

## WELL-POSEDNESS FOR A MULTIDIMENSIONAL VISCOUS LIQUID-GAS TWO-PHASE FLOW MODEL\*

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**Abstract.** The Cauchy problem of a multidimensional ( $d \geq 2$ ) compressible viscous liquid-gas two-phase flow model is studied in this paper. We investigate the global existence and uniqueness of the strong solution for the initial data close to a stable equilibrium and the local-in-time existence and uniqueness of the solution with general initial data in the framework of Besov spaces. A continuation criterion is also obtained for the local solution.

**Key words.** compressible liquid-gas two-phase flow model, global well-posedness, local well-posedness, Besov spaces

**AMS subject classifications.** 76T10, 76N10, 35E15

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**1. Introduction.** Models of two-phase or multiphase flows have a very broad application to hydrodynamics in industry, for example, in manufacturing, engineering, and biomedicine, where the fluids under investigation contain more than one component. Indeed, it has been estimated that over half of everything produced in a modern industrial society depends, to some extent, on a multiphase flow process for its optimum design and safe operation. In nature, there is a variety of different multiphase flow phenomena, such as sediment transport, geysers, volcanic eruptions, clouds, and rain. In addition, models of multiphase flows also naturally appear in many contexts within biology, ranging from tumor biology and anticancer therapies to developmental biology and plant physiology. The principles of single-phase flow fluid dynamics and heat transfer are relatively well understood; however, the thermofluid dynamics of two-phase flows is an order of magnitude more complicated than that of the single-phase flows due to the existence of a moving and deformable interface and its interactions with two phases [3, 14, 15, 16].

In the present paper, we consider the drift-flux model of two-phase flows, which was principally developed by Zuber and Findlay [25], Wallis [19], and Ishii [13]. The basic idea behind drift-flux models is that both phases are well mixed, but the relative motion between the phases is governed by a particular subset of the flow parameters. In the case of liquid-gas fluids, it relates the liquid-gas velocity difference to the drift flux (or “drift velocity”) of the vapor relative to the liquid due to buoyancy effects. In general, the drift-flux models consist of two mass conservation equations corresponding to each of the two phases, and one equation for the conservation of

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the mixture momentum, and are particularly useful in the analysis of sedimentation, fluidization (batch, cocurrent, and countercurrent), and so on [13, 19, 24, 25].

The Cauchy problem to a simplified version of the viscous compressible liquid-gas two-phase flow model of drift-flux type in  $\mathbb{R}^d$  ( $d \geq 2$ ), where the gas phase has not been taken into account in the momentum equation except that the pressure term and the equal velocity of the liquid and gas flows have been assumed, reads

$$(1.1) \quad \begin{cases} \tilde{m}_t + \operatorname{div}(\tilde{m}\mathbf{u}) = 0, \\ \tilde{n}_t + \operatorname{div}(\tilde{n}\mathbf{u}) = 0, \\ (\tilde{m}\mathbf{u})_t + \operatorname{div}(\tilde{m}\mathbf{u} \otimes \mathbf{u}) + \nabla P(\tilde{m}, \tilde{n}) = \tilde{\mu}\Delta\mathbf{u} + (\tilde{\mu} + \tilde{\lambda})\nabla\operatorname{div}\mathbf{u}, \end{cases}$$

with the initial data

$$(1.2) \quad (\tilde{m}, \tilde{n}, \mathbf{u})|_{t=0} = (\tilde{m}_0, \tilde{n}_0, \mathbf{u}_0)(x) \quad \text{in } \mathbb{R}^d,$$

where  $\tilde{m} = \alpha_l \rho_l$  and  $\tilde{n} = \alpha_g \rho_g$  denote the liquid mass and the gas mass, respectively. The unknowns  $\alpha_l, \alpha_g \in [0, 1]$  denote the liquid and gas volume fractions, satisfying the fundamental relation  $\alpha_l + \alpha_g = 1$ . The unknown variables  $\rho_l$  and  $\rho_g$  denote the liquid and gas densities, satisfying the equations of states  $\rho_l = \rho_{l,0} + (P - P_{l,0})/a_l^2$ ,  $\rho_g = P/a_g^2$ , where  $a_l$  and  $a_g$  denote the sonic speeds of the liquid and the gas, respectively, and  $P_{l,0}$  and  $\rho_{l,0}$  are the reference pressure and density given as constants.  $\mathbf{u}$  denotes the mixed velocity of the liquid and the gas, and  $P$  is the common pressure for both phases, which satisfies

$$(1.3) \quad P(\tilde{m}, \tilde{n}) = C_0 \left( -b(\tilde{m}, \tilde{n}) + \sqrt{b^2(\tilde{m}, \tilde{n}) + c(\tilde{m}, \tilde{n})} \right),$$

with  $C_0 = a_l^2/2$ ,  $k_0 = \rho_{l,0} - P_{l,0}/a_l^2 > 0$ ,  $a_0 = a_g^2/a_l^2$ , and

$$b(\tilde{m}, \tilde{n}) = k_0 - \tilde{m} - a_0\tilde{n}, \quad c(\tilde{m}, \tilde{n}) = 4k_0a_0\tilde{n}.$$

$\tilde{\mu}$  and  $\tilde{\lambda}$  are the viscosity constants, satisfying

$$(1.4) \quad \tilde{\mu} > 0, \quad 2\tilde{\mu} + d\tilde{\lambda} \geq 0.$$

For the one-dimensional case, the existence and/or uniqueness of the global weak solution to the free boundary value problem was studied in [9, 11, 22, 23], where the liquid is incompressible and the gas is polytropic, and in [10], where both of two fluids are compressible. However, there are few results for multidimensional cases except for some computational results [18]. As a generalization of the results in [10], the existence of the global solution to the two-dimensional model was obtained in [21] for small initial energies. In [20], a blow-up criterion for the two-dimensional model was proved in terms of the upper bound of the liquid mass for the strong solution in a smooth bounded domain.

One of the main results of the present paper is the existence and uniqueness of the global strong solution to the Cauchy problem (1.1)–(1.2) within the framework of Besov spaces for all multidimensions  $d \geq 2$ , provided that the initial data are close to a constant equilibrium state. The other result is the local well-posedness and the continuation criterion to the Cauchy problem with general initial data. Because of the similarity of the viscous liquid-gas two-phase flow model to the compressible Navier–Stokes equations, we can apply some ideas adopted in the proof of well-posedness

for the compressible Navier–Stokes equations to deal with the two-phase flow model. It was Danchin who first made important progress in applying the Littlewood–Paley theory and Besov spaces to solve the existence and uniqueness for the compressible Navier–Stokes equations or barotropic viscous fluid [5, 8] and for the flows of compressible viscous and heat-conductive gases [6, 7]. However, it is nontrivial to apply directly the ideas used in single-phase models to two-phase models because the momentum equation is given only for the mixture, and the pressure involves the masses of two phases in a nonlinear way, which makes it rather difficult to obtain the estimates of the masses and the mixed velocity  $(\tilde{m}, \tilde{n}, \mathbf{u})$  in Lebesgue spaces  $L^p$  with respect to time. In addition, it seems impossible to get the estimates of  $\tilde{m}$  and  $\tilde{n}$  from the system simultaneously due to the strong coupling among the corresponding terms, even if we change the variables  $(\tilde{m}, \tilde{n})$  linearly.

To overcome these difficulties in global well-posedness theory, we make use of a nonlinear variable transform so as to separate the two mass variables from each other, which enables us to decompose the original system into a transport equation and a coupled hyperbolic-parabolic system. More precisely, we first divide the momentum equation by  $\tilde{m}$  (which supplies additional information) and take a new variable  $n = a_0(\tilde{n}/\tilde{m} - \bar{n}/\bar{m})$  for some constants  $\bar{n}$  and  $\bar{m}$ . This makes the resulting equation for  $n$  a homogeneous transport equation with velocity  $\mathbf{u}$ , and the expected estimates of the new variable depend only on the mixed velocity. Then we remove the linear terms involving  $n$  from the momentum equation so as to separate linearly the equation about  $n$  from the others, which can be done by virtue of the variable changes with a careful choice of coefficient factors. Finally, to establish the a priori estimates for the global existence theory, we deal with the linearized system directly instead of separating the velocity into compressible and incompressible parts.

As for the local well-posedness theory for general data, we need to reformulate the original system and deal with the resulting nonlinear system directly, and in terms of the improvement of the a priori estimates on the densities, we can generalize the local well-posedness result in [1, 8] to the two-phase flow model (1.1) with the specified pressure function.

Before stating the main results, we introduce some notation. Throughout the paper,  $C$  (or  $c$ ) stands for a harmless constant, and we sometimes use  $A \lesssim B$  to stand for  $A \leq CB$ .  $B^s$  and  $B^{s,t}$  denote usual homogeneous Besov spaces and hybrid Besov spaces, respectively;  $\tilde{L}^\infty(B^{s,t})$  and  $\tilde{C}(B^{s,t})$  are mixed time-spatial spaces; see the appendix for details. Let us now introduce the functional spaces which appear in the theorems.

DEFINITION 1.1. *For  $T > 0$  and  $s \in \mathbb{R}$ , we denote*

$$E_T^s = \left\{ (m, n, \mathbf{u}) : \begin{aligned} n &\in \mathcal{C}([0, T]; B^{s-1,s}(\mathbb{R}^d)) \\ m &\in \mathcal{C}([0, T]; B^{s-1,s}(\mathbb{R}^d)) \cap L^1([0, T]; B^{s+1,s}(\mathbb{R}^d)) \\ \mathbf{u} &\in (\mathcal{C}([0, T]; B^{s-1}(\mathbb{R}^d)) \cap L^1([0, T]; B^{s+1}(\mathbb{R}^d)))^d \end{aligned} \right\}$$

and

$$\begin{aligned} \|(m, n, \mathbf{u})\|_{E_T^s} &= \|n\|_{\tilde{L}^\infty([0, T]; B^{s-1,s})} + \|m\|_{\tilde{L}^\infty([0, T]; B^{s-1,s})} + \|\mathbf{u}\|_{\tilde{L}^\infty([0, T]; B^{s-1})} \\ &\quad + \|m\|_{L^1([0, T]; B^{s+1,s})} + \|\mathbf{u}\|_{L^1([0, T]; B^{s+1})}. \end{aligned}$$

We use the notation  $E^s$  if  $T = +\infty$ , changing  $[0, T]$  to  $[0, \infty)$  in the definition above.

DEFINITION 1.2. Let  $\alpha \in [0, 1]$  and  $T > 0$ , and denote

$$F_T^\alpha := (\tilde{\mathcal{C}}([0, T]; B^{d/2, d/2+\alpha})^{1+1} \times (\tilde{\mathcal{C}}([0, T]; B^{d/2-1, d/2-1+\alpha}) \cap L^1([0, T]; B^{d/2+1, d/2+1+\alpha}))^d.$$

Now, we state the global well-posedness results briefly as follows. For more information about the solution, see Theorem 2.1 in the second section.

THEOREM 1.3 (global well-posedness for small data). Let  $d \geq 2$ ,  $\bar{n} \geq 0$ ,  $\bar{m} > (1 - \text{sgn} \bar{n})k_0$ ,  $\tilde{\mu} > 0$ , and  $2\tilde{\mu} + d\tilde{\lambda} \geq 0$ ; in addition,  $\tilde{\mu} + \tilde{\lambda} > 0$  if  $d = 2$ . There exist two positive constants  $\sigma$  and  $Q$  such that if  $\tilde{m}_0 - \bar{m}$ ,  $\tilde{n}_0 - \bar{n} \in B^{d/2-1, d/2}$  and  $\mathbf{u}_0 \in B^{d/2-1}$  satisfy

$$(1.5) \quad \|\tilde{m}_0 - \bar{m}\|_{B^{d/2-1, d/2}} + \|\tilde{n}_0 - \bar{n}\|_{B^{d/2-1, d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}} \leq \sigma,$$

then the following results hold:

(i) Existence: The system (1.1) has a solution  $(\tilde{m}, \tilde{n}, \mathbf{u})$  satisfying

$$\tilde{m} - \bar{m}, \tilde{n} - \bar{n} \in \mathcal{C}(\mathbb{R}^+; B^{d/2-1, d/2}), \quad \mathbf{u} \in \mathcal{C}(\mathbb{R}^+; B^{d/2-1}),$$

and, moreover,

$$\begin{aligned} & \| (a(\tilde{m} - \bar{m}) + ba_0(\tilde{n}/\tilde{m} - \bar{n}/\bar{m}), \tilde{n}/\tilde{m} - \bar{n}/\bar{m}, \mathbf{u}) \|_{E^{d/2}} \\ & \leq Q (\|\tilde{m}_0 - \bar{m}\|_{B^{d/2-1, d/2}} + \|\tilde{n}_0 - \bar{n}\|_{B^{d/2-1, d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}}), \end{aligned}$$

where the constants  $a$  and  $b$  are defined by

$$(1.6) \quad \begin{aligned} a &= \frac{1}{\bar{m}^2} \left( a_0\bar{n} + \bar{m} + \frac{(\bar{m} - a_0\bar{n})(\bar{m} - a_0\bar{n} - k_0)}{\sqrt{(\bar{m} + a_0\bar{n} - k_0)^2 + 4k_0a_0\bar{n}}} \right) > 0, \\ b &= 1 + \frac{(\bar{m} + a_0\bar{n} + k_0)}{\sqrt{(\bar{m} + a_0\bar{n} - k_0)^2 + 4k_0a_0\bar{n}}} > 0. \end{aligned}$$

(ii) Uniqueness: Uniqueness holds in  $\mathcal{C}(\mathbb{R}^+; (B^{d/2-1, d/2})^{1+1} \times (B^{d/2})^d)$  if  $d \geq 3$ . If  $d = 2$ , one should also suppose that  $\tilde{m}_0 - \bar{m}$ ,  $\tilde{n}_0 - \bar{n} \in B^{\varepsilon, 1+\varepsilon}$  and  $\mathbf{u}_0 \in B^\varepsilon$  for  $\varepsilon \in (0, 1)$  to get uniqueness in  $\mathcal{C}(\mathbb{R}^+; (B^{0,1})^{1+1} \times (B^1)^d)$ .

For the general data bounded away from infinity and the vacuum, we have the following local well-posedness theory (refer to Theorem 3.1 for the corresponding statement in terms of new variables).

THEOREM 1.4 (local well-posedness for general data). Let  $d \geq 2$ ,  $\tilde{\mu} > 0$ ,  $2\tilde{\mu} + d\tilde{\lambda} \geq 0$ , with constants  $\bar{m} > 0$  and  $\bar{n} \geq 0$ . Assume that  $\tilde{m}_0^{-1} - \bar{m}^{-1} \in B^{d/2, d/2+1}$ ,  $\tilde{n}_0 - \bar{n} \in B^{d/2, d/2+1}$ , and  $\mathbf{u}_0 \in B^{d/2-1, d/2}$ . In addition,  $\sup_{x \in \mathbb{R}^d} \tilde{m}_0(x) < \infty$ . Then there exists a positive time  $T$  such that system (1.1) has a unique solution  $(\tilde{m}, \tilde{n}, \mathbf{u})$  on  $[0, T] \times \mathbb{R}^d$  and that  $(\tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n}, \mathbf{u})$  belongs to  $F_T^1$  and satisfies  $\sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \tilde{m}(t, x) < \infty$ .

We also have the following continuation criterion for the local existence of the solution (see also Proposition 3.6).

THEOREM 1.5 (continuation criterion). Under the hypotheses of Theorem 1.4, assume that system (1.1) has a solution  $(\tilde{m}, \tilde{n}, \mathbf{u})$  on  $[0, T] \times \mathbb{R}^d$  such that  $(\tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n}, \mathbf{u})$  belongs to  $F_{T'}^1$ , for all  $T' < T$  and satisfies

$$\tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n} \in L^\infty([0, T]; B^{d/2, d/2+1}),$$

$$\sup_{(t,x) \in [0,T) \times \mathbb{R}^d} \tilde{m}(t,x) < \infty, \quad \int_0^T \|\nabla \mathbf{u}\|_\infty dt < \infty.$$

Then there exists some  $T^* > T$  such that  $(\tilde{m}, \tilde{n}, \mathbf{u})$  may be continued on  $[0, T^*] \times \mathbb{R}^d$  to a solution of (1.1) such that  $(\tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n}, \mathbf{u})$  belongs to  $F_{T^*}^1$ .

*Remark 1.6.* The results of the present paper are independent of the special structure (1.3) of the nonlinear pressure term  $P$ . Indeed, the similar results of Theorems 1.3, 1.4, and 1.5 hold true as long as the term  $\nabla P/\tilde{m}$  can be decomposed into a linear term involving the modified mass and some nonlinear terms, similar to (2.1) in the next section.

The rest of this paper is organized as follows. In section 2, we investigate the global well-posedness of the Cauchy problem. We first reformulate the system through changing variables in order to obtain a priori estimates in subsection 2.1. In subsection 2.2, we are devoted to deriving a priori estimates for the transport equation and the linear coupled parabolic-hyperbolic system with convection terms. Subsection 2.3 involves the proof of the existence and uniqueness of the solution. In section 3, we prove the local well-posedness of the problem through some subsections similar to the global ones. An appendix is devoted to recalling some properties of the Littlewood–Paley decomposition and Besov spaces which we have used in this paper.

