bd} \gamma^{ae} (D_t \nabla_a u_b) \nabla_e u_d \\
&= -2g^{bc} h_{cf} g^{fd} \gamma^{ae} \nabla_a u_b \nabla_e u_d - 2g^{bd} \gamma^{ac} h_{cf} \gamma^{fe} \nabla_a u_b \nabla_e u_d \\
& \quad - 2g^{bd} \gamma^{ae} \nabla_e u_d \nabla_a \nabla_b P + 2g^{bd} \gamma^{ae} \nabla_e u_d \nabla_a u^c \nabla_b u_c \\
& \quad + \frac{1}{2\pi} g^{bd} \gamma^{ae} \nabla_e u_d \left(\nabla_a \beta^d \nabla_d \beta_b + \beta^d \nabla_a \nabla_d \beta_b \right) \\
&= -\gamma^{ae} (\nabla_c u_f + \nabla_f u_c) \nabla_a u^c \nabla_e u^f - \gamma^{ac} \gamma^{fe} (\nabla_c u_f + \nabla_f u_c) \nabla_a u^d \nabla_e u_d \\
& \quad - 2\gamma^{ae} \nabla_e u^b \nabla_a \nabla_b P + 2\gamma^{ae} \nabla_e u^b \nabla_a u^c \nabla_b u_c \\
& \quad + \frac{1}{2\pi} \gamma^{ae} \nabla_e u^b \left(\nabla_a \beta^d \nabla_d \beta_b + \beta^d \nabla_a \nabla_d \beta_b \right)
\end{aligned}$$

$$\begin{aligned}
 &= -2\gamma^{ae}\nabla_c u_f \nabla_a u^c \nabla_e u^f - 2\gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a u^d \nabla_c u_d + 2\gamma^{ae}\nabla_e u^b \nabla_a u^c \nabla_b u_c \\
 &\quad - 2\gamma^{ae}\nabla_e u^b \nabla_a \nabla_b P + \frac{1}{2\pi}\gamma^{ae}\nabla_e u^b \left(\nabla_a \beta^d \nabla_d \beta_b + \beta^d \nabla_a \nabla_d \beta_b \right) \\
 &= -2\gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a u^d \nabla_c u_d - 2\gamma^{ae}\nabla_e u^b \nabla_a \nabla_b P \\
 &\quad + \frac{1}{2\pi}\gamma^{ae}\nabla_e u^b \left(\nabla_a \beta^d \nabla_d \beta_b + \beta^d \nabla_a \nabla_d \beta_b \right). \tag{4.3}
 \end{aligned}$$

Similarly, from (4.2), we have

$$\begin{aligned}
 &D_t \left(g^{bd}\gamma^{ae}\nabla_a \beta_b \nabla_e \beta_d \right) \\
 &= -\gamma^{ae}(\nabla_c u_f + \nabla_f u_c)\nabla_a \beta^c \nabla_e \beta^f - \gamma^{ac}\gamma^{fe}(\nabla_c u_f + \nabla_f u_c)\nabla_a \beta^d \nabla_e \beta_d \\
 &\quad + 2\gamma^{ae}\nabla_e \beta^b \left(\nabla_a \beta^c \nabla_c u_b + \nabla_a \beta^c \nabla_b u_c + \beta^d \nabla_d \nabla_a u_b \right) \\
 &= -2\gamma^{ae}\nabla_c u_f \nabla_a \beta^c \nabla_e \beta^f - 2\gamma^{ac}\gamma^{fe}\nabla_c u_f \nabla_a \beta^d \nabla_e \beta_d \\
 &\quad + 2\gamma^{ae}\nabla_e \beta^b \nabla_a \beta^c \nabla_c u_b + 2\gamma^{ae}\nabla_b u_c \nabla_e \beta^b \nabla_a \beta^c + 2\gamma^{ae}\beta^d \nabla_e \beta^b \nabla_d \nabla_a u_b \\
 &= -2\gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a \beta^d \nabla_c \beta_d + 2\gamma^{ae}\nabla_e \beta^b \nabla_a \beta^c \nabla_c u_b \\
 &\quad + 2\gamma^{ae}\beta^d \nabla_a \beta^b \nabla_d \nabla_e u_b. \tag{4.4}
 \end{aligned}$$

Thus, by combining (4.3) with (4.4), we obtain

$$\begin{aligned}
 &D_t \left(g^{bd}\gamma^{ae}\nabla_a u_b \nabla_e u_d + \frac{1}{4\pi}g^{bd}\gamma^{ae}\nabla_a \beta_b \nabla_e \beta_d \right) \\
 &= -2\gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a u^d \nabla_c u_d - \frac{1}{2\pi}\gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a \beta^d \nabla_c \beta_d \\
 &\quad - 2\nabla_b \left(\gamma^{ae}\nabla_e u^b \nabla_a P - \frac{1}{4\pi}\gamma^{ae}\beta^b \nabla_e u^d \nabla_a \beta_d \right) \\
 &\quad + 2(\nabla_b \gamma^{ae}) \left(\nabla_e u^b \nabla_a P - \frac{1}{4\pi}\beta^b \nabla_e u^d \nabla_a \beta_d \right) \\
 &\quad + \frac{1}{2\pi}\gamma^{ae}\nabla_e u^b \nabla_a \beta^d \nabla_d \beta_b + \frac{1}{2\pi}\gamma^{ae}\nabla_e \beta^b \nabla_a \beta^c \nabla_c u_b. \tag{4.5}
 \end{aligned}$$

Now, we calculate the material derivatives of $|\text{curl } u|^2$ and $|\text{curl } \beta|^2$. We have

$$\begin{aligned}
 D_t |\text{curl } u|^2 &= D_t \left(g^{ac}g^{bd}(\text{curl } u)_{ab}(\text{curl } u)_{cd} \right) \\
 &= 2(D_t g^{ac})g^{bd}(\text{curl } u)_{ab}(\text{curl } u)_{cd} + 4g^{ac}g^{bd}(D_t \nabla_a u_b)(\text{curl } u)_{cd} \\
 &= -2g^{ae}g^{fc}g^{bd}(\nabla_e u_f + \nabla_f u_e)(\text{curl } u)_{ab}(\text{curl } u)_{cd} \\
 &\quad + 4g^{ac}g^{bd}(\text{curl } u)_{cd}\nabla_a u^e \nabla_b u_e \\
 &\quad - 4g^{ac}g^{bd}(\text{curl } u)_{cd}\nabla_a \nabla_b P \\
 &\quad + \frac{1}{\pi}g^{ac}g^{bd}(\text{curl } u)_{cd}(\nabla_a \beta^e \nabla_e \beta_b + \beta^e \nabla_a \nabla_e \beta_b) \\
 &= -4g^{ae}g^{bd}\nabla_e u^c(\text{curl } u)_{ab}(\text{curl } u)_{cd} \\
 &\quad + \frac{1}{\pi}g^{ac}(\text{curl } u)_{cd} \left(\nabla_a \beta^e \nabla_e \beta^d + \beta^e \nabla_a \nabla_e \beta^d \right).
 \end{aligned}$$

Similarly,

$$\begin{aligned} D_t |\operatorname{curl} \beta|^2 &= 2(D_t g^{ac})g^{bd} (\operatorname{curl} \beta)_{ab} (\operatorname{curl} \beta)_{cd} + 4g^{ac} g^{bd} (D_t \nabla_a \beta_b) (\operatorname{curl} \beta)_{cd} \\ &= -4g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} \beta)_{ab} (\operatorname{curl} \beta)_{cd} \\ &\quad + 4g^{ac} g^{bd} (\operatorname{curl} \beta)_{cd} (\nabla_a \beta^e (\nabla_e u_b + \nabla_b u_e) + \beta^e \nabla_e \nabla_a u_b). \end{aligned}$$

Thus, we can get

$$\begin{aligned} &D_t \left(|\operatorname{curl} u|^2 + \frac{1}{4\pi} |\operatorname{curl} \beta|^2 \right) \\ &= -4g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} u)_{ab} (\operatorname{curl} u)_{cd} + \frac{1}{\pi} g^{ac} (\operatorname{curl} u)_{cd} \nabla_a \beta^e \nabla_e \beta^d \\ &\quad - \frac{1}{\pi} g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} \beta)_{ab} (\operatorname{curl} \beta)_{cd} \\ &\quad + \frac{1}{\pi} g^{ac} g^{bd} (\operatorname{curl} \beta)_{cd} \nabla_a \beta^e (\nabla_e u_b + \nabla_b u_e) \\ &\quad + \frac{1}{\pi} \nabla_e \left(g^{ac} (\operatorname{curl} u)_{cd} \beta^e \nabla_a \beta^d \right). \end{aligned} \tag{4.6}$$

Define the first order energy as

$$\begin{aligned} E_1(t) &= \int_{\Omega} \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + \frac{1}{4\pi} g^{bd} \gamma^{ae} \nabla_a \beta_b \nabla_e \beta_d \right) d\mu_g \\ &\quad + \int_{\Omega} \left(|\operatorname{curl} u|^2 + \frac{1}{4\pi} |\operatorname{curl} \beta|^2 \right) d\mu_g. \end{aligned} \tag{4.7}$$

Let us recall the Gauss formula for Ω and $\partial\Omega$:

$$\int_{\Omega} \nabla_a w^a d\mu_g = \int_{\partial\Omega} N_a w^a d\mu_{\gamma}, \quad \text{and} \quad \int_{\partial\Omega} \bar{\nabla}_a \bar{f}^a d\mu_{\gamma} = 0 \tag{4.8}$$

if \bar{f} is tangential to $\partial\Omega$ and (N_a) denotes the unit conormal to $\partial\Omega$.

Then, we get the following estimates.

Theorem 4.1. *For any smooth solution of MHD (3.3) for $0 \leq t \leq T$ satisfying*

$$|\nabla P| \leq M, \quad |\nabla u| \leq M, \quad \text{in } [0, T] \times \Omega, \tag{4.9}$$

$$|\theta| + |\nabla u| + \frac{1}{t_0} \leq K, \quad \text{on } [0, T] \times \partial\Omega, \tag{4.10}$$

we have for $t \in [0, T]$

$$E_1(t) \leq 2e^{CMt} E_1(0) + CK^2 (\operatorname{Vol} \Omega + E_0(0)) \left(e^{CMt} - 1 \right). \tag{4.11}$$

Proof. By (4.5), (4.6) and Gauss' formula, we have

$$\begin{aligned}
 \frac{d}{dt} E_1(t) &= \int_{\Omega} D_t \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + \frac{1}{4\pi} g^{bd} \gamma^{ae} \nabla_a \beta_b \nabla_e \beta_d \right) d\mu_g \\
 &+ \int_{\Omega} D_t \left(|\operatorname{curl} u|^2 + \frac{1}{4\pi} |\operatorname{curl} \beta|^2 \right) d\mu_g \\
 &+ \int_{\Omega} \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + \frac{1}{4\pi} g^{bd} \gamma^{ae} \nabla_a \beta_b \nabla_e \beta_d \right) \operatorname{tr} h d\mu_g \\
 &+ \int_{\Omega} \left(|\operatorname{curl} u|^2 + \frac{1}{4\pi} |\operatorname{curl} \beta|^2 \right) \operatorname{tr} h d\mu_g \\
 &= -2 \int_{\Omega} \gamma^{ae} \gamma^{fc} \nabla_e u_f \nabla_a u^d \nabla_c u_d d\mu_g - \frac{1}{2\pi} \int_{\Omega} \gamma^{ae} \gamma^{fc} \nabla_e u_f \nabla_a \beta^d \nabla_c \beta_d d\mu_g \\
 &- 2 \int_{\partial\Omega} N_b \left(\gamma^{ae} \nabla_e u^b \nabla_a P d\mu_g - \frac{1}{4\pi} \gamma^{ae} \beta^b \nabla_e u^d \nabla_a \beta_d \right) d\mu_{\gamma} \tag{4.12}
 \end{aligned}$$

$$+ 2 \int_{\Omega} (\nabla_b \gamma^{ae}) \left(\nabla_e u^b \nabla_a P - \frac{1}{4\pi} \beta^b \nabla_e u^d \nabla_a \beta_d \right) d\mu_g \tag{4.13}$$

