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Cauchy problem for viscous rotating shallow water equations

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ABSTRACT

We consider the Cauchy problem for a viscous compressible rotating shallow water system with a third-order surface-tension term involved, derived recently in the modeling of motions for shallow water with free surface in a rotating sub-domain Marche (2007) [19]. The global existence of the solution in the space of Besov type is shown for initial data close to a constant equilibrium state away from the vacuum. Unlike the previous analysis about the compressible fluid model without Coriolis forces, see for instance Danchin (2000) [10], Haspot (2009) [16], the rotating effect causes a coupling between two parts of Hodge's decomposition of the velocity vector field, and additional regularity is required in order to carry out the Friedrichs' regularization and compactness arguments.

1. Introduction

The nonlinear shallow water equation is used to model the motion of a shallow layer of homogeneous incompressible fluid in a three-dimensional rotating sub-domain and, in particular, to simulate the vertical average dynamics of the fluid in terms of the horizontal velocity and depth variation. In general, it is modeled by the three-dimensional incompressible Navier–Stokes–Coriolis system in a rotating sub-domain of $\mathbb{R}^3$ together with a (nonlinear) free moving surface boundary condition for which the stress tension is evolved at the air–fluid interface from above and the Navier boundary condition of wall-law type holds at the bottom. Under a large-scale assumption and hydrostatic ap-

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proximation, the nonlinear shallow water equation has been derived recently in [14,19]. Usually, the nonlinear shallow water equations take the following form of compressible Navier–Stokes equations

$$\begin{cases} h_t + \operatorname{div}(h\mathbf{u}) = 0, \\ (h\mathbf{u})_t + \operatorname{div}(h\mathbf{u}\otimes\mathbf{u}) + \mathrm{g}h\nabla h + \mathrm{f}(h\mathbf{u})^{\perp} = \operatorname{div}\big(2\xi(h)D(\mathbf{u})\big) + \nabla\big(\lambda(h)\operatorname{div}\mathbf{u}\big), \\ h(0) = h_0, \quad \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1}$$

where $h(t,x)$ is the height of the fluid surface, $\mathbf{u}(t,x) = (u^1(t,x), u^2(t,x))^{\top}$ is the horizontal velocity field, $\mathbf{u}^{\perp}(t,x) = (-u^2(t,x), u^1(t,x))$, $x = (x_1, x_2) \in \mathbb{R}^2$, $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\top})$, $\mathrm{g} > 0$ is the gravitational acceleration, $\mathrm{f} > 0$ is the Coriolis frequency, $\xi \geqslant 0$ and λ are the dynamical viscosities satisfying $\lambda + \xi \geqslant 0$.

For the shallow water system (1), there is a mount of work to deal with the global well-posedness of strong solutions subject to some small initial perturbation of a constant state or the global existence of weak solutions for large initial data. When the viscosities satisfy $\xi(h) = h$ and $\lambda = 0$, and the effect of the Coriolis force and/or third-order surface tension term is omitted ($f = 0$, $\beta = 0$), the local existence and uniqueness of classical solutions to the Cauchy–Dirichlet problem for the shallow water equations with initial data in $C^{2+\alpha}$ was studied in [4] using Lagrangian coordinates and Hölder space estimates. Kloeden and Sundbye [17,23] proved the global existence and uniqueness of classical solutions to the Cauchy–Dirichlet problem using Sobolev space estimates by following the energy method of Matsumura and Nishida [20]. Sundbye [24] proved also the existence and uniqueness of classical solutions to the Cauchy problem using the method of [20]. Wang and Xu, in [25], obtained local solutions for any initial data and global solutions for small initial data $h_0 - \bar{h}_0, \mathbf{u}_0 \in H^{2+s}(\mathbb{R}^2)$ with $s > 0$. The result was improved by Haspot to get global existence in time for small initial data $h_0 - \bar{h}_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}$ and $\mathbf{u}_0 \in \dot{B}^0_{2,1}$ as a special case in [16], and by Chen, Miao and Zhang in [8] to prove the local existence in time for general initial data and the global existence in time for small initial data where $h_0 - \bar{h}_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}$ and $\mathbf{u}_0 \in \dot{B}^0_{2,1}$ with additional conditions that $h \geqslant \bar{h}_0$ and $\bar{h}_0$ is a strictly positive constant. Cheng and Tadmor discussed the long time existence of approximate periodic solutions for the rapidly rotating shallow water for initial data $(h_0, \mathbf{u}_0) \in H^m(\mathbb{T}^2)$ with $m > 5$ where the viscous terms are absent (i.e. $\xi = \lambda = 0$) in [9]. The global existence of weak solutions for arbitrarily large initial data is established in one dimension [18], where the vanishing of vacuum states in finite time is shown, and in multi-dimensional bounded domain with spherical symmetry [15] with the help of the Bresch–Desjardins entropy [2] and the L^1-stability compactness argument [21]. The global existence of weak solutions for arbitrarily large initial data is shown by Bresch and Desjardins [3] where additional drag friction and capillary terms are involved to construct a global approximate solutions. The related systems with a third-order term stemming from the capillary tensor also have been considered by Danchin–Desjardins for a compressible fluid model of Korteweg type [13] with constant viscosity coefficients, and the global existence of strong solution is shown.