**2. Global well-posedness for small data.**

**2.1. Reformulation of the system.** Letting  $\bar{n} \geq 0$  and  $\bar{m} > (1 - \text{sgn}\bar{n})k_0$ , we introduce new variables  $n = a_0(\tilde{n}/\tilde{m} - \bar{n}/\bar{m})$  and  $m = a(\tilde{m} - \bar{m}) + bn$ , i.e.,  $\tilde{m} = \bar{m} + (m - bn)/a$ , in order to cancel the linear terms involving one modified mass from the momentum equation, where  $a$  and  $b$  are positive constants defined in (1.6). We also denote  $n_0 = a_0(\tilde{n}_0/\tilde{m}_0 - \bar{n}/\bar{m})$  and  $m_0 = a(\tilde{m}_0 - \bar{m}) + bn_0$  throughout the paper for the global well-posedness theory. Then we have

$$\frac{P}{C_0} = \left(1 + \frac{a_0\bar{n}}{\bar{m}} + n\right) \tilde{m} - k_0 + \sqrt{\left(\left(1 + \frac{a_0\bar{n}}{\bar{m}} + n\right) \tilde{m} - k_0\right)^2 + 4k_0 \left(n + \frac{a_0\bar{n}}{\bar{m}}\right) \tilde{m}}.$$

Taking the gradient of both sides, we get

$$(2.1) \quad \frac{\nabla P}{C_0\tilde{m}} = \nabla m + \mathbf{H}(m, n),$$

where the nonlinear term is

$$\begin{aligned} \mathbf{H}(m, n) := & \frac{\nabla m - b\nabla n}{a^2\bar{m}^2\tilde{m}} \left( - (a_0\bar{n} + \bar{m})m + (a\bar{m}^2 + b(a_0\bar{n} + \bar{m}))n \right) \\ & + (K(m, n) - K(0, 0)) \left\{ \left( n + \frac{a_0\bar{n}}{\bar{m}} + 1 \right) \tilde{m} \nabla n + \left( n + \frac{a_0\bar{n}}{\bar{m}} + 1 \right)^2 \nabla \tilde{m} \right. \\ & \quad \left. + k_0 \left( n + \frac{a_0\bar{n}}{\bar{m}} - 1 \right) \frac{\nabla m - b\nabla n}{a\tilde{m}} + k_0 \nabla n \right\} \\ & + K(0, 0) \left\{ \left( n + \frac{a_0\bar{n}}{\bar{m}} + 1 \right) \frac{\nabla n(m - bn)}{a} + \bar{m}n \nabla n \right. \\ & \quad \left. + \left[ n^2 + 2n \left( \frac{a_0\bar{n}}{\bar{m}} + 1 \right) \right] \frac{\nabla m - b\nabla n}{a} + \frac{k_0}{a\tilde{m}} n (\nabla m - b\nabla n) \right. \\ & \quad \left. - k_0 \left( \frac{a_0\bar{n}}{\bar{m}} - 1 \right) \frac{(m - bn)(\nabla m - b\nabla n)}{a^2\bar{m}\tilde{m}} \right\}. \end{aligned}$$

Here,

$$K(m, n) = \frac{1}{\sqrt{\left[ \left( \bar{m} + \frac{m-bn}{a} \right) \left( n + \frac{a_0 \bar{n}}{\bar{m}} + 1 \right) - k_0 \right]^2 + 4k_0 \left( n + \frac{a_0 \bar{n}}{\bar{m}} \right) \left( \bar{m} + \frac{m-bn}{a} \right)}}$$

and  $K(0, 0) = 1/\sqrt{(\bar{m} + a_0 \bar{n} - k_0)^2 + 4k_0 a_0 \bar{n}} > 0$ .

Therefore, with the new unknowns, we can rewrite the Cauchy problem of system (1.1) as follows:

$$(2.2) \quad \begin{cases} n_t + \mathbf{u} \cdot \nabla n = 0, \\ m_t + \mathbf{u} \cdot \nabla m + a\bar{m} \operatorname{div} \mathbf{u} = F(m, n, \mathbf{u}), \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + C_0 \nabla m = \mathbf{G}(m, n, \mathbf{u}), \\ (m, n, \mathbf{u})|_{t=0} = (m_0, n_0, \mathbf{u}_0), \end{cases}$$

where

$$\begin{aligned} \mu &= \tilde{\mu}/\bar{m}, \quad \lambda = \tilde{\lambda}/\bar{m}, \quad F(m, n, \mathbf{u}) = -(m - bn) \operatorname{div} \mathbf{u}, \\ \mathbf{G}(m, n, \mathbf{u}) &= -C_0 \mathbf{H}(m, n) - \frac{m - bn}{a\bar{m}} (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}). \end{aligned}$$

Note here that the first equation in (2.2) is a homogeneous transport equation; the estimates of  $n$  depend only on those of the velocity  $\mathbf{u}$ . The second and the third equations in (2.2) consist of a coupled parabolic-hyperbolic system involving the modified mass  $m$  and mixed velocity  $\mathbf{u}$ . Thus, with the help of the decomposition (2.1), the original system is decoupled into a transport equation for the modified gas flow and a coupled system for the motion of the modified liquid fluid.

We can get the following result for the reformulated system.

**THEOREM 2.1.** *Let  $d \geq 2$ ,  $\bar{n} \geq 0$ ,  $\bar{m} > (1 - \operatorname{sgn} \bar{n})k_0$ ,  $\mu > 0$ , and  $2\mu + d\lambda \geq 0$ ; in addition,  $\mu + \lambda > 0$  if  $d = 2$ . There exist two positive constants  $\eta$  and  $Q$  such that if  $m_0, n_0 \in B^{d/2-1, d/2}$  and  $\mathbf{u}_0 \in B^{d/2-1}$  satisfy*

$$(2.3) \quad \|m_0\|_{B^{d/2-1, d/2}} + \|n_0\|_{B^{d/2-1, d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}} \leq \eta,$$

then the following results hold:

(i) *Existence: The system (2.2) has a solution  $(m, n, \mathbf{u})$  in  $E^{d/2}$  which satisfies*

$$\|(m, n, \mathbf{u})\|_{E^{d/2}} \leq Q (\|m_0\|_{B^{d/2-1, d/2}} + \|n_0\|_{B^{d/2-1, d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}}).$$

It also belongs to the affine space

$$(m_L, n_0, \mathbf{u}_L) + (C^{1/2}(\mathbb{R}^+; B^{d/2-1}))^{1+1} \times (C^{1/8}(\mathbb{R}^+; B^{d/2-5/4}))^d,$$

where  $(m_L, \mathbf{u}_L)$  is the solution of the linear system

$$(2.4) \quad \begin{cases} \partial_t m_L + a\bar{m} \operatorname{div} \mathbf{u}_L = 0, \\ \partial_t \mathbf{u}_L - \mu \Delta \mathbf{u}_L - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_L + C_0 \nabla m_L = 0, \\ (m_L, \mathbf{u}_L)|_{t=0} = (m_0, \mathbf{u}_0). \end{cases}$$

(ii) *Uniqueness: Uniqueness holds in  $E^{d/2}$  if  $d \geq 3$ . If  $d = 2$ , one should also suppose that  $n_0, m_0 \in B^{\varepsilon, 1+\varepsilon}$  and  $\mathbf{u}_0 \in B^\varepsilon$  for  $\varepsilon \in (0, 1)$  to get uniqueness in  $E^1$ .*

With the help of Theorem 2.1, we can prove Theorem 1.3 as follows.

*Proof of Theorem 1.3.* From the conditions, we have  $n_0 \in B^{d/2-1, d/2}$ . In addition, from  $\tilde{m}_0 - \bar{m} \in B^{d/2-1, d/2}$ , we can derive  $m_0 = a(\tilde{m}_0 - \bar{m})/\bar{m} + bn_0 \in B^{d/2-1, d/2}$ . Since (1.5) implies (2.3), the conclusion of Theorem 2.1 follows for  $(m, n, \mathbf{u})$ . Changing back to the original variables  $(\tilde{m}, \tilde{n}, \mathbf{u})$ , it leads to Theorem 1.3. By Lemma A.6, it is easy to see that  $\tilde{m} - \bar{m}$  and  $\tilde{n} - \bar{n}$  also belong to  $\mathcal{C}(\mathbb{R}^+; B^{d/2-1} \cap B^{d/2})$ .  $\square$

**2.2. A priori estimates for the linear system with convection terms.** We first investigate some a priori estimates for the linear system with convection terms:

$$(2.5) \quad \begin{cases} n_t + \mathbf{v} \cdot \nabla n = 0, \\ m_t + \mathbf{v} \cdot \nabla m + a\bar{m}\operatorname{div}\mathbf{u} = F, \\ \mathbf{u}_t + \mathbf{v} \cdot \nabla \mathbf{u} - \mu\Delta\mathbf{u} - (\mu + \lambda)\nabla\operatorname{div}\mathbf{u} + C_0\nabla m = \mathbf{G}, \\ (m, n, \mathbf{u})|_{t=0} = (m_0, n_0, \mathbf{u}_0). \end{cases}$$

We do not need to separate the velocity into compressible and incompressible parts. In fact, we can prove the following proposition.

**PROPOSITION 2.2.** *Let  $a > 0$ ,  $\bar{m} > 0$ ,  $s \in (1 - d/2, d/2 + 1]$ , and  $s_1, s_2 \in (-d/2, d/2 + 1]$  be constants. Assume  $\mathbf{v} \in L^1([0, T]; B^{d/2+1})$  and denote  $V(t) = \int_0^t \|\mathbf{v}(\tau)\|_{B^{d/2+1}} d\tau$ . Let  $(m, n, \mathbf{u})$  be a solution of (2.5) on  $[0, T]$ ; then the following estimates hold:*

$$(2.6) \quad \|n\|_{\tilde{L}^\infty([0, T]; B^{s_1, s_2})} \leq e^{CV(T)} \|n_0\|_{B^{s_1, s_2}}$$

and

$$(2.7) \quad \begin{aligned} & \|m\|_{\tilde{L}^\infty([0, T]; B^{s-1, s})} + \|\mathbf{u}\|_{\tilde{L}^\infty([0, T]; B^{s-1})} \\ & + \|m\|_{L^1([0, T]; B^{s+1, s})} + \|\mathbf{u}\|_{L^1([0, T]; B^{s+1})} \\ & \lesssim e^{CV(T)} (\|m_0\|_{B^{s-1, s}} + \|\mathbf{u}_0\|_{B^{s-1}} + \|F\|_{L^1([0, T]; B^{s-1, s})} + \|\mathbf{G}\|_{L^1([0, T]; B^{s-1})}). \end{aligned}$$

*Proof. Step 1: Estimates for the homogeneous transport equation.* We derive the estimates for the first equation of (2.5) in Besov spaces.

Applying the Littlewood–Paley operator  $\Delta_k$  to (2.5)<sub>1</sub> yields

$$(2.8) \quad \begin{cases} \partial_t \Delta_k n + \Delta_k(\mathbf{v} \cdot \nabla n) = 0, \\ \Delta_k n|_{t=0} = \Delta_k n_0. \end{cases}$$

Taking the inner product of (2.8) with  $\Delta_k n$ , we get for any  $s_1, s_2 \in (-d/2, 1 + d/2]$

$$\frac{1}{2} \frac{d}{dt} \|\Delta_k n\|_2^2 = -(\Delta_k(\mathbf{v} \cdot \nabla n), \Delta_k n) \lesssim \gamma_k 2^{-k\varphi^{s_1, s_2}(k)} \|\mathbf{v}\|_{B^{d/2+1}} \|n\|_{B^{s_1, s_2}} \|\Delta_k n\|_2.$$

It follows that

$$\sum_k 2^{k\varphi^{s_1, s_2}(k)} \|\Delta_k n\|_2 \leq \|n_0\|_{B^{s_1, s_2}} + C \int_0^t \|\mathbf{v}\|_{B^{d/2+1}} \|n\|_{B^{s_1, s_2}},$$

which implies the desired estimate (2.6) with the help of the Gronwall inequality.

*Step 2: Estimates for  $(m, \mathbf{u})$ .* Applying the Littlewood–Paley operator  $\Delta_k$  to (2.5)<sub>2</sub> and (2.5)<sub>3</sub>, we have

$$(2.9) \quad \begin{cases} \partial_t \Delta_k m + \Delta_k(\mathbf{v} \cdot \nabla m) + a\bar{m}\operatorname{div}\Delta_k \mathbf{u} = \Delta_k F, \\ \partial_t \Delta_k \mathbf{u} + \Delta_k(\mathbf{v} \cdot \nabla \mathbf{u}) - \mu\Delta\Delta_k \mathbf{u} - (\mu + \lambda)\nabla\operatorname{div}\Delta_k \mathbf{u} + C_0\nabla\Delta_k m = \Delta_k \mathbf{G}. \end{cases}$$

Taking the inner product of (2.9)<sub>1</sub> with  $\Delta_k m$  and  $-\Delta \Delta_k m$ , and (2.9)<sub>2</sub> with  $\Delta_k \mathbf{u}$ , we obtain

$$(2.10) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_k m\|_2^2 + (\Delta_k(\mathbf{v} \cdot \nabla m), \Delta_k m) + a\bar{m}(\operatorname{div} \Delta_k \mathbf{u}, \Delta_k m) = (\Delta_k F, \Delta_k m),$$

$$(2.11) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \Delta_k m\|_2^2 + (\Delta_k(\mathbf{v} \cdot \nabla m), -\Delta \Delta_k m) - a\bar{m}(\operatorname{div} \Delta_k \mathbf{u}, \Delta \Delta_k m) \\ = -(\Delta_k F, \Delta \Delta_k m),$$

$$(2.12) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_k \mathbf{u}\|_2^2 + (\Delta_k(\mathbf{v} \cdot \nabla \mathbf{u}), \Delta_k \mathbf{u}) + \mu \|\nabla \Delta_k \mathbf{u}\|_2^2 + (\mu + \lambda) \|\operatorname{div} \Delta_k \mathbf{u}\|_2^2 \\ + C_0(\nabla \Delta_k m, \Delta_k \mathbf{u}) = (\Delta_k \mathbf{G}, \Delta_k \mathbf{u}).$$

For the intersected term, we have

$$(2.13) \quad \frac{d}{dt} (\Delta_k \mathbf{u}, \nabla \Delta_k m) + (\Delta_k(\mathbf{v} \cdot \nabla \mathbf{u}), \nabla \Delta_k m) - (\Delta_k(\mathbf{v} \cdot \nabla m), \operatorname{div} \Delta_k \mathbf{u}) \\ - a\bar{m} \|\operatorname{div} \Delta_k \mathbf{u}\|_2^2 + C_0 \|\nabla \Delta_k m\|_2^2 + (2\mu + \lambda)(\operatorname{div} \Delta_k \mathbf{u}, \Delta \Delta_k m) \\ = -(\Delta_k F, \operatorname{div} \Delta_k \mathbf{u}) + (\Delta_k \mathbf{G}, \nabla \Delta_k m).$$

Let

$$(2.14) \quad \alpha_k^2 := \frac{C_0}{a\bar{m}} \|\Delta_k m\|_2^2 + \|\Delta_k \mathbf{u}\|_2^2 + \frac{(2\mu + \lambda)A}{a\bar{m}} \|\nabla \Delta_k m\|_2^2 + 2A(\Delta_k \mathbf{u}, \nabla \Delta_k m).$$

For  $A = (\mu + \lambda)/(2a\bar{m}) > 0$ , there exist two positive constants  $c_1$  and  $c_2$  such that

$$(2.15) \quad c_1 \alpha_k^2 \leq \|\Delta_k m\|_2^2 + \|\Delta_k \mathbf{u}\|_2^2 + \|\nabla \Delta_k m\|_2^2 \leq c_2 \alpha_k^2,$$

since we have, for  $M \in (a\bar{m}/(2\mu + \lambda), 2a\bar{m}/(\mu + \lambda))$ , that

$$|2(\Delta_k \mathbf{u}, \nabla \Delta_k m)| \leq M \|\Delta_k \mathbf{u}\|_2^2 + \|\nabla \Delta_k m\|_2^2 / M.$$