$$\begin{aligned}
 &+ \frac{1}{2\pi} \int_{\Omega} \gamma^{ae} \nabla_e u^b \nabla_a \beta^d \nabla_d \beta_b d\mu_g + \frac{1}{2\pi} \int_{\Omega} \gamma^{ae} \nabla_e \beta^b \nabla_a \beta^c \nabla_c u_b d\mu_g \\
 &- 4 \int_{\Omega} g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} u)_{ab} (\operatorname{curl} u)_{cd} d\mu_g \\
 &+ \frac{1}{\pi} \int_{\Omega} g^{ac} (\operatorname{curl} u)_{cd} \nabla_a \beta^e \nabla_e \beta^d d\mu_g \\
 &- \frac{1}{\pi} \int_{\Omega} g^{ae} g^{bd} \nabla_e u^c (\operatorname{curl} \beta)_{ab} (\operatorname{curl} \beta)_{cd} d\mu_g \\
 &+ \frac{1}{\pi} \int_{\Omega} g^{ac} g^{bd} (\operatorname{curl} \beta)_{cd} \nabla_a \beta^e (\nabla_e u_b + \nabla_b u_e) d\mu_g \\
 &+ \frac{1}{\pi} \int_{\partial\Omega} N_e \beta^e g^{ac} (\operatorname{curl} u)_{cd} \nabla_a \beta^d d\mu_{\gamma} \tag{4.14}
 \end{aligned}$$

$$+ \int_{\Omega} \left(g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d + \frac{1}{4\pi} g^{bd} \gamma^{ae} \nabla_a \beta_b \nabla_e \beta_d \right) \operatorname{tr} h d\mu_g$$

$$+ \int_{\Omega} \left(|\operatorname{curl} u|^2 + \frac{1}{4\pi} |\operatorname{curl} \beta|^2 \right) \operatorname{tr} h d\mu_g.$$

Since $P = \frac{1}{8\pi} \zeta^2$ on $\partial\Omega$, it follows that $\bar{\nabla} P = 0$, that is, $\gamma_a^d \nabla_d P = 0$, and then $\gamma^{ae} \nabla_a P = g^{ce} \gamma_c^a \nabla_a P = 0$ on the boundary $\partial\Omega$. In addition, $\beta \cdot N = 0$ on $\partial\Omega$. Thus, the integrals in (4.12) and (4.14) vanish.

From (A.5) and (A.3), we get

$$\theta_{ab} = (\delta_a^c - N_a N^c) \nabla_c N_b = \nabla_a N_b - N_a \nabla_N N_b = \nabla_a N_b, \tag{4.15}$$

since in geodesic coordinates $\nabla_N N = 0$. It follows that

$$\begin{aligned}
 \nabla_b \gamma^{ae} &= \nabla_b (g^{ae} - N^a N^e) = -\nabla_b (N^a N^e) = -(\nabla_b N^a) N^e - (\nabla_b N^e) N^a \\
 &= -\theta_b^a N^e - \theta_b^e N^a.
 \end{aligned}$$

Thus, by the Hölder inequality, (1.18) and Lemma A.2, we get

$$\begin{aligned} |(4.13)| &\leq CK \left(\|\nabla u\|_{L^2(\Omega)} \|\nabla P\|_{L^\infty(\Omega)} (\text{Vol } \Omega)^{1/2} \right. \\ &\quad \left. + \|\nabla u\|_{L^\infty(\Omega)} \|\beta\|_{L^2(\Omega)} \|\nabla \beta\|_{L^2(\Omega)} \right) \\ &\leq CKM \left((\text{Vol } \Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t). \end{aligned}$$

For other terms, we can use the Hölder inequality directly. It yields

$$\begin{aligned} \frac{d}{dt} E_1(t) &\leq CKM \left((\text{Vol } \Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t) \\ &\quad + C \|\nabla u\|_{L^\infty(\Omega)} \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla \beta\|_{L^2(\Omega)}^2 \right) \\ &\quad + \|\text{curl } u\|_{L^2(\Omega)}^2 + \|\text{curl } \beta\|_{L^2(\Omega)}^2 \\ &\leq CKM \left((\text{Vol } \Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t) + CM E_1(t). \end{aligned}$$

From the Gronwall inequality, it follows that

$$E_1^{1/2}(t) \leq e^{CMt/2} E_1^{1/2}(0) + CK \left((\text{Vol } \Omega)^{1/2} + E_0^{1/2}(0) \right) \left(e^{CMt/2} - 1 \right),$$

which implies the desired result. □

Remark 4.1. Since (4.12), especially the integral involving P , vanishes, we do not need the boundary integral in the first order energy $E_1(t)$. But in higher order energies estimates, we need to introduce boundary integrals for P in order to absorb the analogy integral to (4.12).

5. The General r th Order Energy Estimates

From (2.12), (2.19), (1.8a), we get

$$\begin{aligned} D_t \nabla^r u_a &= D_t \nabla_{a_1} \dots \nabla_{a_r} u_a = D_t \left(\frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \partial_{i_1} \dots \partial_{i_r} v_i \right) \\ &= \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \left(D_t \partial_{i_1} \dots \partial_{i_r} v_i + \frac{\partial v^\ell}{\partial x^{i_1}} \partial_\ell \dots \partial_{i_r} v_i + \dots \right. \\ &\quad \left. + \frac{\partial v^\ell}{\partial x^{i_r}} \partial_{i_1} \dots \partial_\ell v_i + \frac{\partial v^\ell}{\partial x^i} \partial_{i_1} \dots \partial_{i_r} v_\ell \right) \\ &= \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \left([D_t, \partial^r] v_i + \partial^r D_t v_i + \frac{\partial v^\ell}{\partial x^{i_1}} \partial_\ell \dots \partial_{i_r} v_i + \dots \right. \\ &\quad \left. + \frac{\partial v^\ell}{\partial x^{i_r}} \partial_{i_1} \dots \partial_\ell v_i + \frac{\partial v^\ell}{\partial x^i} \partial_{i_1} \dots \partial_{i_r} v_\ell \right) \\ &= \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \left(- \sum_{s=0}^{r-1} \binom{r}{s+1} (\partial^{1+s} v) \cdot \partial^{r-s} v_i - \partial^r \partial_i P \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4\pi} \partial^r (B^k \partial_k B_i) + \frac{\partial v^\ell}{\partial x^{i_1}} \partial_\ell \dots \partial_{i_r} v_i + \dots \\
 & + \frac{\partial v^\ell}{\partial x^{i_r}} \partial_{i_1} \dots \partial_\ell v_i + \frac{\partial v^\ell}{\partial x^{i_1}} \partial_{i_1} \dots \partial_{i_r} v_\ell \Big) \\
 = & -\nabla^r \nabla_a P - \sum_{s=1}^{r-1} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a \\
 & + \nabla_a u^c \nabla^r u_c + \frac{1}{4\pi} \sum_{s=0}^r \binom{r}{s} \nabla^s \beta^c \nabla^{r-s} \nabla_c \beta_a,
 \end{aligned}$$

where

$$(\nabla^s \beta^c \nabla^{r-s} \nabla_c \beta_a)_{a_1 \dots a_r} = \sum_{\Sigma_r} \nabla_{a_{\sigma_1} \dots a_{\sigma_s}}^s \beta^c \nabla_{a_{\sigma_{s+1}} \dots a_{\sigma_r}}^{r-s} \nabla_c \beta_a. \tag{5.1}$$

Thus, due to $\text{div } \beta = 0$, we get for $r \geq 2$

$$\begin{aligned}
 & D_t \nabla^r u_a + \nabla^r \nabla_a P \\
 = & (\text{curl } u)_{ac} \nabla^r u^c + \text{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a \\
 & + \frac{1}{4\pi} \nabla_c (\beta^c \nabla^r \beta_a) + \frac{1}{4\pi} \sum_{s=1}^r \binom{r}{s} \nabla^s \beta^c \nabla^{r-s} \nabla_c \beta_a, \tag{5.2}
 \end{aligned}$$

where $\text{sgn}(s)$ is the signum function of the real number s , that is, $\text{sgn}(s) = 1$ for $s > 0$, $\text{sgn}(s) = 0$ for $s = 0$, and $\text{sgn}(s) = -1$ for $s < 0$. Of course, we use this notation $\text{sgn}(2-r)$ to indicate that the related term vanishes for $r = 2$.

Similarly, by noticing that $\text{div } \beta = 0$, we have

$$\begin{aligned}
 D_t \nabla^r \beta_a = & \nabla_a u^c \nabla^r \beta^c - \nabla^r u^c \nabla_c \beta_a \\
 & - \text{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} \beta_a \\
 & + \nabla_c (\beta^c \nabla^r u_a) + \sum_{s=1}^r \binom{r}{s} \nabla^s \beta^c \nabla^{r-s} \nabla_c u_a. \tag{5.3}
 \end{aligned}$$

Define the r -th order energy for $r \geq 2$ as

$$\begin{aligned}
 E_r(t) = & \int_{\Omega} g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \, d\mu_g + \int_{\Omega} |\nabla^{r-1} \text{curl } u|^2 \, d\mu_g \\
 & + \frac{1}{4\pi} \int_{\Omega} g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \beta_b \nabla_F^{r-1} \nabla_f \beta_d \, d\mu_g \\
 & + \frac{1}{4\pi} \int_{\Omega} |\nabla^{r-1} \text{curl } \beta|^2 \, d\mu_g + \int_{\partial\Omega} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a P \nabla_F^{r-1} \nabla_f P \vartheta \, d\mu_\gamma,
 \end{aligned}$$

where $\vartheta = 1/(-\nabla_N P)$ as before.

Theorem 5.1. *Let $r \in \{2, \dots, n + 1\}$, then there exists a $T > 0$ such that the following holds: for any smooth solution of MHD (3.3) for $0 \leq t \leq T$ satisfying*

$$|\beta| \leq M_1 \text{ for } r = 2, \quad \text{in } [0, T] \times \Omega, \quad (5.4)$$

$$|\nabla P| \leq M, \quad |\nabla u| \leq M, \quad |\nabla \beta| \leq M, \quad \text{in } [0, T] \times \Omega, \quad (5.5)$$

$$|\theta| + 1/\iota_0 \leq K, \quad \text{on } [0, T] \times \partial\Omega, \quad (5.6)$$

$$-\nabla_N P \geq \varepsilon > 0, \quad \text{on } [0, T] \times \partial\Omega, \quad (5.7)$$

$$|\nabla^2 P| + |\nabla_N D_t P| \leq L, \quad \text{on } [0, T] \times \partial\Omega, \quad (5.8)$$

we have, for $t \in [0, T]$,

$$E_r(t) \leq e^{C_1 t} E_r(0) + C_2 \left(e^{C_1 t} - 1 \right), \quad (5.9)$$

where C_1 and C_2 depend on $K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0), E_1(0), \dots,$ and $E_{r-1}(0)$.