Combining (2.10)–(2.13), with the help of Lemma A.7, yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \alpha_k^2 + \mu \|\nabla \Delta_k \mathbf{u}\|_2^2 + (\mu + \lambda - a\bar{m}A) \|\operatorname{div} \Delta_k \mathbf{u}\|_2^2 + C_0 A \|\nabla \Delta_k m\|_2^2 \\ &= \frac{C_0}{a\bar{m}} (\Delta_k F, \Delta_k m) + (\Delta_k \mathbf{G}, \Delta_k \mathbf{u}) - \frac{(2\mu + \lambda)A}{a\bar{m}} (\Delta_k F, \Delta \Delta_k m) - A(\Delta_k F, \operatorname{div} \Delta_k \mathbf{u}) \\ & \quad + A(\Delta_k \mathbf{G}, \nabla \Delta_k m) - \frac{C_0}{a\bar{m}} (\Delta_k(\mathbf{v} \cdot \nabla m), \Delta_k m) - (\Delta_k(\mathbf{v} \cdot \nabla \mathbf{u}), \Delta_k \mathbf{u}) \\ & \quad + \frac{(2\mu + \lambda)A}{a\bar{m}} (\Delta_k(\mathbf{v} \cdot \nabla m), \Delta \Delta_k m) + A(\Delta_k(\mathbf{v} \cdot \nabla \mathbf{u}), \nabla \Delta_k m) \\ & \quad + A(\nabla \Delta_k(\mathbf{v} \cdot \nabla m), \Delta_k \mathbf{u}) \\ & \lesssim (\|\Delta_k F\|_2 + \|\nabla \Delta_k F\|_2 + \|\Delta_k \mathbf{G}\|_2) (\|\Delta_k m\|_2 + \|\Delta_k \mathbf{u}\|_2 + \|\nabla \Delta_k m\|_2) \\ & \quad + \gamma_k 2^{-k\varphi^{s-1, s}(k)} \|\mathbf{v}\|_{B^{d/2+1}} \|m\|_{B^{s-1, s}} \|\Delta_k m\|_2 \\ & \quad + \gamma_k 2^{-k(s-1)} \|\mathbf{v}\|_{B^{d/2+1}} \|\mathbf{u}\|_{B^{s-1}} \|\Delta_k \mathbf{u}\|_2 \\ & \quad + \gamma_k 2^{-k(\varphi^{s-1, s}(k)-1)} \|\mathbf{v}\|_{B^{d/2+1}} \|m\|_{B^{s-1, s}} \|\nabla \Delta_k m\|_2 \\ & \quad + \gamma_k \|\mathbf{v}\|_{B^{d/2+1}} \left( 2^{-k(s-1)} \|\nabla \Delta_k m\|_2 \|\mathbf{u}\|_{B^{s-1}} + 2^{-k(\varphi^{s-1, s}(k)-1)} \|m\|_{B^{s-1, s}} \|\Delta_k \mathbf{u}\|_2 \right) \\ & \lesssim \left( \|\Delta_k F\|_2 + \|\nabla \Delta_k F\|_2 + \|\Delta_k \mathbf{G}\|_2 + \gamma_k 2^{-k(s-1)} \|\mathbf{v}\|_{B^{d/2+1}} (\|m\|_{B^{s-1, s}} + \|\mathbf{u}\|_{B^{s-1}}) \right) \\ & \quad \times (\|\Delta_k m\|_2 + \|\Delta_k \mathbf{u}\|_2 + \|\nabla \Delta_k m\|_2). \end{aligned}$$



Thus, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \alpha_k^2 + c_0 \min(2^{2k}, 1) \alpha_k^2 \\ & \lesssim \gamma_k 2^{-k(s-1)} \left[ \|F\|_{B^{s-1,s}} + \|\mathbf{G}\|_{B^{s-1}} + \|\mathbf{v}\|_{B^{d/2+1}} \left( \|m\|_{B^{s-1,s}} + \|\mathbf{u}\|_{B^{s-1}} \right) \right] \alpha_k, \end{aligned}$$

which implies

$$\begin{aligned} & 2^{k(s-1)} \alpha_k + c_0 \int_0^t \min(2^{2k}, 1) 2^{k(s-1)} \alpha_k(\tau) d\tau \\ & \leq 2^{k(s-1)} \alpha_k(0) + C \gamma_k \int_0^t \left[ \|F\|_{B^{s-1,s}} + \|\mathbf{G}\|_{B^{s-1}} + \|\mathbf{v}\|_{B^{d/2+1}} \sum_k 2^{k(s-1)} \alpha_k \right]. \end{aligned}$$

Thus, by the Gronwall inequality, we have

$$\begin{aligned} (2.16) \quad & \|m\|_{\tilde{L}^\infty([0,T];B^{s-1,s})} + \|\mathbf{u}\|_{\tilde{L}^\infty([0,T];B^{s-1})} \\ & + \|m\|_{L^1([0,T];B^{s+1,s})} + \|\mathbf{u}\|_{L^1([0,T];B^{s+1,s-1})} \\ & \lesssim e^{CV(T)} \left( \|m_0\|_{B^{s-1,s}} + \|\mathbf{u}_0\|_{B^{s-1}} + \|F\|_{L^1([0,T];B^{s-1,s})} + \|\mathbf{G}\|_{L^1([0,T];B^{s-1})} \right). \end{aligned}$$

*Step 3: The smoothing effect for u.* By (2.12), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_k \mathbf{u}\|_2^2 + C 2^{2k} \|\Delta_k \mathbf{u}\|_2^2 \\ & \lesssim \|\Delta_k \mathbf{u}\|_2 (2^k \|\Delta_k m\|_2 + \|\Delta_k \mathbf{G}\|_2 + \gamma_k 2^{-k(s-1)} \|\mathbf{v}\|_{B^{d/2+1}} \|\mathbf{u}\|_{B^{s-1}}). \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{d}{dt} \sum_{k \geq 0} 2^{k(s-1)} \|\Delta_k \mathbf{u}\|_2 + C \sum_{k \geq 0} 2^{k(s+1)} \|\Delta_k \mathbf{u}\|_2 \\ & \lesssim \sum_{k \geq 0} 2^{k(s-1)} \left[ 2^k \|\Delta_k m\|_2 + \|\Delta_k \mathbf{G}\|_2 + \gamma_k 2^{-k(s-1)} \|\mathbf{u}\|_{B^{d/2+1}} \|\mathbf{u}\|_{B^{s-1}} \right] \\ & \lesssim \sum_{k \geq 0} 2^{ks} \|\Delta_k m\|_2 + \|\mathbf{G}\|_{B^{s-1}} + \|\mathbf{v}\|_{B^{d/2+1}} \|\mathbf{u}\|_{B^{s-1}}, \end{aligned}$$

which implies, with the help of (2.16), that

$$\begin{aligned} & \int_0^t \sum_{k \geq 0} 2^{k(s+1)} \|\Delta_k \mathbf{u}(\tau)\|_2 \\ & \lesssim \|\mathbf{u}_0\|_{B^{s-1}} + \int_0^t \sum_{k \geq 0} 2^{ks} \|\Delta_k m(\tau)\|_2 d\tau + \int_0^t \|\mathbf{G}(\tau)\|_{B^{s-1}} d\tau \\ & \quad + \sup_{\tau \in [0,t]} \|\mathbf{u}(\tau)\|_{B^{s-1}} \int_0^t \|\mathbf{v}(\tau)\|_{B^{d/2+1}} d\tau \\ & \lesssim e^{CV(t)} \left( \|m_0\|_{B^{s-1,s}} + \|\mathbf{u}_0\|_{B^{s-1}} \int_0^t [\|F(\tau)\|_{B^{s-1,s}} + \|\mathbf{G}(\tau)\|_{B^{s-1}}] d\tau \right). \end{aligned}$$

Combining this with (2.16), we get (2.7). □

From the proof of Proposition 2.2, we immediately have the following.

COROLLARY 2.3. *If a bounded operator  $\mathcal{B}$  acts on the convection terms in (2.5), then the same estimates hold for the refined system*

$$(2.17) \quad \begin{cases} n_t + \mathcal{B}(\mathbf{v} \cdot \nabla n) = 0, \\ m_t + \mathcal{B}(\mathbf{v} \cdot \nabla m) + a\bar{m}\operatorname{div}\mathbf{u} = F, \\ \mathbf{u}_t + \mathcal{B}(\mathbf{v} \cdot \nabla \mathbf{u}) - \mu\Delta\mathbf{u} - (\mu + \lambda)\nabla\operatorname{div}\mathbf{u} + C_0\nabla m = \mathbf{G}. \end{cases}$$

**2.3. Global existence and uniqueness of the solution.**

*Step 1: Friedrich’s approximation.* Let  $L^2_\ell$  be the set of  $L^2$  functions spectrally supported in the annulus  $\mathcal{C}_\ell := \{\xi \in \mathbb{R}^d : 1/\ell \leq |\xi| \leq \ell\}$  endowed with the standard  $L^2$  topology. In order to construct the classical Friedrichs approximation, we first define the Friedrichs projectors  $(\mathbb{F}_\ell)_{\ell \in \mathbb{N}}$  by

$$\mathbb{F}_\ell f := \mathcal{F}^{-1}\mathbf{1}_{\mathcal{C}_\ell}(\xi)\mathcal{F}f$$

for any  $f \in L^2(\mathbb{R}^d)$ , where  $\mathbf{1}_{\mathcal{C}_\ell}(\xi)$  denotes the characteristic function on the annulus  $\mathcal{C}_\ell$ . Then we can define the following approximate system:

$$(2.18) \quad \begin{cases} n_t^\ell + \mathbb{F}_\ell(\mathbb{F}_\ell \mathbf{u}^\ell \cdot \nabla \mathbb{F}_\ell n^\ell) = 0, \\ m_t^\ell + \mathbb{F}_\ell(\mathbb{F}_\ell \mathbf{u}^\ell \cdot \nabla \mathbb{F}_\ell m^\ell) + a\bar{m}\operatorname{div}\mathbb{F}_\ell \mathbf{u}^\ell = F^\ell, \\ \mathbf{u}_t^\ell + \mathbb{F}_\ell(\mathbb{F}_\ell \mathbf{u}^\ell \cdot \nabla \mathbb{F}_\ell \mathbf{u}^\ell) - \mu\Delta\mathbb{F}_\ell \mathbf{u}^\ell - (\mu + \lambda)\nabla\operatorname{div}\mathbb{F}_\ell \mathbf{u}^\ell + C_0\nabla\mathbb{F}_\ell m^\ell = \mathbf{G}^\ell, \\ (m^\ell, n^\ell, \mathbf{u}^\ell)|_{t=0} = (m_\ell, n_\ell, \mathbf{u}_\ell), \end{cases}$$

where

$$m_\ell = \mathbb{F}_\ell m_0, \quad n_\ell = \mathbb{F}_\ell n_0, \quad \mathbf{u}_\ell = \mathbb{F}_\ell \mathbf{u}_0, \\ F^\ell = \mathbb{F}_\ell F(\mathbb{F}_\ell m^\ell, \mathbb{F}_\ell n^\ell, \mathbb{F}_\ell \mathbf{u}^\ell), \quad \mathbf{G}^\ell = \mathbb{F}_\ell \mathbf{G}(\mathbb{F}_\ell m^\ell, \mathbb{F}_\ell n^\ell, \mathbb{F}_\ell \mathbf{u}^\ell).$$

It is easy to check that it is an ordinary differential equation in  $L^2_\ell \times L^2_\ell \times (L^2_\ell)^d$  for every  $\ell \in \mathbb{N}$ . By the usual Cauchy–Lipschitz theorem, there is a strictly positive maximal time  $T_\ell^*$  such that a unique solution  $(m^\ell, n^\ell, \mathbf{u}^\ell)$  exists in  $[0, T_\ell^*]$  which is continuous in time with value in  $L^2_\ell \times L^2_\ell \times (L^2_\ell)^d$ , i.e.,  $(m^\ell, n^\ell, \mathbf{u}^\ell) \in \mathcal{C}([0, T_\ell^*]; L^2_\ell \times L^2_\ell \times (L^2_\ell)^d)$ . As  $\mathbb{F}_\ell^2 = \mathbb{F}_\ell$ , we see that  $\mathbb{F}_\ell(m^\ell, n^\ell, \mathbf{u}^\ell)$  is also a solution, so the uniqueness implies that  $\mathbb{F}_\ell(m^\ell, n^\ell, \mathbf{u}^\ell) = (m^\ell, n^\ell, \mathbf{u}^\ell)$ . Thus, this system can be rewritten as the following system:

$$(2.19) \quad \begin{cases} n_t^\ell + \mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla n^\ell) = 0, \\ m_t^\ell + \mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla m^\ell) + a\bar{m}\operatorname{div}\mathbf{u}^\ell = F_1^\ell, \\ \mathbf{u}_t^\ell + \mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla \mathbf{u}^\ell) - \mu\Delta\mathbf{u}^\ell - (\mu + \lambda)\nabla\operatorname{div}\mathbf{u}^\ell + C_0\nabla m^\ell = \mathbf{G}_1^\ell, \\ (m^\ell, n^\ell, \mathbf{u}^\ell)|_{t=0} = (m_\ell, n_\ell, \mathbf{u}_\ell), \end{cases}$$

where

$$F_1^\ell = \mathbb{F}_\ell F(m^\ell, n^\ell, \mathbf{u}^\ell) \quad \text{and} \quad \mathbf{G}_1^\ell = \mathbb{F}_\ell \mathbf{G}(m^\ell, n^\ell, \mathbf{u}^\ell).$$

*Step 2: Uniform estimates.* Denote

$$E_0 = \|m_0\|_{B^{d/2-1, d/2}} + \|n_0\|_{B^{d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}}$$

and

$$T_\ell := \sup\{T \in [0, T_\ell^*) : \|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}} \leq A\bar{C}E_0\},$$

where  $\bar{C}$  corresponds to the constant in Proposition 2.2 and  $A > \max(2, 1/\bar{C})$  is a constant. Thus, by the continuity, we have  $T_\ell > 0$ .

Let  $M_0$  be the continuity modulus of the embedding relation  $B^{d/2}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ . We make the assumption

$$2(1 + b)A\bar{C}M_0E_0 \leq a\bar{m}.$$

Then it implies

$$\begin{aligned} \|m^\ell\|_{L^\infty([0, T] \times \mathbb{R}^d)} &\leq M_0 \|m^\ell\|_{L^\infty([0, T]; B^{d/2})} \leq M_0 \|m^\ell\|_{L^\infty([0, T]; B^{d/2-1, d/2})} \\ &\leq A\bar{C}M_0E_0 \leq \frac{a\bar{m}}{2(1 + b)}. \end{aligned}$$

Similarly, we have

$$\|n^\ell\|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq \frac{a\bar{m}}{2(1 + b)}.$$

Then

$$\tilde{m}(m^\ell, n^\ell) = \bar{m} + \frac{m - bn}{a} \in \left[ \frac{\bar{m}}{2}, \frac{3\bar{m}}{2} \right].$$

By Proposition 2.2, we have

$$\begin{aligned} &\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{S_T} \\ &\lesssim e^{C\|\mathbf{u}^\ell\|_{L^1([0, T]; B^{d/2+1})}} \left( E_0 + \|F_1^\ell\|_{L^1([0, T]; B^{d/2-1, d/2})} + \|\mathbf{G}_1^\ell\|_{L^1([0, T]; B^{d/2-1})} \right). \end{aligned}$$

From Lemmas A.3 and A.6, we get

$$\begin{aligned} &\|F_1^\ell\|_{L^1([0, T]; B^{d/2-1, d/2})} \\ &\lesssim \|m^\ell \operatorname{div} \mathbf{u}^\ell\|_{L^1([0, T]; B^{d/2-1, d/2})} + \|n^\ell \operatorname{div} \mathbf{u}^\ell\|_{L^1([0, T]; B^{d/2-1, d/2})} \\ &\lesssim (\|m^\ell\|_{L^\infty([0, T]; B^{d/2-1, d/2})} + \|n^\ell\|_{L^\infty([0, T]; B^{d/2-1, d/2})}) \|\operatorname{div} \mathbf{u}^\ell\|_{L^1([0, T]; B^{d/2})} \\ &\lesssim (\|m^\ell\|_{L^\infty([0, T]; B^{d/2-1, d/2})} + \|n^\ell\|_{L^\infty([0, T]; B^{d/2-1, d/2})}) \|\mathbf{u}^\ell\|_{L^1([0, T]; B^{d/2+1})} \\ &\lesssim \|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}}^2. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\| \frac{m^\ell - bn^\ell}{a\tilde{m}(m^\ell, n^\ell)} (\mu \Delta \mathbf{u}^\ell + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}^\ell) \right\|_{L^1([0, T]; B^{d/2-1})} \\ &\lesssim (\|n^\ell\|_{L^\infty([0, T]; B^{d/2})} + \|m^\ell\|_{L^\infty([0, T]; B^{d/2})}) \|\mathbf{u}^\ell\|_{L^1([0, T]; B^{d/2+1})} \\ &\lesssim \|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{S_T}^2. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} \|\mathbf{H}(m^\ell, n^\ell)\|_{L^1([0, T]; B^{d/2-1})} &\lesssim (\|n^\ell\|_{L^\infty([0, T]; B^{d/2})} + \|m^\ell\|_{L^\infty([0, T]; B^{d/2})})^2 \\ &\lesssim \|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}}^2. \end{aligned}$$

Hence

$$\begin{aligned} \|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}} &\leq \bar{C} e^{\bar{C}\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}}} \left( E_0 + C\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}}^2 \right) \\ &\leq \bar{C} e^{\bar{C}^2 A E_0} (1 + C A^2 \bar{C}^2 E_0) E_0. \end{aligned}$$

Thus, we can choose  $E_0$  so small that

(2.20)

$$1 + C A^2 \bar{C}^2 E_0 \leq \frac{A^2}{A+2}, \quad e^{\bar{C}^2 A E_0} \leq \frac{A+1}{A}, \quad \text{and} \quad 2(1+b)A\bar{C}M_0E_0 \leq a\bar{m},$$

which yields  $\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}} \leq \frac{A+1}{A+2} A\bar{C}E_0$  for any  $T < T_\ell$ .