Proof. We have

$$\frac{d}{dt} E_r(t) = \int_{\Omega} D_t \left(g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \right) d\mu_g \quad (5.10)$$

$$+ \frac{1}{4\pi} \int_{\Omega} D_t \left(g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \beta_b \nabla_F^{r-1} \nabla_f \beta_d \right) d\mu_g \quad (5.11)$$

$$+ \int_{\Omega} D_t |\nabla^{r-1} \text{curl } u|^2 d\mu_g + \frac{1}{4\pi} \int_{\Omega} D_t |\nabla^{r-1} \text{curl } \beta|^2 d\mu_g \quad (5.12)$$

$$+ \int_{\Omega} g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \text{tr } h d\mu_g \quad (5.13)$$

$$+ \int_{\Omega} |\nabla^{r-1} \text{curl } u|^2 \text{tr } h d\mu_g + \frac{1}{4\pi} \int_{\Omega} |\nabla^{r-1} \text{curl } \beta|^2 \text{tr } h d\mu_g \quad (5.14)$$

$$+ \frac{1}{4\pi} \int_{\Omega} g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a \beta_b \nabla_F^{r-1} \nabla_f \beta_d \text{tr } h d\mu_g \quad (5.15)$$

$$+ \int_{\partial\Omega} D_t \left(\gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a P \nabla_F^{r-1} \nabla_f P \right) \vartheta d\mu_{\gamma} \quad (5.16)$$

$$+ \int_{\partial\Omega} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a P \nabla_F^{r-1} \nabla_f P \left(\frac{\vartheta_t}{\vartheta} + \text{tr } h - h_{NN} \right) \vartheta d\mu_{\gamma}. \quad (5.17)$$

We first estimate (5.10), (5.11) and (5.16). From Lemmas 2.1 and A.1, and (5.2), we get

$$\begin{aligned} & D_t \left(g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \right) \\ &= \left(D_t g^{bd} \right) \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d + r g^{bd} \left(D_t \gamma^{af} \right) \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \\ &\quad + 2 g^{bd} \gamma^{af} \gamma^{AF} D_t \left(\nabla_A^{r-1} \nabla_a u_b \right) \nabla_F^{r-1} \nabla_f u_d \\ &= -2 \nabla_c u_e \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u^c \nabla_F^{r-1} \nabla_f u^e - 2 r \nabla_c u_e \gamma^{ac} \gamma^{ef} \gamma^{AF} \nabla_A^{r-1} \nabla_a u^d \nabla_F^{r-1} \nabla_f u_d \end{aligned}$$

$$\begin{aligned}
 & -2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a\nabla_b P + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b(\operatorname{curl} u)_{bc}\nabla_A^{r-1}\nabla_a u^c \\
 & + 2\operatorname{sgn}(2-r)\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d \sum_{s=1}^{r-2} \binom{r}{s+1} \left((\nabla^{s+1}u) \cdot \nabla^{r-s}u^d \right)_{Aa} \\
 & + \frac{1}{2\pi}\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d \nabla_c \left(\beta^c \nabla_{Aa}^r \beta^d \right) \\
 & + \frac{1}{2\pi}\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d \sum_{s=1}^r \binom{r}{s} \left(\nabla^s \beta^c \nabla^{r-s} \nabla_c \beta^d \right)_{Aa}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & D_t \left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta_b\nabla_F^{r-1}\nabla_f\beta_d \right) \\
 & = -2\nabla_c u_e \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta^c\nabla_F^{r-1}\nabla_f\beta^e - 2r\nabla_c u_e \gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta^d\nabla_F^{r-1}\nabla_f\beta_d \\
 & + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^b\nabla_b u_c \nabla_A^{r-1}\nabla_a\beta^c - 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^b\nabla_c\beta_b\nabla_A^{r-1}\nabla_a u^c \\
 & + 2\operatorname{sgn}(2-r)\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^b \sum_{s=1}^{r-2} \binom{r}{s+1} \left((\nabla^{1+s}u) \cdot \nabla^{r-s}\beta_b \right)_{Aa} \\
 & + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^d \nabla_c \left(\beta^c \nabla_{Aa}^r u_d \right) \\
 & + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^d \sum_{s=1}^r \binom{r}{s} \left(\nabla^s \beta^c \nabla^{r-s} \nabla_c u_d \right)_{Aa},
 \end{aligned}$$

and

$$\begin{aligned}
 & D_t \left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P \nabla_F^{r-1}\nabla_f P \right) \\
 & = -2r\nabla_c u_e \gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a P \nabla_F^{r-1}\nabla_f P \\
 & + 2\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P D_t \left(\nabla_F^{r-1}\nabla_f P \right).
 \end{aligned}$$

Thus, we get

$$\begin{aligned}
 & (5.10) + (5.11) + (5.16) \\
 & \leq C \left(\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)} \right) E_r(t)
 \end{aligned}$$

$$+ C E_r^{1/2}(t) \sum_{s=1}^{r-2} \left\| \nabla^{s+1}u \right\|_{L^4(\Omega)} \left(\left\| \nabla^{r-s}u \right\|_{L^4(\Omega)} + \left\| \nabla^{r-s}\beta \right\|_{L^4(\Omega)} \right) \tag{5.18}$$

$$+ C E_r^{1/2}(t) \sum_{s=2}^{r-1} \left\| \nabla^s\beta \right\|_{L^4(\Omega)} \left(\left\| \nabla^{r-s+1}u \right\|_{L^4(\Omega)} + \left\| \nabla^{r-s+1}\beta \right\|_{L^4(\Omega)} \right) \tag{5.19}$$

$$+ 2 \int_{\partial\Omega} \gamma^{af}\gamma^{AF}\nabla_{Aa}^r P \left(D_t \nabla_{Ff}^r P - \frac{1}{\vartheta} N_b \nabla_{Ff}^r u^b \right) \vartheta \, d\mu_\gamma \tag{5.20}$$

$$+ 2 \int_{\Omega} \nabla_b \left(\gamma^{af}\gamma^{AF} \right) \nabla_F^{r-1}\nabla_f u^b \nabla_A^{r-1}\nabla_a P \, d\mu_g \tag{5.21}$$

$$+ \frac{1}{2\pi} \int_{\partial\Omega} N_c \gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d \beta^c \nabla_{Aa}^r \beta^d \, d\mu_\gamma \tag{5.22}$$

$$- \frac{1}{2\pi} \int_{\Omega} \nabla_c \left(\gamma^{af}\gamma^{AF} \right) \nabla_F^{r-1}\nabla_f u_d \beta^c \nabla_{Aa}^r \beta^d \, d\mu_g. \tag{5.23}$$

Due to $\beta \cdot N = 0$ on $\partial\Omega$, (5.22) vanishes. From Lemma A.9, we have, for $t_1 \geq 1/K_1$, that

$$\|\beta\|_{L^\infty(\Omega)} \leq C \sum_{0 \leq s \leq 2} K_1^{n/2-s} \|\nabla^s \beta\|_{L^2(\Omega)} \leq C(K_1) \sum_{s=0}^2 E_s^{1/2}(t). \tag{5.24}$$

Thus, for the last integral, by the Hölder inequality and the assumption (5.5), we have for any $r \geq 3$

$$(5.23) \leq CK \|\beta\|_{L^\infty(\Omega)} E_r(t) \leq C(K, K_1) \left(\sum_{s=0}^2 E_s^{1/2}(t) \right) E_r(t). \tag{5.25}$$

For $r = 2$, we have to assume the a priori bound $|\beta| \leq M_1$ on $[0, T] \times \Omega$, that is, (5.4), in order to get a bound that is linear in the highest-order derivative or energy. Then, we have by (5.4)

$$(5.23) \leq CK \|\beta\|_{L^\infty(\Omega)} E_r(t) \leq C(K, M_1) E_r(t), \quad \text{for } r = 2. \tag{5.26}$$

By the Hölder inequality, we have

$$(5.21) \leq CK E_r^{1/2}(t) \|\nabla^r P\|_{L^2(\Omega)}. \tag{5.27}$$

From (1.8a), we have

$$\partial_j(D_t v^j) + \Delta P = \frac{1}{4\pi} \partial_j \left(B^k \partial_k B^j \right),$$

which yields from (2.18)

$$\Delta P = -\partial_j v^k \partial_k v^j + \frac{1}{4\pi} \partial_j B^k \partial_k B^j.$$

Since Δ is invariant, we have

$$\Delta P = -\nabla_a u^b \nabla_b u^a + \frac{1}{4\pi} \nabla_a \beta^b \nabla_b \beta^a. \tag{5.28}$$

It follows that for $r \geq 2$

$$\begin{aligned} \nabla^{r-2} \Delta P &= \nabla^{r-2} \left(-\nabla_a u^b \nabla_b u^a + \frac{1}{4\pi} \nabla_a \beta^b \nabla_b \beta^a \right) \\ &= -\sum_{s=0}^{r-2} \binom{r-2}{s} \nabla^s \nabla_a u^b \nabla^{r-2-s} \nabla_b u^a \\ &\quad + \frac{1}{4\pi} \sum_{s=0}^{r-2} \binom{r-2}{s} \nabla^s \nabla_a \beta^b \nabla^{r-2-s} \nabla_b \beta^a. \end{aligned}$$

From (5.24), we have for $s \geq 0$

$$\|\nabla^s \beta\|_{L^\infty(\Omega)} \leq C \sum_{\ell=0}^2 K_1^{n/2-\ell} \|\nabla^{\ell+s} \beta\|_{L^2(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 E_{s+\ell}^{1/2}(t), \tag{5.29}$$

and, similarly,

$$\|\nabla^s u\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 E_{s+\ell}^{1/2}(t). \tag{5.30}$$

By Hölder’s inequality, (5.29) and (5.30), we get for $r \in \{3, 4\}$,

$$\begin{aligned} & \left\| \nabla^{r-2} \Delta P \right\|_{L^2(\Omega)} \\ & \leq C \sum_{s=0}^{r-2} \left\| \nabla^s \nabla_a u^b \nabla^{r-2-s} \nabla_b u^a \right\|_{L^2(\Omega)} + C \sum_{s=0}^{r-2} \left\| \nabla^s \nabla_a \beta^b \nabla^{r-2-s} \nabla_b \beta^a \right\|_{L^2(\Omega)} \\ & \leq C \|\nabla u\|_{L^\infty(\Omega)} \left\| \nabla^{r-1} u \right\|_{L^2(\Omega)} + C \|\nabla \beta\|_{L^\infty(\Omega)} \left\| \nabla^{r-1} \beta \right\|_{L^2(\Omega)} \\ & \quad + (r-3)C \left(\left\| \nabla^2 u \right\|_{L^\infty(\Omega)} \left\| \nabla^2 u \right\|_{L^2(\Omega)} + \left\| \nabla^2 \beta \right\|_{L^\infty(\Omega)} \left\| \nabla^2 \beta \right\|_{L^2(\Omega)} \right) \\ & \leq C(K_1) E_{r-1}^{1/2}(t) \sum_{\ell=1}^3 E_\ell^{1/2}(t) + (r-3)C(K_1) E_2^{1/2}(t) \sum_{\ell=2}^4 E_\ell^{1/2}(t) \\ & \leq C(K_1) \sum_{\ell=1}^{r-1} E_\ell(t) + C(K_1) E_2^{1/2}(t) E_r^{1/2}(t). \end{aligned} \tag{5.31}$$

For $r = 2$, we have a simple estimate from the assumption (5.5) and Hölder’s inequality, that is,

$$\begin{aligned} \|\Delta P\|_{L^2(\Omega)} & \leq LC \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^\infty(\Omega)} + C \|\nabla \beta\|_{L^2(\Omega)} \|\nabla \beta\|_{L^\infty(\Omega)} \\ & \leq CME_1^{1/2}(t), \end{aligned} \tag{5.32}$$

which is a lower energy term. Thus, by (A.17), (5.31) and (5.32), we obtain for any $\delta_r > 0$

$$\begin{aligned} \|\nabla^r P\|_{L^2(\Omega)} & \leq \delta_r \|\Pi \nabla^r P\|_{L^2(\partial\Omega)} \\ & \quad + C(1/\delta_r, K, \text{Vol } \Omega) \sum_{s \leq r-2} \|\nabla^s \Delta P\|_{L^2(\Omega)} \\ & \leq \delta_r \|\Pi \nabla^r P\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, K_1, M, \text{Vol } \Omega) \sum_{\ell=1}^{r-1} E_\ell(t) \\ & \quad + (r-2)C(1/\delta_r, K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_r^{1/2}(t). \end{aligned} \tag{5.33}$$

Now we estimate the boundary terms. Since $P = \frac{1}{8\pi} \zeta^2$ on $\partial\Omega$, by (A.18), we have for $r \geq 1$

$$\begin{aligned} \|\Pi \nabla^r P\|_{L^2(\partial\Omega)} & \leq C(K, K_1) \left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2) \sum_{k \leq r-3} \|\bar{\nabla}^k \theta\|_{L^2(\partial\Omega)} \right) \\ & \quad \times \sum_{k \leq r-1} \|\nabla^k P\|_{L^2(\partial\Omega)}. \end{aligned} \tag{5.34}$$