We claim that  $T_\ell = T_\ell^*$ . Indeed, if  $T_\ell < T_\ell^*$ , we have seen that  $\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}} \leq \frac{A+1}{A+2} A\bar{C}E_0$ . So by continuity, for a sufficiently small constant  $s > 0$ , we can obtain  $\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_{(T+s)^{d/2}}} \leq A\bar{C}E_0$ , which contradicts the definition of  $T_\ell$ .

Now, we show that the approximate solution is a global one; i.e.,  $T_\ell^* = \infty$ . We assume  $T_\ell^* < \infty$ ; then we have shown  $\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{E_T^{d/2}} \leq A\bar{C}E_0$ . As

$$m^\ell \in L^\infty([0, T_\ell^*]; B^{d/2-1, d/2}), \quad n^\ell \in L^\infty([0, T_\ell^*]; B^{d/2}), \quad \text{and} \quad \mathbf{u}^\ell \in L^\infty([0, T_\ell^*]; B^{d/2-1}),$$

it implies that

$$\|(m^\ell, n^\ell, \mathbf{u}^\ell)\|_{L^\infty([0, T_\ell^*]; L^2)} < \infty.$$

Thus, we may extend the solution continuously beyond the time  $T_\ell^*$  by the Cauchy–Lipschitz theorem. This contradicts the definition of  $T_\ell^*$ . Therefore, the solution  $(m^\ell, n^\ell, \mathbf{u}^\ell)_{\ell \in \mathbb{N}}$  exists global in time.

*Step 3: Time derivatives.* For convenience, we split the approximate solution  $(m^\ell, n^\ell, \mathbf{u}^\ell)$  into a solution of the linear system with initial data  $(m_\ell, n_\ell, \mathbf{u}_\ell)$  and the discrepancy to that linear solution. More precisely, we denote by  $(m_L^\ell, n_L^\ell, \mathbf{u}_L^\ell)$  the solution to the linear system

$$(2.21) \quad \begin{cases} m_t^\ell + a\bar{m}\operatorname{div}\mathbf{u}^\ell = 0, \\ \mathbf{u}_t^\ell - \mu\Delta\mathbf{u}^\ell - (\mu + \lambda)\nabla\operatorname{div}\mathbf{u}^\ell + C_0\nabla m^\ell = 0, \\ (m^\ell, \mathbf{u}^\ell)|_{t=0} = (m_\ell, \mathbf{u}_\ell), \end{cases}$$

and  $(m_D^\ell, n_D^\ell, \mathbf{u}_D^\ell) = (m^\ell - m_L^\ell, n^\ell - n_L^\ell, \mathbf{u}^\ell - \mathbf{u}_L^\ell)$ .

It is clear that the definition of  $(m_\ell, n_\ell, \mathbf{u}_\ell)$  implies

$$m_\ell \rightarrow m_0 \text{ in } B^{d/2-1, d/2}, \quad n_\ell \rightarrow n_0 \text{ in } B^{d/2-1, d/2}, \quad \mathbf{u}_\ell \rightarrow \mathbf{u}_0 \text{ in } B^{d/2-1}.$$

From Corollary 2.3, we have

$$(m_L^\ell, n_L^\ell, \mathbf{u}_L^\ell) \rightarrow (m_L, n_0, \mathbf{u}_L) \text{ in } E^{d/2},$$

where  $m_L$  and  $\mathbf{u}_L$  satisfy the linear system (2.4).

Now, we derive the uniform boundedness of the time derivatives of the discrepancy  $(m_D^\ell, n_D^\ell, \mathbf{u}_D^\ell)$ .

LEMMA 2.4.  $((m_D^\ell, n_D^\ell, \mathbf{u}_D^\ell))_{\ell \in \mathbb{N}}$  is uniformly bounded in

$$(C^{1/2}(\mathbb{R}^+; B^{d/2-1}))^{1+1} \times (C^{1/8}(\mathbb{R}^+; B^{d/2-5/4}))^d.$$

*Proof.* Since

$$\partial_t n_D^\ell = -\mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla n^\ell),$$

we have  $\partial_t n_D^\ell \in L^2(\mathbb{R}^+; B^{d/2-1})$  because  $n^\ell \in L^\infty(\mathbb{R}^+; B^{d/2})$  and  $\mathbf{u}^\ell \in L^2(\mathbb{R}^+; B^{d/2})$  with the help of the interpolation theorem.

From the equation

$$\partial_t m_D^\ell = -\mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla m^\ell) - a\bar{m}\operatorname{div}\mathbf{u}^\ell + a\bar{m}\operatorname{div}\mathbf{u}_L^\ell - \mathbb{F}_\ell((m^\ell - bn^\ell)\operatorname{div}\mathbf{u}^\ell),$$

it follows that  $\partial_t m_D^\ell \in L^2(\mathbb{R}^+; B^{d/2-1})$ .

Recalling that

$$\begin{aligned} \partial_t \mathbf{u}_D^\ell &= -\mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla \mathbf{u}^\ell) + \mu\Delta \mathbf{u}^\ell + \mu\Delta \mathbf{u}_L^\ell + (\mu + \lambda)\nabla \operatorname{div}\mathbf{u}^\ell - (\mu + \lambda)\nabla \operatorname{div}\mathbf{u}_L^\ell \\ &\quad - C_0\nabla m^\ell - C_0\nabla m_L^\ell + \mathbf{G}_1^\ell, \end{aligned}$$

we can obtain  $\partial_t \mathbf{u}_D^\ell \in (L^\infty + L^{8/3} + L^{8/7})(\mathbb{R}^+; B^{d/2-5/4})$  through straightforward but tedious computations with the help of Lemmas A.3, A.6, and A.9.

Applying the Morrey embedding relation  $W^{1,p}(\mathbb{R}) \subset C^{1-1/p}(\mathbb{R})$  to the time variable for  $1 < p \leq \infty$ , we obtain the desired result.  $\square$

*Step 4: Compactness and convergence.* The proof of the existence of a solution is now standard. Indeed, we can use the Arzelà–Ascoli theorem to get strong convergence of the approximate solutions. We need to localize the spatial space in order to utilize some compactness results of local Besov spaces (see [1, Chapter 2]). Let  $(\chi_p)_{p \in \mathbb{N}}$  be a sequence of  $\mathcal{D}(\mathbb{R}^d)$  cut-off functions supported in the ball  $B(0, p + 1)$  of  $\mathbb{R}^d$  and equal to 1 in a neighborhood of  $B(0, p)$ . In view of Lemma 2.4 and uniform estimates obtained in Step 2, we see that  $((\chi_p m_D^\ell, \chi_p n_D^\ell, \chi_p \mathbf{u}_D^\ell))_{\ell \in \mathbb{N}}$  is bounded in  $E^{d/2}$  and uniformly equicontinuous in

$$\mathcal{C}([0, T]; (B^{d/2-1})^{1+1} \times (B^{d/2-5/4})^d)$$

for any  $p \in \mathbb{N}$  and  $T > 0$ . Moreover, the mapping  $f \mapsto \chi_p f$  is compact from  $B^{d/2-1, d/2}$  into  $B^{d/2-1}$  and from  $B^{d/2-1}$  into  $B^{d/2-5/4}$ .

Applying the Arzelà–Ascoli theorem to the family  $((\chi_p m_D^\ell, \chi_p n_D^\ell, \chi_p \mathbf{u}_D^\ell))_{\ell \in \mathbb{N}}$  on the time interval  $[0, p]$ , we then use the Cantor diagonal process. This finally provides us with a distribution  $(m_D, n_D, \mathbf{u}_D)$  continuous in time with values in  $(B^{d/2-1})^{1+1} \times (B^{d/2-5/4})^d$  and a subsequence (which we still denote by the same notation) such that we have for all  $p \in \mathbb{N}$

$$(\chi_p m_D^\ell, \chi_p n_D^\ell, \chi_p \mathbf{u}_D^\ell) \rightarrow (\chi_p m_D, \chi_p n_D, \chi_p \mathbf{u}_D) \text{ as } \ell \rightarrow \infty$$

in  $\mathcal{C}([0, p]; (B^{d/2-1})^{1+1} \times (B^{d/2-5/4})^d)$ . This obviously implies that  $(m_D^\ell, n_D^\ell, \mathbf{u}_D^\ell)$  tends to  $(m_D, n_D, \mathbf{u}_D)$  in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ .

Coming back to the uniform estimates and Lemma 2.4, we further obtain that  $(m_D, n_D, \mathbf{u}_D)$  belongs to  $E^{d/2}$  and to  $(C^{1/2}(\mathbb{R}^+; B^{d/2-1}))^{1+1} \times (C^{1/8}(\mathbb{R}^+; B^{d/2-5/4}))^d$ . The convergence results stemming from this last result and the interpolation argument enable us to pass to the limit in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$  in system (2.18) and to prove that

$(m, n, \mathbf{u}) := (m_L, n_L, \mathbf{u}_L) + (m_D, n_D, \mathbf{u}_D)$  is indeed a solution of (2.2) with the initial data. Since this is just a matter of doing tedious verifications, we omit the details.

*Step 5: Continuities in time.* The continuity of  $\mathbf{u}$  is straightforward. Indeed, from the third equation of (2.2), we have  $\mathbf{u}_t \in (L^1 + L^2)(\mathbb{R}^+; (B^{d/2-1})^d)$ , which implies  $\mathbf{u} \in \mathcal{C}(\mathbb{R}^+; (B^{d/2-1})^d)$  in view of the Morrey embedding and the embedding relation  $W^{1,1}(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$ . Consequently, the continuity of  $n$  in time is obtained from (2.6). For  $m$ , it is easy to see that  $m_t \in L^2(\mathbb{R}^+; B^{d/2-1}) \cap L^1(\mathbb{R}^+; B^{d/2})$  from the second equation of (2.2), which yields  $m \in \mathcal{C}(\mathbb{R}^+; B^{d/2-1, d/2})$  by the embeddings mentioned above.

*Step 6: Uniqueness.* Next, we prove the uniqueness of solutions. Let  $(m_1, n_1, \mathbf{u}_1)$  and  $(m_2, n_2, \mathbf{u}_2)$  be two solutions of (2.2) in  $E_T^{d/2}$  with the same initial data. Denote  $(\delta m, \delta n, \delta \mathbf{u}) = (m_2 - m_1, n_2 - n_1, \mathbf{u}_2 - \mathbf{u}_1)$ . Then they satisfy

$$(2.22) \quad \begin{cases} \partial_t \delta n + \mathbf{u}_2 \cdot \nabla \delta n = -\delta \mathbf{u} \cdot \nabla n_1, \\ \partial_t \delta m + \mathbf{u}_2 \cdot \nabla \delta m + a \bar{m} \operatorname{div} \delta \mathbf{u} = -\delta \mathbf{u} \cdot \nabla m_1 + \delta F, \\ \partial_t \delta \mathbf{u} + \mathbf{u}_2 \cdot \nabla \delta \mathbf{u} - \mu \Delta \delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \delta \mathbf{u} + C_0 \nabla \delta m = -\delta \mathbf{u} \cdot \nabla \mathbf{u}_1 + \delta \mathbf{G}, \\ (\delta m, \delta n, \delta \mathbf{u})|_{t=0} = (0, 0, \mathbf{0}), \end{cases}$$

where  $\delta F = F(m_2, n_2, \mathbf{u}_2) - F(m_1, n_1, \mathbf{u}_1)$  and  $\delta \mathbf{G} = \mathbf{G}(m_2, n_2, \mathbf{u}_2) - \mathbf{G}(m_1, n_1, \mathbf{u}_1)$ .

We first consider the case  $d \geq 3$ . Similar to the derivation of (2.6), we can get for  $t \in [0, T]$  with the help of (2.6)

$$(2.23) \quad \begin{aligned} \|\delta n\|_{\tilde{L}^\infty([0, T]; B^{d/2-2, d/2-1})} &\leq e^C \int_0^T \|\mathbf{u}_2\|_{B^{d/2+1}} d\tau \|\delta \mathbf{u}\|_{L^1([0, T]; B^{d/2})} \|n_1\|_{L^\infty([0, T]; B^{d/2-1, d/2})} \\ &\lesssim e^C \int_0^T \|(\mathbf{u}_1, \mathbf{u}_2)\|_{B^{d/2+1}} d\tau \|n_0\|_{B^{d/2-1, d/2}} \|\delta \mathbf{u}\|_{L^1([0, T]; B^{d/2})}. \end{aligned}$$

By Lemmas A.3 and A.6, we have

$$\begin{aligned} &\|\delta m\|_{\tilde{L}^\infty([0, T]; B^{d/2-2, d/2-1})} + \|\delta \mathbf{u}\|_{\tilde{L}^\infty([0, T]; B^{d/2-2})} \\ &\quad + \|\delta m\|_{L^1([0, T]; B^{d/2, d/2-1})} + \|\delta \mathbf{u}\|_{L^1([0, T]; B^{d/2})} \\ &\lesssim e^C \int_0^T \|\mathbf{u}_2\|_{B^{d/2+1}} d\tau \left[ \|\delta \mathbf{u} \cdot \nabla m_1\|_{L^1([0, T]; B^{d/2-2, d/2-1})} + \|\delta F(\tau)\|_{L^1([0, T]; B^{d/2-2, d/2-1})} \right. \\ &\quad \left. + \|\delta \mathbf{u} \cdot \nabla \mathbf{u}_1\|_{L^1([0, T]; B^{d/2-2})} + \|\delta \mathbf{G}(\tau)\|_{L^1([0, T]; B^{d/2-2})} \right] \\ &\lesssim e^C \int_0^T \|\mathbf{u}_2\|_{B^{d/2+1}} d\tau \left( \|\delta \mathbf{u}\|_{L^1([0, T]; B^{d/2})} \|m_1\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})} \right. \\ &\quad + \|(\delta m, \delta n)\|_{\tilde{L}^\infty([0, T]; B^{d/2-2, d/2-1})} \|\mathbf{u}_2\|_{L^1([0, T]; B^{d/2+1})} \\ &\quad + \|(m_1, n_1)\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})} \|\delta \mathbf{u}\|_{L^1([0, T]; B^{d/2})} \\ &\quad \left. + \|\delta \mathbf{u}\|_{\tilde{L}^\infty([0, T]; B^{d/2-2})} \|\mathbf{u}_1\|_{L^1([0, T]; B^{d/2+1})} + \|\delta \mathbf{G}(\tau)\|_{L^1([0, T]; B^{d/2-2})} \right) \\ &\lesssim e^C \|\mathbf{u}_2\|_{L^1([0, T]; B^{d/2+1})} \left( (1 + \|(m_2, n_2)\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})}) \right. \\ &\quad \left. \times \|(m_1, n_1)\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})} + Z(T) \right) \|(\delta m, \delta n, \delta \mathbf{u})\|_{S_T^{d/2-1}}, \end{aligned}$$

where  $\limsup_{T \rightarrow 0^+} Z(T) = 0$ . Thus,

$$\begin{aligned} &\|(\delta m, \delta n, \delta \mathbf{u})\|_{E_T^{d/2-1}} \\ &\leq C e^C \|(\mathbf{u}_1, \mathbf{u}_2)\|_{L^1([0, T]; B^{d/2+1})} \left( (1 + \|(m_2, n_2)\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})}) \right. \\ &\quad \left. \times \|(m_1, n_1)\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})} + E_0 + Z(T) \right) \|(\delta m, \delta n, \delta \mathbf{u})\|_{E_T^{d/2-1}}. \end{aligned}$$

We take  $E_0$  small enough such that it satisfies the condition  $2C(1 + CAC\bar{E}_0)A\bar{C}E_0 + E_0 < 1/4$  and (2.20), and we choose  $T > 0$  so small that  $C\|(\mathbf{u}_1, \mathbf{u}_2)\|_{L^1([0, T]; B^{d/2+1})} \leq \ln 2$  and  $Z(T) < 1/2$ . It follows that  $\|(\delta m, \delta n, \delta \mathbf{u})\|_{E_T^{d/2-1}} \equiv 0$ . Hence,  $(m_1, n_1, \mathbf{u}_1)(t) = (m_2, n_2, \mathbf{u}_2)(t)$  on  $[0, T]$ . By a standard argument (see, e.g., [5]), we can conclude that  $(m_1, n_1, \mathbf{u}_1)(t) = (m_2, n_2, \mathbf{u}_2)(t)$  on  $\mathbb{R}^+$ .

For the case  $d = 2$ , we have to raise the regularity of the spaces. Thus, we also suppose that  $m_0, n_0 \in B^{\varepsilon, 1+\varepsilon}$  and  $\mathbf{u}_0 \in B^\varepsilon$  for  $\varepsilon \in (0, 1)$ . By the same process, we can prove the existence of solution  $(m, n, \mathbf{u})$  in the space  $E^{1+\varepsilon}$ , provided the norms of initial data are sufficiently small. Then, in the same way as in the case  $d \geq 3$ , we may prove the uniqueness of solutions in the space  $E^\varepsilon$  (which, of course, holds in  $E^1$ ). We omit the details.