From (A.7), we get $\Pi \nabla^2 P = \theta \nabla_N P$ and then, by (5.7), (5.6), (A.31), (5.5) and (5.33), we get

$$\|\theta\|_{L^2(\partial\Omega)} = \left\| \frac{\Pi \nabla^2 P}{\nabla_N P} \right\|_{L^2(\partial\Omega)} \leq \frac{1}{\varepsilon} \|\Pi \nabla^2 P\|_{L^2(\partial\Omega)}, \quad (5.35)$$

$$\begin{aligned} \|\Pi \nabla^2 P\|_{L^2(\partial\Omega)} &\leq \|\theta\|_{L^\infty(\partial\Omega)} \|\nabla P\|_{L^2(\partial\Omega)} \\ &\leq C(K, \text{Vol } \Omega) \left(\|\nabla^2 P\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \right) \\ &\leq C(K, \text{Vol } \Omega) \delta_2 \|\Pi \nabla^2 P\|_{L^2(\partial\Omega)} + C(K, \text{Vol } \Omega) (\text{Vol } \Omega)^{1/2} M \\ &\quad + C(1/\delta_2, K, K_1, M, \text{Vol } \Omega) E_1(t), \end{aligned} \quad (5.36)$$

where the first term of the right hand side of (5.36) can be absorbed by the left hand side if we take δ_2 so small that, example, $C(K, \text{Vol } \Omega) \delta_2 \leq 1/2$. Thus, it follows that

$$\|\Pi \nabla^2 P\|_{L^2(\partial\Omega)} \leq C(K, K_1, M, \text{Vol } \Omega) (1 + E_1(t)), \quad (5.37)$$

$$\|\nabla^2 P\|_{L^2(\Omega)} \leq C(K, K_1, M, \text{Vol } \Omega) (1 + E_1(t)), \quad (5.38)$$

$$\|\theta\|_{L^2(\partial\Omega)} \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon) (1 + E_1(t)). \quad (5.39)$$

By Theorem 4.1, there exists a $T > 0$ such that $E_1(t)$ can be controlled by the initial energy $E_1(0)$ for $t \in [0, T]$, example, $E_1(t) \leq 2E_1(0)$. Thus, from (5.34), (5.39), (5.5) and (5.38) we have

$$\begin{aligned} \|\Pi \nabla^3 P\|_{L^2(\partial\Omega)} &\leq C(K, K_1) (K + \|\theta\|_{L^2(\partial\Omega)}) \sum_{k \leq 2} \|\nabla^k P\|_{L^2(\partial\Omega)} \\ &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon) (1 + E_1(t)) \sum_{k \leq 3} \|\nabla^k P\|_{L^2(\Omega)} \\ &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \|\nabla^3 P\|_{L^2(\Omega)} \\ &\quad + C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)). \end{aligned} \quad (5.40)$$

From (5.33),

$$\begin{aligned} \|\nabla^3 P\|_{L^2(\Omega)} &\leq \delta_3 C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \|\nabla^3 P\|_{L^2(\Omega)} \\ &\quad + \delta_3 C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \\ &\quad + C(1/\delta_3, K, K_1, M, \text{Vol } \Omega) \sum_{\ell=1}^2 E_\ell(t) \\ &\quad + C(1/\delta_3, K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_3^{1/2}(t), \end{aligned} \quad (5.41)$$

which, if we choose $\delta_3 > 0$ so small that

$$\delta_3 C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \leq 1/2,$$

yields

$$\begin{aligned} \|\nabla^3 P\|_{L^2(\Omega)} &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) + C(K, K_1, M, \text{Vol } \Omega) \sum_{\ell=1}^2 E_\ell(t) \\ &\quad + C(K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_3^{1/2}(t), \end{aligned} \tag{5.42}$$

and then

$$\begin{aligned} \|\Pi \nabla^3 P\|_{L^2(\partial\Omega)} &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \\ &\quad \times \left(1 + \sum_{\ell=1}^2 E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t) \right). \end{aligned} \tag{5.43}$$

Since

$$\begin{aligned} \bar{\nabla}_b \nabla_N P &= \gamma_b^d \nabla_d (N^a \nabla_a P) = (\delta_b^d - N_b N^d) ((\nabla_d N^a) \nabla_a P + N^a \nabla_d \nabla_a P) \\ &= \theta_b^a \nabla_a P + N^a \nabla_b \nabla_a P - N_b N^d (\theta_d^a \nabla_a P + N^a \nabla_d \nabla_a P), \end{aligned}$$

from (A.31), it follows that

$$\begin{aligned} \|\bar{\nabla} \nabla_N P\|_{L^2(\partial\Omega)} &\leq C \|\theta\|_{L^\infty(\partial\Omega)} \|\nabla P\|_{L^2(\partial\Omega)} + C \|\nabla^2 P\|_{L^2(\partial\Omega)} \\ &\leq C(K, \text{Vol } \Omega) \left(\|\nabla^3 P\|_{L^2(\Omega)} + \|\nabla^2 P\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \right) \\ &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) + C(K, K_1, M, \text{Vol } \Omega) \sum_{\ell=1}^2 E_\ell(t) \\ &\quad + C(K, K_1, M, \text{Vol } \Omega) E_2^{1/2}(t) E_3^{1/2}(t). \end{aligned}$$

Thus, by (A.8), it follows that $(\bar{\nabla}\theta)\nabla_N P = \Pi \nabla^3 P - 3\theta \tilde{\otimes} \bar{\nabla} \nabla_N P$ and

$$\begin{aligned} \|\bar{\nabla}\theta\|_{L^2(\partial\Omega)} &\leq \frac{1}{\varepsilon} \left(\|\Pi \nabla^3 P\|_{L^2(\partial\Omega)} + C \|\theta\|_{L^\infty(\partial\Omega)} \|\bar{\nabla} \nabla_N P\|_{L^2(\partial\Omega)} \right) \\ &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \\ &\quad \times \left(1 + \sum_{\ell=1}^2 E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t) \right). \end{aligned} \tag{5.44}$$

Hence, from (5.34), (A.31), it yields

$$\begin{aligned} \|\Pi \nabla^4 P\|_{L^2(\partial\Omega)} &\leq C(K, K_1) \left(K + \|\theta\|_{L^2(\partial\Omega)} + \|\bar{\nabla}\theta\|_{L^2(\partial\Omega)} \right) \\ &\quad \times \sum_{k \leq 4} \|\nabla^k P\|_{L^2(\Omega)}. \end{aligned} \tag{5.45}$$

Then, from (5.33), we can absorb the highest order term $\|\nabla^4 P\|_{L^2(\Omega)}$ by the left hand side for $\delta_4 > 0$ small enough which is independent of the highest energy $E_4(t)$, and get

$$\begin{aligned} \|\nabla^4 P\|_{L^2(\Omega)} &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \\ &\quad \times \left(1 + \sum_{\ell=1}^3 E_\ell(t) + E_2^{1/2}(t)E_4^{1/2}(t)\right), \end{aligned} \tag{5.46}$$

$$\begin{aligned} \|\Pi \nabla^4 P\|_{L^2(\partial\Omega)} &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \\ &\quad \times \left(1 + \sum_{\ell=1}^3 E_\ell(t) + E_2^{1/2}(t)E_4^{1/2}(t)\right). \end{aligned} \tag{5.47}$$

Therefore, from (5.38), (5.42) and (5.45), we obtain for $r \geq 2$

$$\begin{aligned} \|\nabla^r P\|_{L^2(\Omega)} &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0)) \\ &\quad \times \left(1 + \sum_{\ell=1}^{r-1} E_\ell(t) + (r-2)E_2^{1/2}(t)E_r^{1/2}(t)\right), \end{aligned} \tag{5.48}$$

which, from (5.27), implies

$$\begin{aligned} (5.21) &\leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon, E_1(0))E_r^{1/2}(t) \\ &\quad \times \left(1 + \sum_{\ell=1}^{r-1} E_\ell(t) + (r-2)E_2^{1/2}(t)E_r^{1/2}(t)\right). \end{aligned} \tag{5.49}$$

Now, we turn to the estimates of (5.20). Since $P = \frac{1}{8\pi} \zeta^2$ on $\partial\Omega$ implies $\gamma_b^a \nabla_a P = 0$ on $\partial\Omega$, we get from (A.3), by noticing that $\vartheta = -1/\nabla_N P$, that

$$-\vartheta^{-1} N_b = \nabla_N P N_b = N^a \nabla_a P N_b = \delta_b^a \nabla_a P - \gamma_b^a \nabla_a P = \nabla_b P. \tag{5.50}$$

By the Hölder inequality and (5.50), we have

$$\begin{aligned} (5.20) &\leq C \|\vartheta\|_{L^\infty(\partial\Omega)}^{1/2} E_r^{1/2}(t) \left\| \Pi \left(D_t (\nabla^r P) - \vartheta^{-1} N_b \nabla^r u^b \right) \right\|_{L^2(\partial\Omega)} \\ &= C \|\vartheta\|_{L^\infty(\partial\Omega)}^{1/2} E_r^{1/2}(t) \left\| \Pi \left(D_t (\nabla^r P) + \nabla^r u \cdot \nabla P \right) \right\|_{L^2(\partial\Omega)}. \end{aligned} \tag{5.51}$$

By (2.25), it follows that

$$\begin{aligned} D_t \nabla^r P + \nabla^r u \cdot \nabla P &= [D_t, \nabla^r] P + \nabla^r D_t P + \nabla^r u \cdot \nabla P \\ &= \text{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{s+1} u) \cdot \nabla^{r-s} P + \nabla^r D_t P. \end{aligned} \tag{5.52}$$

We first consider the estimates of the last term in (5.52). By (A.18) and (A.31), we get, for $2 \leq r \leq 4$

$$\begin{aligned} & \left\| \Pi \nabla^r D_t P \right\|_{L^2(\partial\Omega)} \\ & \leq C(K, K_1, \text{Vol } \Omega) \left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2) \sum_{k \leq r-3} \left\| \bar{\nabla}^k \theta \right\|_{L^2(\partial\Omega)} \right) \\ & \quad \times \sum_{k \leq r} \left\| \nabla^k D_t P \right\|_{L^2(\Omega)}. \end{aligned} \tag{5.53}$$

From (A.17), it follows that

$$\begin{aligned} & \left\| \nabla^r D_t P \right\|_{L^2(\Omega)} \\ & \leq \delta \left\| \Pi \nabla^r D_t P \right\|_{L^2(\partial\Omega)} + C(1/\delta, K, \text{Vol } \Omega) \sum_{s \leq r-2} \left\| \nabla^s \Delta D_t P \right\|_{L^2(\Omega)}. \end{aligned} \tag{5.54}$$

By (2.24), (5.28), Lemma 2.1, (4.1), (4.2) and (3.3), it yields

$$\begin{aligned} \Delta D_t P &= 2h^{ab} \nabla_a \nabla_b P + (\Delta u^e) \nabla_e P - D_t \left(g^{bd} g^{ac} \nabla_a u_d \nabla_b u_c \right) \\ & \quad + \frac{1}{4\pi} D_t \left(g^{bd} g^{ac} \nabla_a \beta_d \nabla_b \beta_c \right) \\ &= 2h^{ab} \nabla_a \nabla_b P + (\Delta u^e) \nabla_e P - 2D_t(g^{bd}) \nabla_a u_d \nabla_b u^a \\ & \quad - 2g^{bd} D_t(\nabla_a u_d) \nabla_b u^a + \frac{1}{2\pi} D_t(g^{bd}) g^{ac} \nabla_a \beta_d \nabla_b \beta^a \\ & \quad + \frac{1}{2\pi} g^{bd} D_t(\nabla_a \beta_d) \nabla_b \beta^a \\ &= 2h^{ab} \nabla_a \nabla_b P + (\Delta u^e) \nabla_e P + 4h^{bd} \nabla_a u_d \nabla_b u^a - \frac{1}{\pi} h^{bd} \nabla_a \beta_d \nabla_b \beta^a \\ & \quad + 2g^{bd} \nabla_b u^a \nabla_a \nabla_d P - 2g^{bd} \nabla_b u^a \nabla_a u^c \nabla_d u_c \\ & \quad - \frac{1}{2\pi} \nabla_b u^a \left(\nabla_a \beta^c \nabla_c \beta^b + \beta^c \nabla_a \nabla_c \beta^b \right) \\ & \quad + \frac{1}{2\pi} g^{bd} \nabla_b \beta^a \left(\nabla_a \beta^e (\nabla_e u_d + \nabla_d u_e) + \beta^e \nabla_e \nabla_a u_d \right) \\ &= 4g^{ac} \nabla_c u^b \nabla_a \nabla_b P + (\delta u^e) \nabla_e P + 2\nabla_e u^b \nabla_b u^a \nabla_a u^e \\ & \quad - \frac{1}{2\pi} \nabla_b u^a \nabla_a \beta^c \nabla_c \beta^b - \frac{1}{2\pi} \nabla_b u^a \beta^c \nabla_a \nabla_c \beta^b + \frac{1}{2\pi} \nabla_b \beta^a \beta^e \nabla_e \nabla_a u^b. \end{aligned}$$

By (5.29), (5.33) and Lemma A.9, it follows that for $s \leq 2$

$$\begin{aligned} & \left\| \nabla^s \delta D_t P \right\|_{L^2(\Omega)} \\ & \leq C \left\| \nabla u \right\|_{L^\infty(\Omega)} \left\| \nabla^{s+2} P \right\|_{L^2(\Omega)} + s(s-1)C \left\| \nabla^3 u \right\|_{L^2(\Omega)} \left\| \nabla^2 P \right\|_{L^\infty(\Omega)} \\ & \quad + sC \left\| \nabla^2 u \right\|_{L^4(\Omega)} \left\| \nabla^{s+1} P \right\|_{L^4(\Omega)} + C \left\| \nabla^{s+2} u \right\|_{L^2(\Omega)} \left\| \nabla P \right\|_{L^\infty(\Omega)} \end{aligned}$$

$$\begin{aligned}
 &+C \left(\|\nabla u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)} \|\nabla \beta\|_{L^\infty(\Omega)} \right) \left\| \nabla^{s+1} u \right\|_{L^2(\Omega)} \\
 &+s(s-1)C \|\nabla u\|_{L^\infty(\Omega)} \left\| \nabla^2 u \right\|_{L^4(\Omega)} \left\| \nabla^2 u \right\|_{L^4(\Omega)} \\
 &+C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla \beta\|_{L^\infty(\Omega)} \left\| \nabla^{s+1} \beta \right\|_{L^2(\Omega)} \\
 &+sC \left\| \nabla^2 u \right\|_{L^4(\Omega)} \left\| \nabla^2 \beta \right\|_{L^4(\Omega)} \left((s-1) \|\nabla \beta\|_{L^\infty(\Omega)} + \|\beta\|_{L^\infty(\Omega)} \right) \\
 &+s(s-1)C \|\nabla u\|_{L^\infty(\Omega)} \left\| \nabla^2 \beta \right\|_{L^4(\Omega)} \left\| \nabla^2 \beta \right\|_{L^4(\Omega)} \\
 &+C \|\nabla u\|_{L^\infty(\Omega)} \|\beta\|_{L^\infty(\Omega)} \left\| \nabla^{s+2} \beta \right\|_{L^2(\Omega)} \\
 &+sC \left\| \nabla^3 u \right\|_{L^2(\Omega)} \|\beta\|_{L^\infty(\Omega)} \left((s-1) \left\| \nabla^2 \beta \right\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)} \right) \\
 &+s(s-1)C \left\| \nabla^3 \beta \right\|_{L^2(\Omega)} \|\beta\|_{L^\infty(\Omega)} \left\| \nabla^2 u \right\|_{L^\infty(\Omega)} \\
 &+s(s-1)C \|\nabla \beta\|_{L^\infty(\Omega)} \left\| \nabla^2 \beta \right\|_{L^4(\Omega)} \left\| \nabla^2 u \right\|_{L^4(\Omega)} \\
 &+s(s-1)C \|\nabla \beta\|_{L^\infty(\Omega)} \|\beta\|_{L^\infty(\Omega)} \left\| \nabla^4 u \right\|_{L^2(\Omega)} \\
 &+s(s-1)C \left\| \nabla^2 \beta \right\|_{L^\infty(\Omega)} \|\beta\|_{L^\infty(\Omega)} \left\| \nabla^3 u \right\|_{L^2(\Omega)}. \tag{5.55}
 \end{aligned}$$

From Lemma A.8 and (5.30), it follows that

$$\begin{aligned}
 \left\| \nabla^{s+1} u \right\|_{L^4(\Omega)} &\leq C \|\nabla^s u\|_{L^\infty(\Omega)}^{1/2} \left(\sum_{\ell=0}^2 \left\| \nabla^{s+\ell} u \right\|_{L^2(\Omega)} K_1^{2-\ell} \right)^{1/2} \\
 &\leq C(K_1) \sum_{\ell=0}^2 E_{s+\ell}^{1/2}(t). \tag{5.56}
 \end{aligned}$$

We can estimate all the terms with $L^4(\Omega)$ norms in the same way with the help of (5.29), (5.30), the similar estimate of P and the assumptions. Thus, we obtain the bound which is linear about the highest-order derivative or the highest-order energy $E_r^{1/2}(t)$, that is,

$$\begin{aligned}
 \left\| \nabla^s \delta D_t P \right\|_{L^2(\Omega)} &\leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \\
 &\quad \times \left(1 + \sum_{\ell=0}^{r-1} E_\ell(t) \right) \left(1 + E_r^{1/2}(t) \right). \tag{5.57}
 \end{aligned}$$

Thus, from (5.53), (5.54), (5.57) and taking some small δ 's which are independent of $E_r(t)$, we obtain, by induction argument for r , that

$$\begin{aligned}
 \left\| \Pi \nabla^r D_t P \right\|_{L^2(\partial\Omega)} &\leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \\
 &\quad \times \left(1 + \sum_{\ell=0}^{r-1} E_\ell(t) \right) \left(1 + E_r^{1/2}(t) \right). \tag{5.58}
 \end{aligned}$$

To estimate (5.52), it only remains to estimate

$$\left\| \Pi \left((\nabla^{s+1} u) \cdot \nabla^{r-s} P \right) \right\|_{L^2(\partial\Omega)} \quad \text{for } 1 \leq s \leq r - 2. \tag{5.59}$$

For $r = 3, 4$ and $s = r - 2$, we have, by (5.8) and Lemma A.11, that

$$\begin{aligned} & \left\| \Pi \left((\nabla^{r-1} u) \cdot \nabla^2 P \right) \right\|_{L^2(\partial\Omega)} \\ & \leq \left\| \nabla^{r-1} u \right\|_{L^2(\partial\Omega)} \left\| \nabla^2 P \right\|_{L^\infty(\partial\Omega)} \\ & \leq CL \left\| \nabla^2 u \right\|_{L^{2(n-1)/(n-2)}(\partial\Omega)} \\ & \leq C(K, \text{Vol } \Omega) L \left(\left\| \nabla^r u \right\|_{L^2(\Omega)} + \left\| \nabla^{r-1} u \right\|_{L^2(\Omega)} \right) \\ & \leq C(K, L, \text{Vol } \Omega) \left(E_{r-1}^{1/2}(t) + E_r^{1/2}(t) \right). \end{aligned} \tag{5.60}$$

For $n = 3, r = 4$ and $s = 1$, by (A.6), Lemma A.11 and (5.33), we get

$$\begin{aligned} & \left\| \Pi \left((\nabla^2 u) \cdot \nabla^3 P \right) \right\|_{L^2(\partial\Omega)} \\ & = \left\| \Pi \nabla^2 u \cdot \Pi \nabla^3 P + \Pi (\nabla^2 u \cdot N) \otimes \Pi (N \cdot \nabla^3 P) \right\|_{L^2(\partial\Omega)} \\ & \leq C \left\| \Pi \nabla^2 u \right\|_{L^4(\partial\Omega)} \left\| \Pi \nabla^3 P \right\|_{L^4(\partial\Omega)} \\ & \quad + C \left\| \Pi (N^a \nabla^2 u_a) \right\|_{L^4(\partial\Omega)} \left\| \Pi (\nabla_N \nabla^2 P) \right\|_{L^4(\partial\Omega)} \\ & \leq C \left\| \nabla^2 u \right\|_{L^4(\partial\Omega)} \left\| \nabla^3 P \right\|_{L^4(\partial\Omega)} \\ & \leq C(K, \text{Vol } \Omega) \left(\left\| \nabla^3 u \right\|_{L^2(\Omega)} + \left\| \nabla^2 u \right\|_{L^2(\Omega)} \right) \left(\left\| \nabla^4 P \right\|_{L^2(\Omega)} + \left\| \nabla^3 P \right\|_{L^2(\Omega)} \right) \\ & \leq C(K, K_1, \text{Vol } \Omega) \left(E_3^{1/2}(t) + E_2^{1/2}(t) \right) \left(\sum_{s=0}^3 E_s(t) + \left(\sum_{\ell=0}^2 E_\ell^{1/2}(t) \right) E_4^{1/2}(t) \right) \\ & \leq C(K, K_1, \text{Vol } \Omega) \sum_{s=0}^3 E_s(t) \sum_{\ell=0}^4 E_\ell^{1/2}(t). \end{aligned} \tag{5.61}$$

Hence, we have

$$\begin{aligned} (5.20) & \leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \\ & \quad \times \left(1 + \sum_{s=0}^{r-1} E_s(t) \right) (1 + E_r(t)). \end{aligned} \tag{5.62}$$

By Lemma A.8, we can obtain

$$(5.18) + (5.19) \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon) \left(1 + \sum_{s=0}^{r-1} E_s(t) \right) E_r(t). \tag{5.63}$$

Therefore, we have shown that

$$(5.10) + (5.11) + (5.16) \leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \\ \times \left(1 + \sum_{s=0}^{r-1} E_s(t)\right) (1 + E_r(t)). \quad (5.64)$$

We now calculate the material derivatives of $|\nabla^{r-1} \text{curl } u|^2$ and $|\nabla^{r-1} \text{curl } \beta|^2$. From Lemma 2.1, (5.2) and (5.3), we have

$$\begin{aligned} & D_t \left(|\nabla^{r-1} \text{curl } u|^2 + \frac{1}{4\pi} |\nabla^{r-1} \text{curl } \beta|^2 \right) \\ &= D_t \left(g^{ac} g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } u)_{ab} \nabla_F^{r-1} (\text{curl } u)_{cd} \right) \\ &\quad + \frac{1}{4\pi} D_t \left(g^{ac} g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } \beta)_{ab} \nabla_F^{r-1} (\text{curl } \beta)_{cd} \right) \\ &= (r+1) D_t (g^{ac}) g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } u)_{ab} \nabla_F^{r-1} (\text{curl } u)_{cd} \\ &\quad + 4g^{ac} g^{bd} g^{AF} D_t \left(\nabla_A^{r-1} \nabla_a u_b \right) \nabla_F^{r-1} (\text{curl } u)_{cd} \\ &\quad + \frac{r+1}{4\pi} D_t (g^{ac}) g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } \beta)_{ab} \nabla_F^{r-1} (\text{curl } \beta)_{cd} \\ &\quad + \frac{1}{\pi} g^{ac} g^{bd} g^{AF} D_t \left(\nabla_A^{r-1} \nabla_a \beta_b \right) \nabla_F^{r-1} (\text{curl } \beta)_{cd} \\ &= -2(r+1) g^{ae} \nabla_e u^c g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } u)_{ab} \nabla_F^{r-1} (\text{curl } u)_{cd} \\ &\quad - \frac{r+1}{2\pi} g^{ae} \nabla_e u^c g^{bd} g^{AF} \nabla_A^{r-1} (\text{curl } \beta)_{ab} \nabla_F^{r-1} (\text{curl } \beta)_{cd} \\ &\quad - 4g^{ac} g^{bd} g^{AF} \nabla_F^{r-1} (\text{curl } u)_{cd} \nabla_{Aa} \nabla_b P \quad (\text{this vanishes by symmetry}) \\ &\quad + 4g^{ac} g^{bd} g^{AF} \nabla_F^{r-1} (\text{curl } u)_{cd} (\text{curl } u)_{be} \nabla_{Aa}^r u^e \\ &\quad + 4\text{sgn}(2-r) g^{ac} g^{AF} \nabla_F^{r-1} (\text{curl } u)_{cd} \sum_{s=1}^{r-2} \binom{r}{s+1} \left((\nabla^{1+s} u) \cdot \nabla^{r-s} u^d \right)_{Aa} \\ &\quad + \frac{1}{\pi} \text{sgn}(2-r) g^{ac} g^{AF} \nabla_F^{r-1} (\text{curl } \beta)_{cd} \sum_{s=1}^{r-2} \binom{r}{s+1} \left((\nabla^{1+s} \beta) \cdot \nabla^{r-s} \beta^d \right)_{Aa} \\ &\quad + \frac{1}{\pi} g^{ac} g^{bd} g^{AF} \nabla_F^{r-1} (\text{curl } \beta)_{cd} \nabla_{Aa}^r \beta^e \nabla_b u_e \\ &\quad \quad - \frac{1}{\pi} g^{ac} g^{AF} \nabla_F^{r-1} (\text{curl } \beta)_{cd} \nabla_{Aa}^r u^e \nabla_e \beta^d \\ &\quad + \frac{1}{\pi} \nabla_e \left(g^{ac} g^{AF} \beta^e \nabla_F^{r-1} (\text{curl } u)_{cd} \nabla_{Aa}^r \beta^d \right) \\ &\quad + \frac{1}{\pi} g^{ac} g^{AF} \nabla_F^{r-1} (\text{curl } u)_{cd} \sum_{s=1}^r \binom{r}{s} \left(\nabla^s \beta^e \nabla^{r-s} \nabla_e \beta^d \right)_{Aa} \\ &\quad + \frac{1}{\pi} g^{ac} g^{AF} \nabla_F^{r-1} (\text{curl } \beta)_{cd} \sum_{s=1}^r \binom{r}{s} \left(\nabla^s \beta^e \nabla^{r-s} \nabla_e u^d \right)_{Aa}. \end{aligned}$$