**3. Local well-posedness for large data.**

**3.1. Reformulation of the system.** We change the variables to  $\rho = \bar{m}(\tilde{m}^{-1} - \bar{m}^{-1})$  and  $g = \tilde{n} - \bar{n}$ . Then we can reformulate system (1.1)–(1.2) as

$$(3.1) \quad \begin{cases} \rho_t + \mathbf{u} \cdot \nabla \rho = (\rho + 1)\operatorname{div} \mathbf{u}, \\ g_t + \mathbf{u} \cdot \nabla g = -(g + \bar{n})\operatorname{div} \mathbf{u}, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - (1 + \rho)(\mu \Delta \mathbf{u} + (\mu + \lambda)\nabla \operatorname{div} \mathbf{u}) + Q(\rho, g) = 0, \\ (\rho, g, \mathbf{u})|_{t=0} = (\rho_0, g_0, \mathbf{u}_0), \end{cases}$$

where  $\rho_0 = \bar{m}(\tilde{m}_0^{-1} - \bar{m}^{-1})$ ,  $g_0 = \tilde{n}_0 - \bar{n}$ , and

$$\begin{aligned} Q(\rho, g) &:= \bar{m}^{-1}(1 + \rho)\nabla P(\bar{m}/(1 + \rho), g + \bar{n}) \\ &= \frac{\rho \nabla \rho}{\rho + 1} - \nabla \rho + \frac{a_0}{\bar{m}}(\rho + 1)\nabla g + B(\rho, g) \left[ -\frac{\bar{m}}{(\rho + 1)^2} \nabla \rho + \frac{k_0 - a_0 \bar{n}}{\rho + 1} \nabla \rho \right. \\ &\quad \left. + \frac{a_0 g \nabla \rho}{\rho + 1} + \frac{a_0^2}{\bar{m}}(g + g\rho)\nabla g + \frac{a_0(k_0 + \bar{m} + a_0 \bar{n})}{\bar{m}} \nabla g + \frac{a_0 k_0 + a_0^2 \bar{n}}{\bar{m}} \rho \nabla g \right], \end{aligned}$$

with

$$B(\rho, g) := \left[ \left( \frac{\bar{m}}{\rho + 1} + a_0(g + \bar{n}) - k_0 \right)^2 + 4k_0 a_0(g + \bar{n}) \right]^{-1/2}.$$

We now state the result for the local theory for general data bounded away from the vacuum as follows.

**THEOREM 3.1.** *Let  $d \geq 2$ ,  $\mu > 0$ ,  $2\mu + d\lambda \geq 0$ , and the constants  $\bar{m} > 0$  and  $\bar{n} \geq 0$ . Assume that  $\rho_0 \in B^{d/2, d/2+1}$ ,  $g_0 \in B^{d/2, d/2+1}$ , and  $\mathbf{u}_0 \in B^{d/2-1, d/2}$ . In addition,  $\inf_{x \in \mathbb{R}^d} \rho_0(x) > -1$ . Then there exists a positive time  $T$  such that system (3.1) has a unique solution  $(\rho, g, \mathbf{u})$  on  $[0, T] \times \mathbb{R}^d$  which belongs to*

$$(\tilde{\mathcal{C}}([0, T]; B^{d/2, d/2+1}))^{1+1} \times (\tilde{\mathcal{C}}([0, T]; B^{d/2-1, d/2}) \cap L^1([0, T]; B^{d/2+1, d/2+2}))^d$$

and satisfies  $\inf_{(t,x) \in [0, T] \times \mathbb{R}^d} \rho(t, x) > -1$ .

**3.2. A priori estimates.** Now, let us recall some estimates for the following parabolic system (studied in [1, 8]) which is obtained by linearizing the momentum equation:

$$\begin{cases} \mathbf{u}_t + \mathbf{v} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{w} - b(t, x)(\mu \Delta \mathbf{u} + (\mu + \lambda)\nabla \operatorname{div} \mathbf{u}) = f, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases}$$

Precisely, we have the following lemma (cf. [1, Proposition 10.12]).

LEMMA 3.2. *Let  $\alpha \in (0, 1]$ ,  $s \in (-d/2, d/2]$ ,  $\underline{\nu} = \min(\mu, \lambda + 2\mu)$ , and  $\bar{\nu} = \mu + |\mu + \lambda|$ . Assume that  $b = 1 + \rho$  with  $\rho \in L^\infty([0, T]; B^{d/2+\alpha})$  and that*

$$b_* := \inf_{(t,x) \in [0,T] \times \mathbb{R}^d} b(t, x) > 0.$$

*There exist a universal constant  $\kappa$ , and a constant  $C$  depending only on  $d, \alpha$ , and  $s$ , such that for all  $t \in [0, T]$ ,*

$$\begin{aligned} & \| \mathbf{u} \|_{\tilde{L}^\infty([0,t]; B^s)} + \kappa b_* \underline{\nu} \| \mathbf{u} \|_{L^1([0,t]; B^{s+2})} \\ & \leq \left( \| \mathbf{u}_0 \|_{B^s} + \| f \|_{L^1([0,t]; B^s)} \right) \\ & \quad \times \exp \left( C \int_0^t \left( \| \mathbf{v} \|_{B^{d/2+1}} + \| \mathbf{w} \|_{B^{d/2+1}} + (b_* \underline{\nu})^{1-2/\alpha} \bar{\nu}^{2/\alpha} \| \rho \|_{B^{d/2+\alpha}}^{2/\alpha} \right) d\tau \right). \end{aligned}$$

*If  $\mathbf{v}$  and  $\mathbf{w}$  depend linearly on  $\mathbf{u}$ , then the above inequality is true for all  $s \in (0, d/2 + \alpha]$ , and the argument of the exponential term may be replaced with*

$$C \int_0^t \left( \| \nabla \mathbf{u} \|_\infty + (b_* \underline{\nu})^{1-2/\alpha} \bar{\nu}^{2/\alpha} \| \rho \|_{B^{d/2+\alpha}}^{2/\alpha} \right) d\tau.$$

For the mass equations, we only need to study the following equation with two constants  $\theta \in \mathbb{R}$  and  $\beta > 0$ :

$$(3.2) \quad \begin{cases} h_t + \mathbf{v} \cdot \nabla h = \theta(h + \beta) \operatorname{div} \mathbf{v}, \\ h|_{t=0} = h_0. \end{cases}$$

PROPOSITION 3.3. *Let  $s \in (-d/2, d/2 + 1]$ ,  $T > 0$ ,  $\theta \in \mathbb{R}$ , and  $\beta \geq 0$  be constants. Assume that  $h_0 \in B^{d/2}$ ,  $\mathbf{v} \in L^1([0, T]; B^{d/2+1})$  and that they satisfy (3.2). There exists a constant  $C$  depending only on  $d$  such that for all  $t \in [0, T]$ , we have*

$$(3.3) \quad \| h \|_{\tilde{L}^\infty([0,t]; B^{d/2})} \leq e^{C(1+2|\theta|) \int_0^t \| \mathbf{v} \|_{B^{d/2+1}} d\tau} \left( \| h_0 \|_{B^{d/2}} + \frac{\beta}{1+2|\theta|} \right) - \frac{\beta}{1+2|\theta|}$$

and

$$(3.4) \quad \begin{aligned} & \| h \|_{\tilde{L}^\infty([0,t]; B^s)} \leq e^{C(1+|\theta|) \int_0^t \| \mathbf{v} \|_{B^{d/2+1}} d\tau} \left( \| h_0 \|_{B^s} \right. \\ & \left. + C|\theta| \left[ e^{C(1+2|\theta|) \int_0^t \| \mathbf{v} \|_{B^{d/2+1}} d\tau} \left( \| h_0 \|_{B^{d/2}} + \frac{\beta}{1+2|\theta|} \right) + \frac{2|\theta|\beta}{1+2|\theta|} \right] \int_0^t \| \mathbf{v} \|_{B^{s+1}} d\tau \right). \end{aligned}$$

*Proof.* Applying the operator  $\Delta_k$  to (3.2) yields

$$\partial_t \Delta_k h + \Delta_k (\mathbf{v} \cdot \nabla h) = \theta \Delta_k ((h + \beta) \operatorname{div} \mathbf{v}).$$

Taking the  $L^2$  inner product with  $\Delta_k h$ , we get, with the help of Lemmas A.6 and A.7, that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \| \Delta_k h \|_2^2 \\ & = -(\Delta_k (\mathbf{v} \cdot \nabla a), \Delta_k h) + \theta (\Delta_k ((h + \beta) \operatorname{div} \mathbf{v}), \Delta_k h) \end{aligned}$$



$$\begin{aligned} &\lesssim \gamma_k 2^{-ks} \|\mathbf{v}\|_{B^{d/2+1}} \|h\|_{B^s} \|\Delta_k h\|_2 + |\theta| \gamma_k 2^{-ks} (\|h \operatorname{div} \mathbf{v}\|_{B^s} + \beta \|\operatorname{div} \mathbf{v}\|_{B^s}) \|\Delta_k h\|_2 \\ &\lesssim \gamma_k 2^{-ks} ((1 + |\theta|) \|h\|_{B^s} \|\mathbf{v}\|_{B^{d/2+1}} + (\|h\|_{B^{d/2}} + \beta) |\theta| \|\mathbf{v}\|_{B^{s+1}}) \|\Delta_k h\|_2. \end{aligned}$$

Eliminating the factor  $\|\Delta_k h\|_2$  from both sides and integrating in time, we have

$$\begin{aligned} \|\Delta_k h\|_2 &\leq \|\Delta_k h_0\|_2 \\ &\quad + C \gamma_k \int_0^t 2^{-ks} ((1 + |\theta|) \|h\|_{B^s} \|\mathbf{v}\|_{B^{d/2+1}} + (\|h\|_{B^{d/2}} + \beta) |\theta| \|\mathbf{v}\|_{B^{s+1}}) d\tau. \end{aligned}$$

It follows, for any  $k \in \mathbb{Z}$  and any  $t \in [0, T]$ , that

$$(3.5) \quad \begin{aligned} 2^{ks} \|\Delta_k h\|_2 &\leq 2^{ks} \|\Delta_k h_0\|_2 \\ &\quad + C \gamma_k \int_0^t ((1 + |\theta|) \|h\|_{B^s} \|\mathbf{v}\|_{B^{d/2+1}} + (\|h\|_{B^{d/2}} + \beta) |\theta| \|\mathbf{v}\|_{B^{s+1}}) d\tau. \end{aligned}$$

Summing up on  $k \in \mathbb{Z}$ , this yields

$$\begin{aligned} \|h\|_{\tilde{L}^\infty([0, T]; B^s)} &\leq \|h_0\|_{B^s} \\ &\quad + \int_0^t C [(1 + |\theta|) \|\mathbf{v}\|_{B^{d/2+1}} \|h\|_{B^s} + (\|h\|_{B^{d/2}} + \beta) |\theta| \|\mathbf{v}\|_{B^{s+1}}] d\tau. \end{aligned}$$

By the Gronwall inequality, we have (3.3) for  $s = d/2$ , and then for any  $s \in (-d/2, d/2 + 1]$

$$\begin{aligned} &\|h\|_{\tilde{L}^\infty([0, t]; B^s)} \\ &\leq e^{C(1+|\theta|) \int_0^t \|\mathbf{v}\|_{B^{d/2+1}} d\tau} \left( \|h_0\|_{B^s} + C \int_0^t |\theta| (\|h\|_{B^{d/2}} + \beta) \|\mathbf{v}\|_{B^{s+1}} d\tau \right) \\ &\leq e^{C(1+|\theta|) \int_0^t \|\mathbf{v}\|_{B^{d/2+1}} d\tau} \left( \|h_0\|_{B^s} \right. \\ &\quad \left. + C |\theta| \left[ e^{C(1+2|\theta|) \int_0^t \|\mathbf{v}\|_{B^{d/2+1}} d\tau} \left( \|h_0\|_{B^{d/2}} + \frac{\beta}{1+2|\theta|} \right) + \frac{2|\theta|\beta}{1+2|\theta|} \int_0^t \|\mathbf{v}\|_{B^{s+1}} d\tau \right] \right). \end{aligned}$$

This completes the proof.  $\square$

In general, for the transport equation

$$\begin{cases} h_t + \mathbf{v} \cdot \nabla h = f, \\ h(0) = h_0, \end{cases}$$

we can get, in a way similar to that of Proposition 3.3, the following proposition.

PROPOSITION 3.4. *Let  $s_1, s_2 \in (-d/2, d/2 + 1]$  and  $T > 0$ . Then it holds for  $t \in [0, T]$  that*

$$\|h\|_{\tilde{L}^\infty([0, t]; B^{s_1, s_2})} \leq e^{C \int_0^t \|\mathbf{v}\|_{B^{d/2+1}} d\tau} \left( \|h_0\|_{B^{s_1, s_2}} + \int_0^t \|f\|_{B^{s_1, s_2}} d\tau \right).$$

**3.3. Existence of local solution.**

*Step 1: The Friedrich’s approximation.* For convenience, we introduce the solution  $\mathbf{u}_{1s}$  to the linear system

$$(3.6) \quad \partial_t \mathbf{u}_{1s} - \mu \Delta \mathbf{u}_{1s} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_{1s} = 0, \quad \mathbf{u}_{1s}(0) = \mathbf{u}_0.$$

Denote  $\mathbf{u}_{\text{is}}^\ell := \mathbb{F}_\ell \mathbf{u}_{\text{is}}$  and  $\tilde{\mathbf{u}}^\ell := \mathbf{u}^\ell - \mathbf{u}_{\text{is}}^\ell$ . Then we can construct the approximation  $(\rho^\ell, g^\ell, \tilde{\mathbf{u}}^\ell)$  satisfying

$$(3.7) \quad \begin{cases} \rho_t^\ell + \mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla \rho^\ell) = \mathbb{F}_\ell((\rho^\ell + 1)\text{div} \mathbf{u}^\ell), \\ g_t^\ell + \mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla g^\ell) = -\mathbb{F}_\ell((g^\ell + \bar{n})\text{div} \mathbf{u}^\ell), \\ \partial_t \tilde{\mathbf{u}}^\ell + \mathbb{F}_\ell(\mathbf{u}_{\text{is}}^\ell \cdot \nabla \tilde{\mathbf{u}}^\ell) + \mathbb{F}_\ell(\tilde{\mathbf{u}}^\ell \cdot \nabla \mathbf{u}^\ell) - \mathbb{F}_\ell[(1 + \rho^\ell)(\mu \Delta \tilde{\mathbf{u}}^\ell + (\mu + \lambda)\nabla \text{div} \tilde{\mathbf{u}}^\ell)] \\ \quad = \mathbb{F}_\ell[\rho^\ell(\mu \Delta \mathbf{u}_{\text{is}}^\ell + (\mu + \lambda)\nabla \text{div} \mathbf{u}_{\text{is}}^\ell)] - \mathbb{F}_\ell(\mathbf{u}_{\text{is}}^\ell \cdot \nabla \mathbf{u}_{\text{is}}^\ell) - \mathbb{F}_\ell Q(\rho^\ell, g^\ell), \\ (\rho^\ell, g^\ell, \tilde{\mathbf{u}}^\ell)|_{t=0} = (\rho_0^\ell, g_0^\ell, \mathbf{0}), \end{cases}$$

where  $\rho_0^\ell := \mathbb{F}_\ell \rho_0$ ,  $g_0^\ell := \mathbb{F}_\ell g_0$ , and  $\mathbf{u}^\ell := \mathbf{u}_{\text{is}}^\ell + \tilde{\mathbf{u}}^\ell$ .

Note that if  $1 + \rho_0$  is bounded away from zero, then so is  $1 + \mathbb{F}_\ell \rho_0$  for sufficiently large  $\ell$ . It is easy to check that (3.7) is an ordinary differential equation in  $L_\ell^2 \times L_\ell^2 \times (L_\ell^2)^d$  for every  $\ell \in \mathbb{N}$ . By the usual Cauchy–Lipschitz theorem, there is a strictly positive maximal time  $T_\ell^*$  such that a unique solution  $(\rho^\ell, g^\ell, \tilde{\mathbf{u}}^\ell)$  exists in  $[0, T_\ell^*)$  which is continuous in time with value in  $L_\ell^2 \times L_\ell^2 \times (L_\ell^2)^d$ , i.e.,  $(\rho^\ell, g^\ell, \tilde{\mathbf{u}}^\ell) \in \mathcal{C}([0, T_\ell^*]; L_\ell^2 \times L_\ell^2 \times (L_\ell^2)^d)$ , and  $1 + \rho^\ell$  is bounded away from zero.