Noticing that $\beta \cdot N = 0$ on $\partial\Omega$, then by the Hölder inequality and the Gauss formula, we get

$$(5.12) \leq C(K, K_1, M, \text{Vol } \Omega, 1/\varepsilon) \left(1 + \sum_{s=0}^{r-1} E_s(t) \right) E_r(t). \tag{5.65}$$

Thus, by (A.12) and (2.23), we get

$$\begin{aligned} D_t(\nabla_N P) &= D_t(N^a \nabla_a P) = (D_t N^a) \nabla_a P + N^a D_t \nabla_a P \\ &= \left(-2h_d^a N^d + h_{NN} N^a \right) \nabla_a P + N^a \nabla_a D_t P \\ &= -2h_d^a N^d \nabla_a P + h_{NN} \nabla_N P + \nabla_N D_t P, \end{aligned}$$

which yields

$$\frac{\vartheta_t}{\vartheta} = -\frac{D_t \nabla_N P}{\nabla_N P} = \frac{2h_d^a N^d \nabla_a P}{\nabla_N P} - h_{NN} + \frac{\nabla_N D_t P}{\nabla_N P}. \tag{5.66}$$

Thus, we can easily obtain that the remainder integrals, that is, (5.13), (5.14), (5.15) and (5.17), can be controlled by $C(K, M, L, 1/\varepsilon) E_r(t)$.

Therefore, we obtain

$$\begin{aligned} \frac{d}{dt} E_r(t) &\leq C(K, K_1, M, M_1, L, 1/\varepsilon, \text{Vol } \Omega, E_0(0)) \\ &\quad \times \left(1 + \sum_{s=0}^{r-1} E_s(t) \right) (1 + E_r(t)), \end{aligned} \tag{5.67}$$

which implies the desired result (5.9) by Gronwall’s inequality and the induction argument for $r \in \{2, \dots, n + 1\}$. □

6. Justification of a Priori Assumptions

Let $\mathcal{K}(t)$ and $\varepsilon(t)$ be the maximum and minimum values, respectively, such that (5.6) and (5.7) hold at time t :

$$\mathcal{K}(t) = \max \left(\|\theta(t, \cdot)\|_{L^\infty(\partial\Omega)}, 1/\iota_0(t) \right), \tag{6.1}$$

$$\varepsilon(t) = \|1/(\nabla_N P(t, \cdot))\|_{L^\infty(\partial\Omega)} = 1/\varepsilon(t). \tag{6.2}$$

Lemma 6.1. *Let $K_1 \geq 1/\iota_1$ be as in Definition A.3, $\mathcal{E}(t)$ as in (6.2). Then there are continuous functions G_j , $j = 1, 2, 3, 4$, such that*

$$\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)} + \|\beta\|_{L^\infty(\Omega)} \leq G_1(K_1, E_0, \dots, E_{n+1}), \tag{6.3}$$

$$\|\nabla P\|_{L^\infty(\Omega)} + \left\| \nabla^2 P \right\|_{L^\infty(\partial\Omega)} \leq G_2(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \text{Vol } \Omega), \tag{6.4}$$

$$\|\theta\|_{L^\infty(\partial\Omega)} \leq G_3(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \text{Vol } \Omega), \tag{6.5}$$

$$\|\nabla D_t P\|_{L^\infty(\partial\Omega)} \leq G_4(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \text{Vol } \Omega). \tag{6.6}$$

Proof. (6.3) follows from (5.30), (5.29) and (5.24). From Lemmas A.9 and A.7, we have

$$\|\nabla P\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 \left\| \nabla^{\ell+1} P \right\|_{L^2(\Omega)}, \tag{6.7}$$

$$\left\| \nabla^2 P \right\|_{L^\infty(\partial\Omega)} \leq C(K_1) \sum_{\ell=0}^{n+1} \left\| \nabla^\ell P \right\|_{L^2(\partial\Omega)}. \tag{6.8}$$

Thus, (6.4) follows from (6.7), (6.8), Lemmas A.10, A.11, (5.32), (5.38) and (5.42). Since, from (A.7),

$$|\nabla^2 P| \geq |\Pi \nabla^2 P| = |\nabla_N P| |\theta| \geq \mathcal{E}^{-1} |\theta|, \tag{6.9}$$

so (6.5) follows from (6.4). (6.6) follows from Lemma A.7, (5.54), (5.57) and (5.58). \square

Lemma 6.2. *Let $K_1 \geq 1/\iota_1$ and ε_1 be as in Definition A.3. Then*

$$\left| \frac{d}{dt} E_r \right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \text{Vol } \Omega) \sum_{s=0}^r E_s, \tag{6.10}$$

and

$$\left| \frac{d}{dt} \mathcal{E} \right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{n+1}, \text{Vol } \Omega). \tag{6.11}$$

Proof. (6.10) is a consequence of Lemma 6.1 and the estimates in the proof of Theorems 4.1 and 5.1. (6.11) follows from

$$\left| \frac{d}{dt} \left\| \frac{1}{-\nabla_N P(t, \cdot)} \right\|_{L^\infty(\partial\Omega)} \right| \leq C \left\| \frac{1}{-\nabla_N P(t, \cdot)} \right\|_{L^\infty(\partial\Omega)}^2 \|\nabla_N D_t P(t, \cdot)\|_{L^\infty(\partial\Omega)}$$

and (6.6). \square

As a result of Lemma 6.2, we have the following:

Lemma 6.3. *There exists a continuous function $\mathcal{T} > 0$ depending on $K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega$ such that for*

$$0 \leq t \leq \mathcal{T}(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega), \tag{6.12}$$

the following statements hold: We have

$$E_s(t) \leq 2E_s(0), \quad 0 \leq s \leq n + 1, \quad \mathcal{E}(t) \leq 2\mathcal{E}(0). \tag{6.13}$$

Furthermore,

$$\frac{g_{ab}(0, y) Y^a Y^b}{2} \leq g_{ab}(t, y) Y^a Y^b \leq 2g_{ab}(0, y) Y^a Y^b, \tag{6.14}$$

and with ε_1 as in Definition A.3,

$$|\mathcal{N}(x(t, \bar{y})) - \mathcal{N}(x(0, \bar{y}))| \leq \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega, \tag{6.15}$$

$$|x(t, y) - x(0, y)| \leq \frac{l_1}{16}, \quad y \in \Omega, \tag{6.16}$$

$$\left| \frac{\partial x(t, \bar{y})}{\partial y} - \frac{\partial(0, \bar{y})}{\partial y} \right| \leq \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega. \tag{6.17}$$

Proof. We get (6.13) from Lemma 6.2 if $\mathcal{S}(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega) > 0$ is sufficiently small. Then from (6.13) and Lemma 6.1, we have

$$\begin{aligned} \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)} + \|\beta\|_{L^\infty(\Omega)} + \|\nabla P\|_{L^\infty(\Omega)} \\ \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0)), \end{aligned} \tag{6.18}$$

$$\begin{aligned} \left\| \nabla^2 P \right\|_{L^\infty(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)} \\ \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega), \end{aligned} \tag{6.19}$$

$$\|\nabla D_t P\|_{L^\infty(\partial\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega). \tag{6.20}$$

By (4.1) and (4.2), we have

$$|D_t \nabla u| \leq \left| \nabla^2 P \right| + |\nabla u|^2 + |\nabla \beta|^2 + |\beta| \left| \nabla^2 \beta \right|, \tag{6.21}$$

$$|D_t \nabla \beta| \leq |\nabla \beta| |\nabla u| + |\beta| \left| \nabla^2 u \right|. \tag{6.22}$$

By (A.25), (A.31), Lemma 6.1 and (6.13), we have

$$\|\nabla u\|_{L^\infty(\partial\Omega)} + \|\nabla \beta\|_{L^\infty(\partial\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega).$$

Thus, by noticing that $|\beta| = \zeta$ on $\partial\Omega$, it follows, from (6.18), (6.19), Lemmas A.7 and A.11, (5.30) and (5.29), that

$$\begin{aligned} & \|D_t \nabla u\|_{L^\infty(\partial\Omega)} + \|D_t \nabla \beta\|_{L^\infty(\partial\Omega)} \\ & \leq \left\| \nabla^2 P \right\|_{L^\infty(\partial\Omega)} + (\|\nabla u\|_{L^\infty(\partial\Omega)} + \|\nabla \beta\|_{L^\infty(\partial\Omega)})^2 \\ & \quad + \zeta \left(\left\| \nabla^2 u \right\|_{L^\infty(\partial\Omega)} + \left\| \nabla^2 \beta \right\|_{L^\infty(\partial\Omega)} \right) \\ & \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega) \\ & \quad \times (1 + \|\nabla u\|_{L^\infty(\partial\Omega)} + \|\nabla \beta\|_{L^\infty(\partial\Omega)}), \end{aligned}$$

which yields, with the help of Gronwall’s inequality, for $0 \leq t \leq T$

$$\begin{aligned} & \|\nabla u(t, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla \beta(t, \cdot)\|_{L^\infty(\partial\Omega)} \\ & \leq e^{C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega)t} (\|\nabla u(0, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla \beta(0, \cdot)\|_{L^\infty(\partial\Omega)}) \\ & \quad + e^{C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega)t} - 1. \end{aligned} \tag{6.23}$$

If T is sufficiently small, it follows, after possibly making $\mathcal{T} > 0$ smaller, that

$$\begin{aligned} & \|\nabla u(T, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla\beta(T, \cdot)\|_{L^\infty(\partial\Omega)} \\ & \leq 2 (\|\nabla u(0, \cdot)\|_{L^\infty(\partial\Omega)} + \|\nabla\beta(0, \cdot)\|_{L^\infty(\partial\Omega)}), \end{aligned} \tag{6.24}$$

which also guarantee the a priori assumption of (3.1).

By (2.23), (A.28), (5.54), (5.57) and (5.58), we have

$$\begin{aligned} \|D_t \nabla P\|_{L^\infty(\Omega)} &= \|\nabla D_t P\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{\ell=0}^2 \left\| \nabla^{\ell+1} D_t P \right\|_{L^2(\Omega)} \\ &\leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega), \end{aligned}$$

which implies for sufficiently small $T > 0$

$$\|\nabla P(t, \cdot)\|_{L^\infty(\Omega)} \leq 2 \|\nabla P(0, \cdot)\|_{L^\infty(\Omega)}. \tag{6.25}$$

By (1.8) and (6.18), we have

$$\|D_t v\|_{L^\infty(\mathcal{D}_t)} \leq \|\partial P\|_{L^\infty(\mathcal{D}_t)} + \|B\|_{L^\infty(\mathcal{D}_t)} \|\partial B\|_{L^\infty(\mathcal{D}_t)} \tag{6.26}$$

$$\leq \|\nabla P\|_{L^\infty(\Omega)} + \|\beta\|_{L^\infty(\Omega)} \|\nabla\beta\|_{L^\infty(\Omega)} \tag{6.27}$$

$$\leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0)), \tag{6.28}$$

which yields

$$\|v(t, \cdot)\|_{L^\infty(\mathcal{D}_t)} \leq 2 \|v(0, \cdot)\|_{L^\infty(\Omega)}. \tag{6.29}$$

(6.14) follows from the same argument since $D_t g_{ab} = \nabla_a u_b + \nabla_b u_a$ and by (6.18)

$$\left| g_{ab}(T, y) Y^a Y^b - g_{ab}(0, y) Y^a Y^b \right| \leq \int_0^T |D_t g_{ab}(s, y)| \, ds Y^a Y^b \tag{6.30}$$

$$\leq 2 \int_0^T \|\nabla_a u_b(s)\|_{L^\infty(\Omega)} \, ds Y^a Y^b \leq \frac{1}{2} g_{ab}(0, y) Y^a Y^b, \tag{6.31}$$

if T is sufficiently small. Now the estimate for \mathcal{N} follows from

$$D_t n_a = h_{NN} n_a,$$

and the estimates for x and $\partial x / \partial y$ from

$$D_t x(t, y) = v(t, x(t, y)), \tag{6.32}$$

$$D_t \frac{\partial x}{\partial y} = \frac{\partial v(t, x(t, y))}{\partial y} = \frac{\partial v(t, x)}{\partial x} \frac{\partial x}{\partial y}, \tag{6.33}$$

and (6.29) and (6.24), respectively. □

Now we use (6.14)–(6.17) to pick a K_1 , that is, ι_1 , which depends only on its value at $t = 0$,

$$\iota_1(t) \geq \iota_1(0)/2. \tag{6.34}$$

Lemma 6.4. *Let \mathcal{T} be as in Lemma 6.2. Pick $t_1 > 0$ such that*

$$|\mathcal{N}(x(0, y_1)) - \mathcal{N}(x(0, y_2))| \leq \frac{\varepsilon_1}{2}, \quad \text{whenever } |x(0, y_1) - x(0, y_2)| \leq 2t_1. \tag{6.35}$$

Then if $t \leq \mathcal{T}$, we have

$$|\mathcal{N}(x(t, y_1)) - \mathcal{N}(x(t, y_2))| \leq \varepsilon_1, \quad \text{whenever } |x(t, y_1) - x(t, y_2)| \leq 2t_1. \tag{6.36}$$

Proof. (6.36) follows from (6.35), (6.15) and (6.16) in view of triangle inequalities. \square

Lemma 6.4 allows us to pick a K_1 depending only on initial conditions, while Lemma 6.3 gives us $\mathcal{T} > 0$, that depends only on the initial conditions and K_1 such that, by Lemma 6.4, $1/t_1 \leq K_1$ for $t \leq \mathcal{T}$. Thus, we immediately obtain the following theorem.

Theorem 6.1. *There exists a continuous function $\mathcal{T} > 0$ such that if*

$$T \leq \mathcal{T}(\mathcal{K}(0), \mathcal{E}(0), E_0(0), \dots, E_{n+1}(0), \text{Vol } \Omega), \tag{6.37}$$

any smooth solution of the free boundary problem for MHD Equations (1.1) and (1.6) for $0 \leq t \leq T$ satisfies

$$\sum_{s=0}^{n+1} E_s(t) \leq 2 \sum_{s=0}^{n+1} E_s(0), \quad 0 \leq t \leq T. \tag{6.38}$$

Acknowledgments The authors would like to thank the referees for the suggestions which have helped to improve the presentation of the paper significantly. HAO was partially supported by the National Natural Science Foundation of China (Grant 11171327) and by the Youth Innovation Promotion Association, Chinese Academy of Sciences. This work was also partially supported by NSFC (11271219). Part of this work originated from the seminars on ‘‘Fluid Interface Problems’’ at Mathematical Sciences Center, Tsinghua University and Morningside Center of Mathematics, Chinese Academy of Sciences.

Appendix A. Preliminaries and Some Estimates

Let N^a denote the unit normal to $\partial\Omega$, $g_{ab}N^aN^b = 1$, $g_{ab}N^aT^b = 0$ if $T \in T(\partial\Omega)$, and let $N_a = g_{ab}N^b$ denote the unit conormal, $g^{ab}N_aN_b = 1$. The induced metric γ on the tangent space to the boundary $T(\partial\Omega)$ extended to be 0 on the orthogonal complement in $T(\Omega)$ is then given by

$$\gamma_{ab} = g_{ab} - N_aN_b, \quad \gamma^{ab} = g^{ab} - N^aN^b. \tag{A.1}$$

The orthogonal projection of an (r, s) tensor S to the boundary is given by

$$(\Pi S)_{b_1 \dots b_s}^{a_1 \dots a_r} = \gamma_{c_1}^{a_1} \dots \gamma_{c_r}^{a_r} \gamma_{b_1}^{d_1} \dots \gamma_{b_s}^{d_s} S_{d_1 \dots d_s}^{c_1 \dots c_r}, \tag{A.2}$$

where

$$\gamma_a^c = \delta_a^c - N_a N^c. \tag{A.3}$$

Covariant differentiation on the boundary $\bar{\nabla}$ is given by

$$\bar{\nabla} S = \Pi \nabla S. \tag{A.4}$$

The second fundamental form of the boundary is given by

$$\theta_{ab} = (\Pi \nabla N)_{ab} = \gamma_a^c \nabla_c N_b. \tag{A.5}$$

Let us now recall some properties of the projection. Since $g^{ab} = \gamma^{ab} + N^a N^b$, we have

$$\Pi(S \cdot R) = \Pi(S) \cdot \Pi(R) + \Pi(S \cdot N) \tilde{\otimes} \Pi(N \cdot R), \tag{A.6}$$

where $S \tilde{\otimes} R$ denotes some partial symmetrization of the tensor product $S \otimes R$, that is, a sum over some subset of the permutations of the indices divided by the number of permutations in that subset. Similarly, we let $S \cdot R$ denote a partial symmetrization of the dot product $S \cdot R$. Now we recall some identities:

$$\Pi \nabla^2 q = \bar{\nabla}^2 q + \theta \nabla_N q, \tag{A.7}$$

$$\Pi \nabla^3 q = \bar{\nabla}^3 q - 2\theta \tilde{\otimes} (\theta \cdot \bar{\nabla} q) + (\bar{\nabla} \theta) \nabla_N q + 3\theta \tilde{\otimes} \bar{\nabla} \nabla_N q, \tag{A.8}$$

$$\begin{aligned} \Pi \nabla^4 q = & \bar{\nabla}^4 q - \theta \tilde{\otimes} \left(5(\bar{\nabla} \theta) \cdot \bar{\nabla} q + 8\theta \cdot \bar{\nabla}^2 q \right) - 2(\bar{\nabla} \theta) \tilde{\otimes} (\theta \cdot \bar{\nabla} q) \\ & + (\bar{\nabla}^2 \theta) \nabla_N q + 4(\bar{\nabla} \theta) \tilde{\otimes} \bar{\nabla} \nabla_N q + 6\theta \tilde{\otimes} \bar{\nabla}^2 \nabla_N q \\ & - 3\theta \tilde{\otimes} (\theta \cdot \theta) \nabla_N q + 3\theta \tilde{\otimes} \theta \nabla_N^2 q. \end{aligned} \tag{A.9}$$

Definition A.1. Let $\mathcal{N}(\bar{x})$ be the outward unit normal to $\partial \mathcal{D}_t$ at $\bar{x} \in \partial \mathcal{D}_t$. Let $\text{dist}(x_1, x_2) = |x_1 - x_2|$ denote the Euclidean distance in \mathbb{R}^n , and for $\bar{x}_1, \bar{x}_2 \in \partial \mathcal{D}_t$, let $\text{dist}_{\partial \mathcal{D}_t}(\bar{x}_1, \bar{x}_2)$ denote the geodesic distance on the boundary.

Definition A.2. Let $\text{dist}(x, \partial \mathcal{D}_t)$ be the Euclidean distance from x to the boundary. Let ι_0 be the injectivity radius of the normal exponential map of $\partial \mathcal{D}_t$, that is, the largest number such that the map

$$\begin{aligned} \partial \mathcal{D}_t \times (-\iota_0, \iota_0) & \rightarrow \{x \in \mathbb{R}^n : \text{dist}(x, \partial \mathcal{D}_t) < \iota\} \\ \text{given by } (\bar{x}, \iota) & \rightarrow x = \bar{x} + \iota \mathcal{N}(\bar{x}) \end{aligned} \tag{A.10}$$

is an injection.

Definition A.3. Let $0 < \varepsilon_1 < 2$ be a fixed number, and let $\iota_1 = \iota_1(\varepsilon_1)$ the largest number such that

$$|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1 \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial \mathcal{D}_t. \tag{A.11}$$

Lemma A.1. ([4, Lemma 3.9]) *Let N be the unit normal to $\partial\Omega$, and let $h_{ab} = \frac{1}{2}D_t g_{ab}$. On $[0, T] \times \partial\Omega$, we have*

$$D_t N_a = h_{NN} N_a, \quad D_t N^c = -2h_d^c N^d + h_{NN} N^c, \tag{A.12}$$

$$D_t \gamma^{ab} = -2\gamma^{ac} h_{cd} \gamma^{db}, \tag{A.13}$$

where $h_{NN} = h_{ab} N^a N^b$. The volume element on $\partial\Omega$ satisfies

$$D_t d\mu_\gamma = (\text{tr } h - h_{NN})d\mu_\gamma = (\text{tr } \theta u \cdot N + \gamma^{ab} \bar{\nabla}_a \bar{u}_b) d\mu_\gamma, \tag{A.14}$$

where \bar{u}_b denotes the tangential component of u_b to the boundary $\partial\Omega$.