*Step 2: Lower bound for lifespan and uniform estimates.* We introduce the following notation:

$$\begin{aligned} M_0 &:= \|\rho_0\|_{B^{d/2, d/2+\alpha}}, & M^\ell(t) &:= \|\rho^\ell\|_{\tilde{L}^\infty([0, t]; B^{d/2, d/2+\alpha})}, \\ N_0 &:= \|g_0\|_{B^{d/2, d/2+\alpha}}, & N^\ell(t) &:= \|g^\ell\|_{\tilde{L}^\infty([0, t]; B^{d/2, d/2+\alpha})}, \\ U_0 &:= \|\mathbf{u}_0\|_{B^{d/2-1, d/2-1+\alpha}}, & U_{\text{is}}^\ell(t) &:= \|\mathbf{u}_{\text{is}}^\ell\|_{L^1([0, t]; B^{d/2+1, d/2+1+\alpha})}, \\ \tilde{U}^\ell(t) &:= \|\tilde{\mathbf{u}}^\ell\|_{\tilde{L}^\infty([0, t]; B^{d/2-1, d/2-1+\alpha})} + b_* \underline{L} \|\tilde{\mathbf{u}}^\ell\|_{L^1([0, t]; B^{d/2+1, d/2+1+\alpha})}. \end{aligned}$$

In view of Lemma 3.2, we take  $\mathbf{v} = \mathbf{w} = f = \mathbf{0}$  and  $\rho = 0$  there to get

$$\|\mathbf{u}_{\text{is}}^\ell\|_{\tilde{L}^\infty([0, t]; B^{d/2-1, d/2-1+\alpha})} \lesssim U_0.$$

Note that for all  $k \in \mathbb{Z}$ , we have

$$\|\Delta_k \rho_0^\ell\|_2 \leq \|\Delta_k \rho_0\|_2, \quad \|\rho_0^\ell\|_{B^s} \leq \|\rho_0\|_{B^s}$$

and similar properties for  $g_0^\ell$  because of the boundedness of the operators  $\Delta_k$ .

From (3.3) and (3.4), we get

$$(3.8) \quad \|\rho^\ell\|_{\tilde{L}^\infty([0, t]; B^{d/2})} \leq e^{3C \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1}} d\tau} \left( \|\rho_0\|_{B^{d/2}} + \frac{1}{3} \right) - \frac{1}{3},$$

$$(3.9) \quad \begin{aligned} \|\rho^\ell\|_{\tilde{L}^\infty([0, t]; B^{d/2+\alpha})} &\leq e^{2C \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1}} d\tau} \left( \|\rho_0\|_{B^{d/2+\alpha}} \right. \\ &\left. + C \left[ e^{3C \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1}} d\tau} \left( \|\rho_0\|_{B^{d/2}} + \frac{1}{3} \right) + \frac{2}{3} \right] \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1+\alpha}} d\tau \right), \end{aligned}$$

$$(3.10) \quad \|g^\ell\|_{\tilde{L}^\infty([0, t]; B^{d/2})} \leq e^{3C \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1}} d\tau} \left( \|g_0\|_{B^{d/2}} + \frac{\bar{n}}{3} \right) - \frac{\bar{n}}{3},$$

and

$$(3.11) \quad \begin{aligned} \|g^\ell\|_{\tilde{L}^\infty([0,t];B^{d/2+\alpha})} &\leq e^{2C \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1}} d\tau} \left( \|g_0\|_{B^{d/2+\alpha}} \right. \\ &\left. + C \left[ e^{3C \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1}} d\tau} \left( \|g_0\|_{B^{d/2}} + \frac{\bar{n}}{3} \right) + \frac{2\bar{n}}{3} \right] \int_0^t \|\mathbf{u}^\ell\|_{B^{d/2+1+\alpha}} d\tau \right). \end{aligned}$$

Let  $b(t, x) = 1 + \rho^\ell$ . From Step 1, we know  $b_* > 0$ . Thus, by Lemmas 3.2, A.3, and A.6, we have

$$(3.12) \quad \begin{aligned} &\|\tilde{\mathbf{u}}^\ell\|_{\tilde{L}^\infty([0,t];B^{d/2-1})} + \kappa b_* \mathcal{L} \|\tilde{\mathbf{u}}^\ell\|_{L^1([0,t];B^{d/2+1})} \\ &\leq \int_0^t \left( \|\rho^\ell(\mu \Delta \mathbf{u}_{\text{is}}^\ell + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_{\text{is}}^\ell)\|_{B^{d/2-1}} \right. \\ &\quad \left. + \|\mathbf{u}_{\text{is}}^\ell \cdot \nabla \mathbf{u}_{\text{is}}^\ell\|_{B^{d/2-1}} + \|Q(\rho^\ell, g^\ell)\|_{B^{d/2-1}} \right) d\tau \\ &\quad \times \exp \left( C \int_0^t \left( \|\mathbf{u}^\ell\|_{B^{d/2+1}} + \|\mathbf{u}_{\text{is}}^\ell\|_{B^{d/2+1}} + (b_* \mathcal{L})^{1-2/\alpha} \bar{\nu}^{2/\alpha} \|\rho^\ell\|_{B^{d/2+\alpha}}^{2/\alpha} \right) d\tau \right), \end{aligned}$$

and similarly

$$(3.13) \quad \begin{aligned} &\|\tilde{\mathbf{u}}^\ell\|_{\tilde{L}^\infty([0,t];B^{d/2-1+\alpha})} + \kappa b_* \mathcal{L} \|\tilde{\mathbf{u}}^\ell\|_{L^1([0,t];B^{d/2+1+\alpha})} \\ &\leq \int_0^t \left( \|\rho^\ell(\mu \Delta \mathbf{u}_{\text{is}}^\ell + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_{\text{is}}^\ell)\|_{B^{d/2-1+\alpha}} \right. \\ &\quad \left. + \|\mathbf{u}_{\text{is}}^\ell \cdot \nabla \mathbf{u}_{\text{is}}^\ell\|_{B^{d/2-1+\alpha}} + \|Q(\rho^\ell, g^\ell)\|_{B^{d/2-1+\alpha}} \right) d\tau \\ &\quad \times \exp \left( C \int_0^t \left( \|\mathbf{u}^\ell\|_{B^{d/2+1}} + \|\mathbf{u}_{\text{is}}^\ell\|_{B^{d/2+1}} + (b_* \mathcal{L})^{1-2/\alpha} \bar{\nu}^{2/\alpha} \|\rho^\ell\|_{B^{d/2+\alpha}}^{2/\alpha} \right) d\tau \right). \end{aligned}$$

By Lemma A.6, we get, for all  $\sigma \in \{0, \alpha\}$ , that

$$\begin{aligned} \|\rho^\ell \Delta \mathbf{u}_{\text{is}}^\ell\|_{B^{d/2-1+\sigma}} &\lesssim \|\rho^\ell\|_{B^{d/2}} \|\Delta \mathbf{u}_{\text{is}}^\ell\|_{B^{d/2-1+\sigma}} \lesssim \|\rho^\ell\|_{B^{d/2}} \|\mathbf{u}_{\text{is}}^\ell\|_{B^{d/2+1+\sigma}}, \\ \|\rho^\ell \nabla \operatorname{div} \mathbf{u}_{\text{is}}^\ell\|_{B^{d/2-1+\sigma}} &\lesssim \|\rho^\ell\|_{B^{d/2}} \|\mathbf{u}_{\text{is}}^\ell\|_{B^{d/2+1+\sigma}}, \\ \|\mathbf{u}_{\text{is}}^\ell \cdot \nabla \mathbf{u}_{\text{is}}^\ell\|_{B^{d/2-1+\sigma}} &\lesssim \|\mathbf{u}_{\text{is}}^\ell\|_{B^{d/2}} \|\nabla \mathbf{u}_{\text{is}}^\ell\|_{B^{d/2-1+\sigma}} \lesssim \|\mathbf{u}_{\text{is}}^\ell\|_{B^{d/2}} \|\mathbf{u}_{\text{is}}^\ell\|_{B^{d/2+\sigma}}. \end{aligned}$$

Recall that

$$\begin{aligned} Q(\rho^\ell, g^\ell) &= \frac{\rho^\ell \nabla \rho^\ell}{\rho^\ell + 1} - \nabla \rho^\ell + \frac{a_0}{\bar{m}} (\rho^\ell + 1) \nabla g^\ell + B(\rho^\ell, g^\ell) \left[ - \frac{\bar{m}}{(\rho^\ell + 1)^2} \nabla \rho^\ell \right. \\ &\quad \left. + \frac{k_0 - a_0 \bar{n}}{\rho^\ell + 1} \nabla \rho^\ell + \frac{a_0 g^\ell \nabla \rho^\ell}{\rho^\ell + 1} + \frac{a_0^2}{\bar{m}} (g^\ell + g^\ell \rho^\ell) \nabla g^\ell + \frac{a_0(k_0 + \bar{m} + a_0 \bar{n})}{\bar{m}} \nabla g^\ell \right. \\ &\quad \left. + \frac{a_0 k_0 + a_0^2 \bar{n}}{\bar{m}} \rho^\ell \nabla g^\ell \right], \end{aligned}$$

where

$$B(\rho^\ell, g^\ell) := \left[ \left( \frac{\bar{m}}{\rho^\ell + 1} + a_0(g^\ell + \bar{n}) - k_0 \right)^2 + 4k_0 a_0 (g^\ell + \bar{n}) \right]^{-1/2}.$$

Similarly, for the third term of  $Q$

$$\|(\rho^\ell + 1)\nabla g^\ell\|_{B^{d/2-1+\sigma}} \lesssim (\|\rho^\ell\|_{B^{d/2}} + 1)\|\nabla g^\ell\|_{B^{d/2-1+\sigma}} \lesssim (\|\rho^\ell\|_{B^{d/2}} + 1)\|g^\ell\|_{B^{d/2+\sigma}}.$$

By Lemmas A.6 and A.3, we get for the first two terms of  $Q$

$$\begin{aligned} \left\| \frac{\rho^\ell \nabla \rho^\ell}{\rho^\ell + 1} - \nabla \rho^\ell \right\|_{B^{d/2-1+\sigma}} &\lesssim \left( \left\| \frac{\rho^\ell}{\rho^\ell + 1} \right\|_{B^{d/2}} + 1 \right) \|\nabla \rho^\ell\|_{B^{d/2-1+\sigma}} \\ &\lesssim (\|\rho^\ell\|_{B^{d/2}} + 1)\|\rho^\ell\|_{B^{d/2+\sigma}} \end{aligned}$$

and

$$\begin{aligned} &\left\| B(\rho^\ell, g^\ell) \frac{\nabla \rho^\ell}{(\rho^\ell + 1)^2} \right\|_{B^{d/2-1+\sigma}} \\ &\lesssim \left\| [B(\rho^\ell, g^\ell) - B(0, 0)] \frac{\nabla \rho^\ell}{(\rho^\ell + 1)^2} \right\|_{B^{d/2-1+\sigma}} + B(0, 0) \left\| \frac{\nabla \rho^\ell}{(\rho^\ell + 1)^2} \right\|_{B^{d/2-1+\sigma}} \\ &\lesssim (\|B(\rho^\ell, g^\ell) - B(0, 0)\|_{B^{d/2}} + 1) \left( \left\| \frac{\rho^\ell \rho^\ell + 2\rho^\ell}{(\rho^\ell + 1)^2} \right\|_{B^{d/2}} + 1 \right) \|\nabla \rho^\ell\|_{B^{d/2-1+\sigma}} \\ &\lesssim (\|\rho^\ell\|_{B^{d/2}} + \|g^\ell\|_{B^{d/2}} + 1)^2 \|\rho^\ell\|_{B^{d/2+\sigma}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| B(\rho^\ell, g^\ell) \frac{\nabla \rho^\ell}{\rho^\ell + 1} \right\|_{B^{d/2-1+\sigma}} &\lesssim (\|\rho^\ell\|_{B^{d/2}} + \|g^\ell\|_{B^{d/2}} + 1)^2 \|\rho^\ell\|_{B^{d/2+\sigma}}, \\ \left\| B(\rho^\ell, g^\ell) \frac{g^\ell \nabla \rho^\ell}{\rho^\ell + 1} \right\|_{B^{d/2-1+\sigma}} &\lesssim (\|\rho^\ell\|_{B^{d/2}} + \|g^\ell\|_{B^{d/2}} + 1)^2 \|g^\ell\|_{B^{d/2}} \|\rho^\ell\|_{B^{d/2+\sigma}}, \\ \|B(\rho^\ell, g^\ell)(1 + \rho^\ell)g^\ell \nabla g^\ell\|_{B^{d/2-1+\sigma}} &\lesssim (\|\rho^\ell\|_{B^{d/2}} + \|g^\ell\|_{B^{d/2}} + 1)^2 \|g^\ell\|_{B^{d/2}} \|g^\ell\|_{B^{d/2+\sigma}}, \\ \|B(\rho^\ell, g^\ell)\nabla g^\ell\|_{B^{d/2-1+\sigma}} &\lesssim (\|\rho^\ell\|_{B^{d/2}} + \|g^\ell\|_{B^{d/2}} + 1) \|g^\ell\|_{B^{d/2+\sigma}}, \\ \|B(\rho^\ell, g^\ell)\rho^\ell \nabla g^\ell\|_{B^{d/2-1+\sigma}} &\lesssim (\|\rho^\ell\|_{B^{d/2}} + \|g^\ell\|_{B^{d/2}} + 1) \|\rho^\ell\|_{B^{d/2}} \|g^\ell\|_{B^{d/2+\sigma}}. \end{aligned}$$

Thus, we get

$$\|Q(\rho^\ell, g^\ell)\|_{B^{d/2-1+\sigma}} \lesssim (\|\rho^\ell\|_{B^{d/2}} + \|g^\ell\|_{B^{d/2}} + 1)^3 (\|\rho^\ell\|_{B^{d/2+\sigma}} + \|g^\ell\|_{B^{d/2+\sigma}}).$$

Therefore, from (3.8)–(3.13), we conclude that

$$\begin{aligned} M^\ell(T) &\lesssim e^{C(U_{\text{is}}^\ell(T) + \tilde{U}^\ell(T)/(b_* \underline{\nu}))} \left( M_0 + \frac{1}{3} \right) \\ &\quad + e^{C(U_{\text{is}}^\ell(T) + \tilde{U}^\ell(T)/(b_* \underline{\nu}))} \left( M_0 + \frac{2}{3} \right) (U_{\text{is}}^\ell(T) + \tilde{U}^\ell(T)/(b_* \underline{\nu})) - \frac{1}{3}, \\ N^\ell(T) &\lesssim e^{C(U_{\text{is}}^\ell(T) + \tilde{U}^\ell(T)/(b_* \underline{\nu}))} \left( N_0 + \frac{\bar{n}}{3} \right) \\ &\quad + e^{C(U_{\text{is}}^\ell(T) + \tilde{U}^\ell(T)/(b_* \underline{\nu}))} \left( N_0 + \frac{2\bar{n}}{3} \right) (U_{\text{is}}^\ell(T) + \tilde{U}^\ell(T)/(b_* \underline{\nu})) - \frac{\bar{n}}{3}, \\ \tilde{U}^\ell(T) &\lesssim ((\bar{\nu} M^\ell(T) + U_0)U_{\text{is}}^\ell(T) + (M^\ell(T) + N^\ell(T) + 1)^3 (M^\ell(T) + N^\ell(T))T) \\ &\quad \times e^{C[U_{\text{is}}^\ell(T) + \tilde{U}^\ell(T)/(b_* \underline{\nu}) + (b_* \underline{\nu})^{1-2/\alpha} \bar{\nu}^{2/\alpha} (M^\ell(T))^{2/\alpha} T]}. \end{aligned}$$

Now, if we take  $T$  so small that

$$\exp(CU_{\text{ls}}^\ell(T)) \leq \sqrt{2}, \quad \exp\left(\frac{C\tilde{U}^\ell(T)}{b_*\underline{\nu}}\right) \leq \sqrt{2},$$

and

$$\exp\left(C(b_*\underline{\nu})^{1-2/\alpha}\bar{\nu}^{2/\alpha}(M^\ell(T))^{2/\alpha}T\right) \leq 2,$$

then we have

$$(3.14) \quad \begin{cases} M^\ell(T) \leq 4M_0 + \frac{5}{3}, & N^\ell(T) \leq 4N_0 + \frac{5\bar{n}}{3}, \\ \tilde{U}^\ell(T) \leq C((M_0 + N_0 + 1)^4(T + \bar{\nu}U_{\text{ls}}^\ell(T)) + U_0U_{\text{ls}}^\ell(T)). \end{cases}$$

Notice that  $(\mathbb{F}_\ell\rho^\ell, \mathbb{F}_\ell g^\ell, \mathbb{F}_\ell\tilde{\mathbf{u}}^\ell) = (\rho^\ell, g^\ell, \tilde{\mathbf{u}}^\ell)$  by the construction of the approximated system. Thus, we have

$$\partial_t(1 + \rho^\ell)^{\pm 1} + \mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla(1 + \rho^\ell)^{\pm 1}) \pm \mathbb{F}_\ell((1 + \rho^\ell)^{\pm 1} \text{div}\mathbf{u}^\ell) = 0.$$

It follows, by noticing that  $|\partial_t|f| = |\partial_t f|$ , that

$$\begin{aligned} \|(1 + \rho^\ell)^{\pm 1}(t)\|_\infty &\leq \|(1 + \rho_0^\ell)^{\pm 1}\|_\infty \\ &\quad + \int_0^t [\|\mathbf{u}^\ell \cdot \nabla(1 + \rho^\ell)^{\pm 1}\|_\infty + \|(1 + \rho^\ell)^{\pm 1} \text{div}\mathbf{u}^\ell\|_\infty] d\tau, \end{aligned}$$

which yields, by the Gronwall inequality, that

$$\begin{aligned} \|(1 + \rho^\ell)^{\pm 1}(t)\|_\infty &\leq e^{\int_0^t \|\text{div}\mathbf{u}^\ell\|_\infty d\tau} \left( \|(1 + \rho_0^\ell)^{\pm 1}\|_\infty + \int_0^t \|\mathbf{u}^\ell\|_\infty \|\nabla\rho^\ell\|_\infty d\tau \right) \\ &\leq e^{\int_0^t \|\text{div}\mathbf{u}^\ell\|_\infty d\tau} \left( \|(1 + \rho_0^\ell)^{\pm 1}\|_\infty + C \int_0^t (\|\mathbf{u}_{\text{ls}}^\ell\|_{B^{d/2}} + \|\tilde{\mathbf{u}}^\ell\|_{B^{d/2}}) \|\rho^\ell\|_{B^{d/2+1}} d\tau \right) \\ &\leq e^{\int_0^t \|\text{div}\mathbf{u}^\ell\|_\infty d\tau} \left( \|(1 + \rho_0^\ell)^{\pm 1}\|_\infty + CT(U_0 + \tilde{U}^\ell(T))M^\ell(T) \right), \end{aligned}$$

where we have to choose  $\alpha = 1$  in the previous estimates. Hence, if we assume that there exist two positive constants  $b_*$  and  $b^*$  such that

$$b_* \leq 1 + \rho_0 \leq b^*,$$

then we can take  $T$  small enough such that

$$\int_0^T \|\text{div}\mathbf{u}^\ell\|_\infty d\tau \leq \ln 2 \quad \text{and} \quad CT(U_0 + \tilde{U}^\ell(T))M^\ell(T) \leq 1,$$

and so

$$(3.15) \quad \frac{b_*}{2(1 + b_*)} \leq 1 + \rho^\ell \leq 2(b^* + 1).$$

Now, by means of a bootstrap argument, we can get that there exist two constants  $\eta$  and  $C$  depending only on  $d$  such that if

$$(3.16) \quad \begin{cases} (b_*\underline{\nu})^{1-2/\alpha}\bar{\nu}^{2/\alpha}(M^\ell(T))^{2/\alpha}T \leq \eta, \\ (M_0 + N_0 + 1)^4(T + \bar{\nu}U_{\text{ls}}^\ell(T)) + U_0U_{\text{ls}}^\ell(T) \leq \eta b_*\underline{\nu}, \end{cases}$$

then we have (3.14) and (3.15).

Therefore,  $T_\ell^*$  may be bounded from below by any time  $T$  satisfying (3.16), and the inequalities (3.14) and (3.15) are satisfied by  $(\rho^\ell, g^\ell, \mathbf{u}^\ell)$ . In particular,  $(\rho^\ell, g^\ell, \mathbf{u}^\ell)_{\ell \in \mathbb{N}}$  is bounded in  $F_T^1$ .

*Step 3: Time derivatives.* In order to pass to the limit in the approximated system, we first give the following lemma.

LEMMA 3.5. *Let  $\tilde{\rho}^\ell := \rho^\ell - \mathbb{F}_\ell \rho_0$ ,  $\tilde{g}^\ell := g^\ell - \mathbb{F}_\ell g_0$ . Then the sequences  $(\tilde{\rho}^\ell)_{\ell \in \mathbb{N}}$  and  $(\tilde{g}^\ell)_{\ell \in \mathbb{N}}$  are uniformly bounded in*

$$\mathcal{C}([0, T]; B^{d/2, d/2+1}) \cap \mathcal{C}^{1/2}([0, T]; B^{d/2-1, d/2}),$$

and the sequence  $(\tilde{\mathbf{u}}^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in

$$(\mathcal{C}([0, T]; B^{d/2-1, d/2}) \cap \mathcal{C}^{1/4}([0, T]; B^{d/2-1, d/2} + B^{d/2-3/2, d/2-1/2}))^d.$$

*Proof.* From  $\partial_t \tilde{\rho}^\ell = -\mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla \rho^\ell) + \mathbb{F}_\ell((\rho^\ell + 1)\operatorname{div} \mathbf{u}^\ell)$ , we have

$$\begin{aligned} & \|\partial_t \tilde{\rho}^\ell\|_{L^2([0, T]; B^{d/2-1, d/2})} \leq \|\partial_t \tilde{\rho}^\ell\|_{\tilde{L}^2([0, T]; B^{d/2-1, d/2})} \\ & \leq \|\mathbf{u}^\ell \cdot \nabla \rho^\ell\|_{\tilde{L}^2([0, T]; B^{d/2-1, d/2})} + \|(\rho^\ell + 1)\operatorname{div} \mathbf{u}^\ell\|_{\tilde{L}^2([0, T]; B^{d/2-1, d/2})} \\ & \lesssim \|\mathbf{u}^\ell\|_{\tilde{L}^2([0, T]; B^{d/2})} \|\rho^\ell\|_{\tilde{L}^\infty([0, T]; B^{d/2, d/2+1})} \\ & \quad + (\|\rho^\ell\|_{\tilde{L}^\infty([0, T]; B^{d/2})} + 1) \|\mathbf{u}^\ell\|_{\tilde{L}^2([0, T]; B^{d/2, d/2+1})}. \end{aligned}$$

Since  $(\mathbf{u}^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in

$$\tilde{L}^\infty([0, T]; B^{d/2-1, d/2}) \cap L^1([0, T]; B^{d/2+1, d/2+2}),$$

it is also bounded in  $\tilde{L}^2([0, T]; B^{d/2, d/2+1})$  by Lemma A.9. Recall that  $(\rho^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in  $\tilde{L}^\infty([0, T]; B^{d/2, d/2+1})$ ; then  $(\partial_t \tilde{\rho}^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in

$$L^2([0, T]; B^{d/2-1, d/2}),$$

and so  $(\tilde{\rho}^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in

$$\mathcal{C}^{1/2}([0, T]; B^{d/2-1, d/2}) \quad \text{and in} \quad \mathcal{C}([0, T]; B^{d/2, d/2-1}).$$

Similarly, we have the same arguments for  $\tilde{g}^\ell$ .

Recall that

$$\begin{aligned} \partial_t \tilde{\mathbf{u}}^\ell &= -\mathbb{F}_\ell(\mathbf{u}^\ell \cdot \nabla \mathbf{u}^\ell) + \mathbb{F}_\ell[(1 + \rho^\ell)(\mu \Delta \tilde{\mathbf{u}}^\ell + (\mu + \lambda)\nabla \operatorname{div} \tilde{\mathbf{u}}^\ell)] \\ & \quad + \mathbb{F}_\ell[\rho^\ell(\mu \Delta \mathbf{u}_{\text{is}}^\ell + (\mu + \lambda)\nabla \operatorname{div} \mathbf{u}_{\text{is}}^\ell)] - \mathbb{F}_\ell Q(\rho^\ell, g^\ell). \end{aligned}$$

Since

$$\begin{aligned} & \|\mathbf{u}^\ell \cdot \nabla \mathbf{u}^\ell\|_{L^2([0, T]; B^{d/2-1, d/2})} \lesssim \|\mathbf{u}^\ell\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})} \|\mathbf{u}^\ell\|_{L^2([0, T]; B^{d/2+1})}, \\ & \|(1 + \rho^\ell)(\mu \Delta \tilde{\mathbf{u}}^\ell + (\mu + \lambda)\nabla \operatorname{div} \tilde{\mathbf{u}}^\ell)\|_{L^{4/3}([0, T]; B^{d/2-3/2, d/2-1/2})} \\ & \lesssim (1 + \|\rho^\ell\|_{\tilde{L}^\infty([0, T]; B^{d/2})}) \|\tilde{\mathbf{u}}^\ell\|_{L^{4/3}([0, T]; B^{d/2+1/2, d/2+3/2})}, \\ & \|\rho^\ell(\mu \Delta \mathbf{u}_{\text{is}}^\ell + (\mu + \lambda)\nabla \operatorname{div} \mathbf{u}_{\text{is}}^\ell)\|_{L^{4/3}([0, T]; B^{d/2-3/2, d/2-1/2})} \\ & \lesssim \|\rho^\ell\|_{\tilde{L}^\infty([0, T]; B^{d/2})} \|\mathbf{u}_{\text{is}}^\ell\|_{L^{4/3}([0, T]; B^{d/2+1/2, d/2+3/2})}, \\ & \|Q(\rho^\ell, g^\ell)\|_{\tilde{L}^\infty([0, T]; B^{d/2-1, d/2})} \end{aligned}$$

$$\lesssim \left( \|\rho^\ell\|_{\tilde{L}^\infty([0,T];B^{d/2})} + \|g^\ell\|_{\tilde{L}^\infty([0,T];B^{d/2})} + 1 \right)^3 \left( \|\rho^\ell\|_{\tilde{L}^\infty([0,T];B^{d/2,d/2+1})} + \|g^\ell\|_{\tilde{L}^\infty([0,T];B^{d/2,d/2+1})} \right),$$

by Lemma A.9, we can conclude that  $(\tilde{\mathbf{u}}^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in

$$\mathcal{C}^{1/4}([0, T]; B^{d/2-1,d/2} + B^{d/2-3/2,d/2-1/2}) \text{ and in } \mathcal{C}([0, T]; B^{d/2-1,d/2}).$$

This completes the proof of the lemma.  $\square$

*Step 4: Compactness and convergence.* The proof is based on the Arzelà–Ascoli theorem and compact embeddings for Besov spaces. Since it is similar to the arguments for global well-posedness, we only give the outline of the proof.

From Lemma 3.5,  $(\tilde{\rho}^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in the space

$$\tilde{L}^\infty([0, T]; B^{d/2,d/2+1})$$

and equicontinuous on  $[0, T]$  with values in  $B^{d/2-1,d/2}$ . Since the embedding

$$B^{d/2-1,d/2} \hookrightarrow B^{d/2-1}$$

is (locally) compact, and  $(\rho_0^\ell)_{\ell \in \mathbb{N}}$  tends to  $\rho_0$  in  $B^{d/2,d/2+1}$ , we conclude that  $(\rho^\ell)_{\ell \in \mathbb{N}}$  tends (up to an extraction) to some distribution  $\rho$ . Given that  $(\rho^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in  $\tilde{L}^\infty([0, T]; B^{d/2})$ , we actually have

$$\rho \in \tilde{L}^\infty([0, T]; B^{d/2}).$$

The same arguments are valid for the sequence  $(g^\ell)_{\ell \in \mathbb{N}}$ .

From the definition of  $(\mathbf{u}_{\text{is}}^\ell)_{\ell \in \mathbb{N}}$ , it is clear that  $(\mathbf{u}_{\text{is}}^\ell)_{\ell \in \mathbb{N}}$  tends to the solution  $\mathbf{u}_{\text{is}}$  of (3.6) in  $\tilde{L}^\infty([0, t]; B^{d/2-1,d/2}) \cap L^1([0, T]; B^{d/2+1,d/2+2})$ .

Since  $(\tilde{\mathbf{u}}^\ell)_{\ell \in \mathbb{N}}$  is uniformly bounded in  $\tilde{L}^\infty([0, T]; B^{d/2-1,d/2})$  and equicontinuous on  $[0, T]$  with values in  $B^{d/2-1,d/2} + B^{d/2-3/2,d/2-1/2}$ , it enables us to conclude that  $(\tilde{\mathbf{u}}^\ell)_{\ell \in \mathbb{N}}$  converges, up to an extraction, to some function  $\tilde{\mathbf{u}} \in \tilde{L}^\infty([0, T]; B^{d/2-1}) \cap L^1([0, T]; B^{d/2+1})$ .

Thus, we can pass to the limit in system (3.7) and set  $\mathbf{u} := \tilde{\mathbf{u}} + \mathbf{u}_{\text{is}}$ . Then  $(\rho, g, \mathbf{u})$  satisfies system (3.1).

*Step 5: Continuities in time.* From the first equation of (3.1), we get  $\partial_t \rho \in L^2([0, T]; B^{d/2-1,d/2})$ , which implies  $\rho \in \mathcal{C}([0, T]; B^{d/2-1,d/2})$ . So does  $g$  in the same space. For  $\mathbf{u}$ , we can derive, from the third equation of (3.1), that  $\partial_t \mathbf{u} \in (L^1 + L^2)([0, T]; B^{d/2-1,d/2})$ , which yields  $\mathbf{u} \in \mathcal{C}([0, T]; B^{d/2-1,d/2})$ .

**3.4. Uniqueness.** Let  $(\rho_1, g_1, \mathbf{u}_1)$  and  $(\rho_2, g_2, \mathbf{u}_2)$  be two solutions in  $F_T^1$  of (3.1) with the same initial data. Without loss of generality, we can assume that  $(\rho_2, g_2, \mathbf{u}_2)$  is the solution constructed in the previous subsection such that

$$1 + \inf_{(t,x) \in [0,T] \times \mathbb{R}^d} \rho_2(t, x) > 0.$$

We want to prove that  $(\rho_2, g_2, \mathbf{u}_2) \equiv (\rho_1, g_1, \mathbf{u}_1)$  on  $[0, T] \times \mathbb{R}^d$ . Toward this goal, we shall estimate the discrepancy  $(\delta\rho, \delta g, \delta\mathbf{u}) := (\rho_2 - \rho_1, g_2 - g_1, \mathbf{u}_2 - \mathbf{u}_1)$  with respect to a suitable norm, satisfying

$$(3.17) \quad \begin{cases} \partial_t \delta\rho + \mathbf{u}_2 \cdot \nabla \delta\rho + \delta\mathbf{u} \cdot \nabla \rho_1 = \delta\rho \operatorname{div} \mathbf{u}_2 + (\rho_1 + 1) \operatorname{div} \delta\mathbf{u}, \\ \partial_t \delta g + \mathbf{u}_2 \cdot \nabla \delta g + \delta\mathbf{u} \cdot \nabla g_1 = -\delta g \operatorname{div} \mathbf{u}_2 - (g_1 + \bar{n}) \operatorname{div} \delta\mathbf{u}, \\ \partial_t \delta\mathbf{u} + \mathbf{u}_2 \cdot \nabla \delta\mathbf{u} + \delta\mathbf{u} \cdot \nabla \mathbf{u}_1 - (1 + \rho_2)(\mu \Delta \delta\mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \delta\mathbf{u}) \\ \quad - \delta\rho(\mu \Delta \mathbf{u}_1 + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_1) + Q(\rho_2, g_2) - Q(\rho_1, g_1) = 0, \\ (\delta\rho, \delta g, \delta\mathbf{u})|_{t=0} = (0, 0, \mathbf{0}). \end{cases}$$

We shall prove the uniqueness in a larger function space

$$F_T := (\mathcal{C}([0, T]; B^{d/2}))^{1+1} \times (\mathcal{C}([0, T]; B^{d/2}) \cap L^1([0, T]; B^{d/2+1}))^d.$$

By Proposition 3.4, we get for all  $T' \in [0, T]$

$$\begin{aligned} & \|\delta\rho\|_{\tilde{L}^\infty([0, T']; B^{d/2})} \\ & \leq e^{C\|\mathbf{u}_2\|_{L^1([0, T']; B^{d/2+1})}} \int_0^{T'} \left( \|\delta\mathbf{u} \cdot \nabla\rho_1\|_{B^{d/2}} + \|\delta\rho\operatorname{div}\mathbf{u}_2\|_{B^{d/2}} \right. \\ & \quad \left. + \|(\rho_1 + 1)\operatorname{div}\delta\mathbf{u}\|_{B^{d/2}} \right) d\tau \\ & \lesssim e^{C\|\mathbf{u}_2\|_{L^1([0, T']; B^{d/2+1})}} \int_0^{T'} \left[ \|\delta\mathbf{u}\|_{B^{d/2}} \|\rho_1\|_{B^{d/2+1}} + \|\delta\rho\|_{B^{d/2}} \|\mathbf{u}_2\|_{B^{d/2+1}} \right. \\ & \quad \left. + (1 + \|\rho_1\|_{B^{d/2}}) \|\delta\mathbf{u}\|_{B^{d/2+1}} \right] d\tau. \end{aligned}$$

Using the Gronwall inequality, this yields

$$\begin{aligned} (3.18) \quad & \|\delta\rho\|_{\tilde{L}^\infty([0, T']; B^{d/2})} \\ & \lesssim e^{C\|\mathbf{u}_2\|_{L^1([0, T']; B^{d/2+1})}} \int_0^{T'} \left[ \|\delta\mathbf{u}\|_{B^{d/2}} \|\rho_1\|_{B^{d/2+1}} + (1 + \|\rho_1\|_{B^{d/2}}) \|\delta\mathbf{u}\|_{B^{d/2+1}} \right] d\tau \\ & \leq C_T \left( \|\delta\mathbf{u}\|_{L^2([0, T']; B^{d/2})} + \|\delta\mathbf{u}\|_{L^1([0, T']; B^{d/2+1})} \right), \end{aligned}$$

where  $C_T$  is independent of  $T'$ .