Lemma A.2. (cf. [4, Lemma 5.5]) *Let $w_a = w_{Aa} = \nabla_A^r f_a$, $\nabla_A^r = \nabla_{a_1} \dots \nabla_{a_r}$, f be a $(0, 1)$ tensor, and $[\nabla_a, \nabla_b] = 0$. Let $\text{div } w = \nabla_a w^a = \nabla^r \text{div } f$, and let $(\text{curl } w)_{ab} = \nabla_a w_b - \nabla_b w_a = \nabla^r (\text{curl } f)_{ab}$. Then,*

$$|\nabla w|^2 \leq C \left(g^{ab} \gamma^{cd} \gamma^{AB} \nabla_c w_{Aa} \nabla_d w_{Bb} + |\text{div } w|^2 + |\text{curl } w|^2 \right). \tag{A.15}$$

Lemma A.3. ([4, Proposition 5.8]) *Let ι_0 and ι_1 be as in Definitions A.2 and A.3, and suppose that $|\theta| + 1/\iota_0 \leq K$ and $1/\iota_1 \leq K_1$. Then with $\tilde{K} = \min(K, K_1)$ we have, for any $r \geq 2$ and $\delta > 0$,*

$$\begin{aligned} & \|\nabla^r q\|_{L^2(\partial\Omega)} + \|\nabla^r q\|_{L^2(\Omega)} \\ & \leq C \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} + C(\tilde{K}, \text{Vol } \Omega) \sum_{s \leq r-1} \|\nabla^s \delta q\|_{L^2(\Omega)}, \end{aligned} \tag{A.16}$$

$$\begin{aligned} & \|\nabla^{r-1} q\|_{L^2(\partial\Omega)} + \|\nabla^r q\|_{L^2(\Omega)} \\ & \leq \delta \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} + C(1/\delta, K, \text{Vol } \Omega) \sum_{s \leq r-2} \|\nabla^s \delta q\|_{L^2(\Omega)}. \end{aligned} \tag{A.17}$$

Lemma A.4. (cf. [4, Proposition 5.9]) *Assume that $0 \leq r \leq 4$. Suppose that $|\theta| \leq K$ and $\iota_1 \geq 1/K_1$, where ι_1 is as in Definition 3.5 of [4]. If $q = 0$ on $\partial\Omega$, then for $m = 0, 1$,*

$$\begin{aligned} \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} & \leq C(K, K_1) \left(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k \leq r-2-m} \|\bar{\nabla}^k \theta\|_{L^2(\partial\Omega)} \right) \\ & \quad \times \sum_{k \leq r-2+m} \|\nabla^k q\|_{L^2(\partial\Omega)}. \end{aligned} \tag{A.18}$$

If, in addition, $|\nabla_N q| \geq \varepsilon > 0$ and $|\nabla_N q| \geq 2\varepsilon \|\nabla_N q\|_{L^\infty(\partial\Omega)}$, then

$$\begin{aligned} & \|\bar{\nabla}^{r-2} \theta\|_{L^2(\partial\Omega)} \\ & \leq C \left(K, K_1, \frac{1}{\varepsilon} \right) \left(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k \leq r-3} \|\bar{\nabla}^k \theta\|_{L^2(\partial\Omega)} \right) \sum_{k \leq r-1} \|\nabla^k q\|_{L^2(\partial\Omega)}. \end{aligned} \tag{A.19}$$

Lemma A.5. (cf. [4, Proposition 5.10]) Assume that $0 \leq r \leq 4$ and that $|\theta| + 1/\iota_0 \leq K$. If $q = 0$ on $\partial\Omega$, then

$$\begin{aligned} \left\| \nabla^{r-1} q \right\|_{L^2(\partial\Omega)} &\leq C \left(\left\| \bar{\nabla}^{r-3} \theta \right\|_{L^2(\partial\Omega)} \left\| \nabla_N q \right\|_{L^\infty(\partial\Omega)} + \left\| \nabla^{r-2} \delta q \right\|_{L^2(\Omega)} \right) \\ &\quad + C(K, \text{Vol } \Omega, \|\theta\|_{L^2(\partial\Omega)}) \\ &\quad \times \left(\left\| \nabla_N q \right\|_{L^\infty(\partial\Omega)} + \sum_{s=0}^{r-3} \left\| \nabla^s \delta q \right\|_{L^2(\Omega)} \right). \end{aligned} \tag{A.20}$$

Lemma A.6. ([4, Lemma A.1]) If α is a $(0, r)$ tensor, then with $a = k/m$ and a constant C that only depends on m and n , such that

$$\left\| \bar{\nabla}^k \alpha \right\|_{L^s(\partial\Omega)} \leq C \|\alpha\|_{L^q(\partial\Omega)}^{1-a} \left\| \bar{\nabla}^m \alpha \right\|_{L^p(\partial\Omega)}^a, \tag{A.21}$$

if

$$\frac{m}{s} = \frac{k}{p} + \frac{m-k}{q}, \quad 2 \leq p \leq s \leq q \leq \infty.$$

Lemma A.7. ([4, Lemma A.2]) Suppose that for $\iota_1 \geq 1/K_1$

$$|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1, \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t, \tag{A.22}$$

and

$$C_0^{-1} \gamma_{ab}^0(y) Z^a Z^b \leq \gamma_{ab}(t, y) Z^a Z^b \leq C_0 \gamma_{ab}^0(y) Z^a Z^b, \quad \text{if } Z \in T(\Omega), \tag{A.23}$$

where $\gamma_{ab}^0(y) = \gamma_{ab}(0, y)$. Then if α is a $(0, r)$ tensor,

$$\|\alpha\|_{L^{(n-1)p/(n-1-kp)}(\partial\Omega)} \leq C(K_1) \sum_{\ell=0}^k \left\| \nabla^\ell \alpha \right\|_{L^p(\partial\Omega)}, \quad 1 \leq p < \frac{n-1}{k}, \tag{A.24}$$

$$\|\alpha\|_{L^\infty(\partial\Omega)} \leq \delta \left\| \nabla^k \alpha \right\|_{L^p(\partial\Omega)} + C_\delta(K_1) \sum_{\ell=0}^{k-1} \left\| \nabla^\ell \alpha \right\|_{L^p(\partial\Omega)}, \quad k > \frac{n-1}{p}, \tag{A.25}$$

for any $\delta > 0$.

Lemma A.8. ([4, Lemma A.3]) With notation as in Lemmas A.6 and A.7, we have

$$\sum_{j=0}^k \left\| \nabla^j \alpha \right\|_{L^s(\Omega)} \leq C \|\alpha\|_{L^q(\Omega)}^{1-a} \left(\sum_{i=0}^m \left\| \nabla^i \alpha \right\|_{L^p(\Omega)} K_1^{m-i} \right)^a. \tag{A.26}$$

Lemma A.9. ([4, Lemma A.4]) *Suppose that $\iota_1 \geq 1/K_1$ and α is a $(0, r)$ tensor. Then*

$$\|\alpha\|_{L^{np/(n-kp)}(\Omega)} \leq C \sum_{\ell=0}^k K_1^{k-\ell} \|\nabla^\ell \alpha\|_{L^p(\Omega)}, \quad 1 \leq p < \frac{n}{k}, \quad (\text{A.27})$$

$$\|\alpha\|_{L^\infty(\Omega)} \leq C \sum_{\ell=0}^k K_1^{n/p-\ell} \|\nabla^\ell \alpha\|_{L^p(\Omega)}, \quad k > \frac{n}{p}. \quad (\text{A.28})$$

Lemma A.10. ([4, Lemma A.5]) *Suppose that $q = 0$ on $\partial\Omega$. Then*

$$\|q\|_{L^2(\Omega)} \leq C(\text{Vol } \Omega)^{1/n} \|\nabla q\|_{L^2(\Omega)}, \quad (\text{A.29})$$

$$\|\nabla q\|_{L^2(\Omega)} \leq C(\text{Vol } \Omega)^{1/2n} \|\delta q\|_{L^2(\Omega)}. \quad (\text{A.30})$$

Lemma A.11. ([4, Lemma A.7]) *Let α be a $(0, r)$ tensor. Assume that*

$$\text{Vol } \Omega \leq V \text{ and } \|\theta\|_{L^\infty(\partial\Omega)} + 1/\iota_0 \leq K,$$

then there is a $C = C(K, V, r, n)$ such that

$$\|\alpha\|_{L^{(n-1)p/(n-p)}(\partial\Omega)} \leq C \|\nabla \alpha\|_{L^p(\Omega)} + C \|\alpha\|_{L^p(\Omega)}, \quad 1 \leq p < n, \quad (\text{A.31})$$

$$\|\nabla^2 \alpha\|_{L^2(\Omega)} \leq C \left(\|\Pi \nabla^2 \alpha\|_{L^{2(n-1)/n}(\partial\Omega)} + \|\delta \alpha\|_{L^2(\Omega)} + \|\nabla \alpha\|_{L^2(\Omega)} \right). \quad (\text{A.32})$$

References

1. AMBROSE, D.M., MASMOUDI, N.: The zero surface tension limit of two-dimensional water waves. *Commun. Pure Appl. Math.* **58**(10), 1287–1315 (2005)
2. BEALE, J.T., HOU, T.Y., LOWENGRUB, J.S.: Growth rates for the linearized motion of fluid interfaces away from equilibrium. *Commun. Pure Appl. Math.* **46**(9), 1269–1301 (1993)
3. CHEN, G.Q., WANG, Y.G.: Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics. *Arch. Rational Mech. Anal.* **187**(3), 369–408 (2008)
4. CHRISTODOULOU, D., LINDBLAD, H.: On the motion of the free surface of a liquid. *Commun. Pure Appl. Math.* **53**(12), 1536–1602 (2000)
5. COUTAND, D., SHKOLLER, S.: Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Am. Math. Soc.* **20**(3), 829–930 (2007)
6. DÍAZ, J.I., LERENA, M.B.: On the inviscid and non-resistive limit for the equations of incompressible magnetohydrodynamics. *Math. Model. Methods Appl. Sci.* **12**(10), 1401–1419 (2002)
7. DUVAUT, G., LIONS, J.L.: Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Rational Mech. Anal.* **46**, 241–279 (1972)
8. EBIN, D.G.: The equations of motion of a perfect fluid with free boundary are not well posed. *Commun. Partial Differ. Equ.* **12**(10), 1175–1201 (1987)
9. FRIEDMAN, A., LIU, Y.: A free boundary problem arising in magnetohydrodynamic system. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22**(3), 375–448 (1995)

10. GOEDBLOED, J.P.H., Poedts, S.: *Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas*. Cambridge University Press, Cambridge, (2004)
11. HE, C., XIN, Z.: Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations. *J. Funct. Anal.* **227**(1), 113–152 (2005)
12. HU, X., WANG, D.: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. *Commun. Math. Phys.* **283**(1), 255–284 (2008)
13. HU, X., WANG, D.: Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. *Arch. Rational Mech. Anal.* **197**(1), 203–238 (2010)
14. LI, X., WANG, D.: Global strong solution to the three-dimensional density-dependent incompressible magnetohydrodynamic flows. *J. Differ. Equ.* **251**(6), 1580–1615 (2011)
15. LINDBLAD, H.: Well-posedness for the linearized motion of an incompressible liquid with free surface boundary. *Commun. Pure Appl. Math.* **56**(2), 153–197 (2003)
16. LINDBLAD, H.: Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. Math. (2)* **162**(1), 109–194 (2005)
17. LINDBLAD, H., NORDGREN, K.H.: A priori estimates for the motion of a self-gravitating incompressible liquid with free surface boundary. *J. Hyperbolic Differ. Equ.* **6**(2), 407–432 (2009)
18. PADULA, M., SOLONNIKOV, V.A.: On the free boundary problem of magnetohydrodynamics. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **385**(41), 135–186, 236 (2010). Translation in *J. Math. Sci. (N. Y.)* **178**(3), 313–344 (2011)
19. SCHMIDT, P.G.: On a magnetohydrodynamic problem of Euler type. *J. Differ. Equ.* **74**(2), 318–335 (1988)
20. SERMANGE, M., TEMAM, R.: Some mathematical questions related to the MHD equations. *Commun. Pure Appl. Math.* **36**(5), 635–664 (1983)
21. SHATAH, J., ZENG, C.: Geometry and a priori estimates for free boundary problems of the Euler equation. *Commun. Pure Appl. Math.* **61**(5), 698–744 (2008)
22. TRAKHININ, Y.: The existence of current-vortex sheets in ideal compressible magnetohydrodynamics. *Arch. Rational Mech. Anal.* **191**(2), 245–310 (2009)
23. TRAKHININ, Y.: On the well-posedness of a linearized plasma–vacuum interface problem in ideal compressible MHD. *J. Differ. Equ.* **249**(10), 2577–2599 (2010)
24. WU, S.: Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.* **130**(1), 39–72 (1997)
25. WU, S.: Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Am. Math. Soc.* **12**(2), 445–495 (1999)
26. YANAGISAWA, T., MATSUMURA, A.: The fixed boundary value problems for the equations of ideal magnetohydrodynamics with a perfectly conducting wall condition. *Commun. Math. Phys.* **136**(1), 119–140 (1991)
27. ZHANG, P., ZHANG, Z.: On the free boundary problem of three-dimensional incompressible Euler equations. *Commun. Pure Appl. Math.* **61**(7), 877–940 (2008)

Institute of Mathematics, Academy of Mathematics and Systems Science,
and Hua Loo-Keng Key Laboratory of Mathematics,
Chinese Academy of Sciences,

Beijing 100190,
China

and

BCMIIS,
Beijing 100048,
China.

e-mail: hcc@amss.ac.cn

and

Department of Mathematics and Statistics,
Georgetown University,
Washington, DC 20057,
USA.
e-mail: tl48@georgetown.edu

(Received June 28, 2012 / Accepted November 29, 2013)

Published online January 15, 2014 – © Springer-Verlag Berlin Heidelberg (2014)