Similarly, we have

$$(3.19) \quad \|\delta g\|_{\tilde{L}^\infty([0, T']; B^{d/2})} \leq C_T \left( \|\delta\mathbf{u}\|_{L^2([0, T']; B^{d/2})} + \|\delta\mathbf{u}\|_{L^1([0, T']; B^{d/2+1})} \right).$$

Applying Lemmas 3.2 and A.6 to the third equation of (3.17) yields

$$\begin{aligned} & \|\delta\mathbf{u}\|_{\tilde{L}^\infty([0, T']; B^{d/2-1})} + \|\delta\mathbf{u}\|_{L^1([0, T']; B^{d/2+1})} \\ & \leq C e^{C \int_0^{T'} [\|\mathbf{u}_1\|_{B^{d/2+1}} + \|\mathbf{u}_2\|_{B^{d/2+1}}] d\tau} \int_0^{T'} \left( \|\delta\rho\|_{B^{d/2}} \|\mathbf{u}_1\|_{B^{d/2+1}} \right. \\ & \quad \left. + \|Q(\rho_2, g_2) - Q(\rho_1, g_1)\|_{B^{d/2-1}} \right) d\tau. \end{aligned}$$

By Lemma A.3, we get

$$\|Q(\rho_2, g_2) - Q(\rho_1, g_1)\|_{B^{d/2-1}} \lesssim (1 + \|(\rho_1, \rho_2, g_1, g_2)\|_{B^{d/2}})^3 (\|\delta\rho\|_{B^{d/2}} + \|\delta g\|_{B^{d/2}}).$$

Thus, it follows that

$$(3.20) \quad \begin{aligned} & \|\delta\mathbf{u}\|_{\tilde{L}^\infty([0, T']; B^{d/2-1})} + \|\delta\mathbf{u}\|_{L^1([0, T']; B^{d/2+1})} \\ & \leq C_T (T' + T'^{1/2}) \left( \|\delta\rho\|_{L^\infty([0, T']; B^{d/2})} + \|\delta g\|_{L^\infty([0, T']; B^{d/2})} \right), \end{aligned}$$

since  $\mathbf{u}_1 \in L^2([0, T]; B^{d/2+1})$  by Lemma A.9.



From (3.18)–(3.20), this yields, with the help of Lemma A.9, that

$$\begin{aligned} & \|\delta \mathbf{u}\|_{\tilde{L}^\infty([0, T']; B^{d/2-1})} + \|\delta \mathbf{u}\|_{L^1([0, T']; B^{d/2+1})} \\ & \leq C_T(T' + T'^{1/2})(\|\delta \mathbf{u}\|_{L^2([0, T']; B^{d/2})} + \|\delta \mathbf{u}\|_{L^1([0, T']; B^{d/2+1})}) \\ & \leq C_T(T' + T'^{1/2})(\|\delta \mathbf{u}\|_{\tilde{L}^\infty([0, T']; B^{d/2-1})} + \|\delta \mathbf{u}\|_{L^1([0, T']; B^{d/2+1})}). \end{aligned}$$

Therefore, if we choose  $T'$  so small that  $C_T(T' + T'^{1/2}) < 1$ , then we obtain that  $(\delta \rho, \delta g, \delta \mathbf{u}) = (0, 0, \mathbf{0})$  on the time interval  $[0, T']$ . As in the proof of uniqueness for global well-posedness, we can extend  $T'$  to  $T$  by the translation with respect to the time variable, i.e.,  $(\delta \rho, \delta g, \delta \mathbf{u}) = (0, 0, \mathbf{0})$  on the time interval  $[0, T]$ .

**3.5. A continuation criterion.**

PROPOSITION 3.6. *Under the hypotheses of Theorem 3.1, assume that system (3.1) has a solution  $(\rho, g, \mathbf{u})$  on  $[0, T) \times \mathbb{R}^d$  which belongs to  $F_{T'}^1$ , for all  $T' < T$  and satisfies*

$$\rho, g \in L^\infty([0, T]; B^{d/2, d/2+1}), \quad \inf_{(t,x) \in [0, T) \times \mathbb{R}^d} \rho(t, x) > -1, \quad \int_0^T \|\nabla \mathbf{u}\|_\infty dt < \infty.$$

Then there exists some  $T^* > T$  such that  $(\rho, g, \mathbf{u})$  may be continued on  $[0, T^*) \times \mathbb{R}^d$  to a solution of (3.1) which belongs to  $F_{T^*}^1$ .

*Proof.* Recall that  $\mathbf{u}$  satisfies

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - (1 + \rho)(\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) + Q(\rho, g) = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0.$$

By Lemma 3.2, we get, for  $T' < T$ , that

$$\begin{aligned} & \|\mathbf{u}\|_{\tilde{L}^\infty([0, T']; B^{d/2-1, d/2})} + \mathcal{L}\|\mathbf{u}\|_{L^1([0, T']; B^{d/2+1, d/2+2})} \\ & \leq C e^{C \int_0^{T'} (\|\nabla \mathbf{u}\|_\infty + \|\rho\|_{B^{d/2+1}}^2) dt} \left( \|\mathbf{u}_0\|_{B^{d/2-1, d/2}} + \int_0^{T'} \|\rho\|_{B^{d/2, d/2+1}} dt \right) \end{aligned}$$

for some constant  $C$  depending only on  $d$  and viscosity coefficients. Thus, there exists a constant  $\varepsilon > 0$  such that (3.1) with initial data  $(\rho(T - \varepsilon), g(T - \varepsilon), \mathbf{u}(T - \varepsilon))$  yields a solution on  $[0, 2\varepsilon]$ . Since the solution  $(\rho, g, \mathbf{u})$  is unique on  $[0, T)$ , this provides a continuation of  $(\rho, g, \mathbf{u})$  beyond  $T$ .  $\square$

**Appendix. Littlewood–Paley theory and Besov spaces.** This section is devoted to recalling some properties of Littlewood–Paley theory and Besov spaces which will be used in this paper. For more details, see [6, 12] and the references therein.

Let  $\psi : \mathbb{R}^d \rightarrow [0, 1]$  be a radial smooth cut-off function valued in  $[0, 1]$  such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leq 3/4, \\ \text{smooth}, & 3/4 < |\xi| < 4/3, \\ 0, & |\xi| \geq 4/3. \end{cases}$$

Let  $\varphi(\xi)$  be the function

$$\varphi(\xi) := \psi(\xi/2) - \psi(\xi).$$

Thus,  $\psi$  is supported in the ball  $\{\xi \in \mathbb{R}^d : |\xi| \leq 4/3\}$ , and  $\varphi$  is also a smooth cut-off function valued in  $[0, 1]$  and supported in the annulus  $\{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}$ . By construction, we have

$$\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1 \quad \forall \xi \neq 0.$$

One can define the dyadic blocks as follows. For  $k \in \mathbb{Z}$ , let

$$\Delta_k f := \mathcal{F}^{-1} \varphi(2^{-k}\xi) \mathcal{F} f.$$

The formal decomposition

$$(A.1) \quad f = \sum_{k \in \mathbb{Z}} \Delta_k f$$

is called the homogeneous Littlewood–Paley decomposition. Nevertheless, (A.1) is true modulo polynomials; in other words (cf. [17]), if  $f \in \mathcal{S}'(\mathbb{R}^d)$ , then  $\sum_{k \in \mathbb{Z}} \Delta_k f$  converges modulo  $\mathcal{P}[\mathbb{R}^d]$  and (A.1) holds in  $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}[\mathbb{R}^d]$ .

DEFINITION A.1. *Let  $s \in \mathbb{R}$ . For  $f \in \mathcal{S}'(\mathbb{R}^d)$ , we write*

$$\|f\|_{B^s} = \sum_{k \in \mathbb{Z}} 2^{ks} \|\Delta_k f\|_2.$$

A difficulty comes from the choice of homogeneous spaces at this point. Indeed,  $\|\cdot\|_{B^s}$  cannot be a norm on  $\{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s} < \infty\}$  because  $\|f\|_{B^s} = 0$  means that  $f$  is a polynomial. This forces us to adopt the following definition for homogeneous Besov spaces (cf. [6]).

DEFINITION A.2. *Let  $s \in \mathbb{R}$  and  $m = -[d/2 + 1 - s]$ . If  $m < 0$ , then we define  $B^s(\mathbb{R}^d)$  as*

$$B^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s} < \infty \text{ and } f = \sum_{k \in \mathbb{Z}} \Delta_k f \text{ in } \mathcal{S}'(\mathbb{R}^d) \right\}.$$

If  $m \geq 0$ , we denote by  $\mathcal{P}_m$  the set of  $d$  variable polynomials of degree less than or equal to  $m$  and define

$$B^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_m : \|f\|_{B^s} < \infty \text{ and } f = \sum_{k \in \mathbb{Z}} \Delta_k f \text{ in } \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_m \right\}.$$

For the composition of functions, we have the following estimates.

LEMMA A.3. *Let  $s > 0$  and  $u \in B^s \cap L^\infty$ . Then the following hold:*

(i) *Let  $F \in W_{loc}^{[s]+2, \infty}(\mathbb{R}^d)$  with  $F(0) = 0$ . Then  $F(u) \in B^s$ . Moreover, there exists a function of one variable  $C_0$  depending only on  $s$  and  $F$ , and such that*

$$\|F(u)\|_{B^s} \leq C_0(\|u\|_{L^\infty}) \|u\|_{B^s}.$$

(ii) *If  $u, v \in B^{d/2}$ ,  $(v - u) \in B^s$  for  $s \in (-d/2, d/2]$ , and  $G \in W_{loc}^{[d/2]+3, \infty}(\mathbb{R}^d)$  satisfies  $G'(0) = 0$ , then  $G(v) - G(u) \in B^s$ , and there exists a function of two variables  $C$  depending only on  $s, N$ , and  $G$ , and such that*

$$\|G(v) - G(u)\|_{B^s} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) (\|u\|_{B^{d/2}} + \|v\|_{B^{d/2}}) \|v - u\|_{B^s}.$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies [6]. We are going to recall the definition of these new spaces and some of their main properties.

DEFINITION A.4. *Let  $s, t \in \mathbb{R}$ . We define*

$$\|f\|_{B^{s,t}} = \sum_{k \leq 0} 2^{ks} \|\Delta_k f\|_2 + \sum_{k > 0} 2^{kt} \|\Delta_k f\|_2.$$

Let  $m = -[d/2 + 1 - s]$ ; we then define

$$\begin{aligned} B^{s,t}(\mathbb{R}^d) &= \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^{s,t}} < \infty\} \quad \text{if } m < 0, \\ B^{s,t}(\mathbb{R}^d) &= \{f \in \mathcal{S}'(\mathbb{R}^d) / \mathcal{P}_m : \|f\|_{B^{s,t}} < \infty\} \quad \text{if } m \geq 0. \end{aligned}$$

LEMMA A.5. *We have the following inclusions for hybrid Besov spaces:*

- (i) *We have  $B^{s,s} = B^s$ .*
- (ii) *If  $s \leq t$ , then  $B^{s,t} = B^s \cap B^t$ . Otherwise,  $B^{s,t} = B^s + B^t$ .*
- (iii) *The space  $B^{0,s}$  coincides with the usual inhomogeneous Besov space  $B_{2,1}^s$ .*
- (iv) *If  $s_1 \leq s_2$  and  $t_1 \geq t_2$ , then  $B^{s_1,t_1} \hookrightarrow B^{s_2,t_2}$ .*

Let us now recall some useful estimates for the product in hybrid Besov spaces.

LEMMA A.6. *Let  $s_1, s_2 > 0$  and  $f, g \in L^\infty \cap B^{s_1,s_2}$ . Then  $fg \in B^{s_1,s_2}$  and*

$$\|fg\|_{B^{s_1,s_2}} \lesssim \|f\|_{L^\infty} \|g\|_{B^{s_1,s_2}} + \|f\|_{B^{s_1,s_2}} \|g\|_{L^\infty}.$$

Let  $s \in (-d/2, d/2]$ ,  $f \in B^{d/2}$ , and  $g \in B^s$ ; then  $fg \in B^s$  and

$$\|fg\|_{B^s} \lesssim \|f\|_{B^{d/2}} \|g\|_{B^s}.$$

Let  $s_1, s_2, t_1, t_2 \leq d/2$  such that  $\min(s_1 + s_2, t_1 + t_2) > 0$ ,  $f \in B^{s_1,t_1}$ , and  $g \in B^{s_2,t_2}$ . Then  $fg \in B^{s_1+s_2-1, t_1+t_2-1}$  and

$$\|fg\|_{B^{s_1+s_2-d/2, t_1+t_2-d/2}} \lesssim \|f\|_{B^{s_1,t_1}} \|g\|_{B^{s_2,t_2}}.$$

For  $\alpha, \beta \in \mathbb{R}$ , let us define the following characteristic function on  $\mathbb{Z}$ :

$$\tilde{\varphi}^{\alpha,\beta}(k) = \begin{cases} \alpha & \text{if } k \leq 0, \\ \beta & \text{if } k \geq 1. \end{cases}$$

Then we can recall the following lemma.

LEMMA A.7. *Let  $F$  be a homogeneous smooth function of degree  $m$ . Suppose that  $-d/2 < s_1, t_1, s_2, t_2 \leq 1 + d/2$ . The following two estimates hold:*

$$\begin{aligned} & |(F(D)\Delta_k(\mathbf{v} \cdot \nabla a), F(D)\Delta_k a)| \\ & \lesssim \gamma_k 2^{-k(\tilde{\varphi}^{s_1,s_2}(k)-m)} \|\mathbf{v}\|_{B^{d/2+1}} \|a\|_{B^{s_1,s_2}} \|F(D)\Delta_k a\|_2, \\ & |(F(D)\Delta_k(\mathbf{v} \cdot \nabla a), \Delta_k b) + (\Delta_k(\mathbf{v} \cdot \nabla b), F(D)\Delta_k a)| \\ & \lesssim \gamma_k \|\mathbf{v}\|_{B^{d/2+1}} \times (2^{-k\tilde{\varphi}^{t_1,t_2}(k)} \|F(D)\Delta_k a\|_2 \|b\|_{B^{t_1,t_2}} \\ & \quad + 2^{-k(\tilde{\varphi}^{s_1,s_2}(k)-m)} \|a\|_{B^{s_1,s_2}} \|\Delta_k b\|_2), \end{aligned}$$

where  $(\cdot, \cdot)$  denotes the 2-inner product,  $\sum_{k \in \mathbb{Z}} \gamma_k \leq 1$ , and the operator  $F(D)$  is defined by  $F(D)f := \mathcal{F}^{-1}F(\xi)\mathcal{F}f$ .

In the context of this paper, we also need to use the interpolation spaces of hybrid Besov spaces together with a time space such as  $L^p([0, T]; B^{s,t})$ . Thus, we have to

introduce the Chemin–Lerner-type space (cf. [4]), which is a refinement of the space  $L^p([0, T]; B^{s,t})$ .

DEFINITION A.8. *Let  $p \in [1, \infty]$ ,  $T \in (0, \infty]$ , and  $s_1, s_2 \in \mathbb{R}$ . Then we define*

$$\|f\|_{\tilde{L}^p([0,T];B^{s,t})} = \sum_{k \leq 0} 2^{ks} \|\Delta_k f\|_{L^p([0,T];L^2)} + \sum_{k > 0} 2^{kt} \|\Delta_k f\|_{L^p([0,T];L^2)}.$$

Noting that Minkowski’s inequality yields  $\|f\|_{L^p([0,T];B^{s,t})} \leq \|f\|_{\tilde{L}^p([0,T];B^{s,t})}$ , we define spaces  $\tilde{L}^p([0, T]; B^{s,t})$  as follows:

$$\tilde{L}^p([0, T]; B^{s,t}) = \{f \in L^p([0, T]; B^{s,t}) : \|f\|_{\tilde{L}^p([0,T];B^{s,t})} < \infty\}.$$

If  $T = \infty$ , then we omit the subscript  $T$  from the notation  $\tilde{L}^p([0, T]; B^{s,t})$ , that is,  $\tilde{L}^p(B^{s,t})$  for simplicity. We will denote by  $\tilde{C}([0, T]; B^{s,t})$  the subset of functions of  $\tilde{L}^\infty([0, T]; B^{s,t})$  which are continuous on  $[0, T]$  with values in  $B^{s,t}$ .

Let us observe that  $L^1([0, T]; B^{s,t}) = \tilde{L}^1([0, T]; B^{s,t})$ , but the embedding

$$\tilde{L}^p([0, T]; B^{s,t}) \subset L^p([0, T]; B^{s,t})$$

is strict if  $p > 1$ .

We will use the following interpolation property which can be verified easily (cf. [1, 2]).

LEMMA A.9. *Let  $s, t, s_1, t_1, s_2, t_2 \in \mathbb{R}$  and  $p, p_1, p_2 \in [1, \infty]$ . We have*

$$\|f\|_{\tilde{L}^p([0,T];B^{s,t})} \leq \|f\|_{\tilde{L}^{p_1}([0,T];B^{s_1,t_1})}^\theta \|f\|_{\tilde{L}^{p_2}([0,T];B^{s_2,t_2})}^{1-\theta}$$

where  $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ ,  $s = \theta s_1 + (1 - \theta)s_2$ , and  $t = \theta t_1 + (1 - \theta)t_2$ .

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