In the present paper, we consider the global existence of the Cauchy problem for the 2D viscous shallow water equations

$$\begin{cases} h_t + \operatorname{div}(h\mathbf{u}) = 0, \\ (h\mathbf{u})_t + \operatorname{div}(h\mathbf{u}\otimes\mathbf{u}) + \mathrm{g}h\nabla h + \mathrm{f}(h\mathbf{u})^{\perp} = 2\mu\operatorname{div}\big(hD(\mathbf{u})\big) + 2\mu\nabla(h\operatorname{div}\mathbf{u}) + \beta h\nabla\Delta h, \\ h(0) = h_0, \quad \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{2}$$

which corresponds to (1) for the case $2\xi(h) = \lambda(h) = 2\mu h$ with $\mu > 0$ a constant, and is derived recently in [19] with a third-order surface tension term involved by considering second-order approximation and parabolic correction where $\beta > 0$ is the capillary coefficient. Although there are many mathematical results about the shallow water equations (1), there is no analysis about Eq. (2). It also should be mentioned that the global existence of weak solutions does not apply here since the Bresch–Desjardins entropy [2] is not satisfied for Eq. (2). In addition, the classical theory does not cover the case with Coriolis force and capillarity term involved.

We investigate the global existence of strong solution in some Besov space. Although we also make use of the Hodge's decomposition to separate the velocity field into a compressible part and an incompressible part, unlike [10,11] we finally obtain a coupled system due to the rotating effect of the Coriolis force. In fact, it cannot be decoupled into a system involving only the compressible part and a heat equation containing only the incompressible part because of the appearance of the Coriolis frequency, which leads to a strong coupling between the gradient vector field part and divergence free part of the fluid velocity in terms of the Hodge's decomposition. Thus, we have to investigate the whole system of the height, the compressible velocity field part and the incompressible velocity field part. With the help of the Littlewood–Paley analysis and hybrid Besov spaces, we obtain the a priori estimates in Chemin–Lerner type time-spatial spaces which are necessary in order to use the interpolation theory of time-spatial spaces involving hybrid Besov spaces. Then we use a classical Friedrichs' regularization method to construct approximate solutions and prove the existence of a solution by compactness arguments. For the uniqueness of solutions, due to the contribution of the third-order surface tension term, we can prove it in a larger space than that for the existence and we do not need more regularity on the spaces.

For the convenience of the statement of main results, we note that $\tilde{B}^{s_1,s_2}_{2,1}$ is a hybrid Besov space defined in the next section, the space E^s is defined by

$$E^s = \left\{(h,\mathbf{u}) \in \tilde{\mathcal{C}}\big([0,\infty);\, \tilde{B}^{s-1,s}_{2,1}\big) \cap L^1\big(0,\infty;\, \tilde{B}^{s+3,s+2}_{2,1}\big) \times \big(\tilde{\mathcal{C}}\big([0,\infty);\, \dot{B}^{s-1}_{2,1}\big) \cap L^1\big(0,\infty;\, \dot{B}^{s+1}_{2,1}\big)\big)^2\right\},$$

and $\tilde{\mathcal{C}}([0,\infty);\, \tilde{B}^{s_1,s_2}_{2,1})$ is the subset of functions of the Chemin–Lerner type space $\tilde{L}^\infty_T(\tilde{B}^{s_1,s_2}_{2,1})$ defined in the next section which are continuous on $[0,\infty)$ with values in $\tilde{B}^{s_1,s_2}_{2,1}$.

For the initial data h_0, we suppose that it is a small perturbation of some positive constant $\bar{h}_0$. The main theorem of this paper reads as follows.

Theorem 1.1. *Let $\varepsilon \in (0,1)$, $h_0 - \bar{h}_0 \in \tilde{B}^{0,1+\varepsilon}_{2,1}$ and $\mathbf{u}_0 \in \tilde{B}^{0,\varepsilon}_{2,1}$. Then, there exist two positive constants α small enough and M such that if*

$$\|h_0 - \bar{h}_0\|_{\tilde{B}^{0,1+\varepsilon}_{2,1}} + \|\mathbf{u}_0\|_{\tilde{B}^{0,\varepsilon}_{2,1}} \leqslant \alpha,$$

then (2) *yields a unique global solution $(h,\mathbf{u})$ in $(\bar{h}_0,\mathbf{0}) + (E^1 \cap E^{1+\varepsilon})$ which satisfies*

$$\big\|(h-\bar{h}_0,\mathbf{u})\big\|_{E^1\cap E^{1+\varepsilon}} \leqslant M\big(\|h_0 - \bar{h}_0\|_{\tilde{B}^{0,1+\varepsilon}_{2,1}} + \|\mathbf{u}_0\|_{\tilde{B}^{0,\varepsilon}_{2,1}}\big),$$

where M is independent of the initial data.

The paper is organized as follows. We recall some Littlewood–Paley theories for homogeneous Besov spaces and give the definitions and some properties of hybrid Besov spaces and Chemin–Lerner type spaces in the second section. In Section 3, we are dedicated into proving of the a priori estimates. In Section 4, we prove the global existence and uniqueness of solution for small initial data by using a classical Friedrichs' regularization method and compactness arguments.

2. Littlewood–Paley theory and Besov spaces

Let $\psi : \mathbb{R}^2 \to [0,1]$ be a radial smooth cut-off function valued in $[0,1]$ such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leqslant 3/4, \\ \text{smooth}, & 3/4 < |\xi| < 4/3, \\ 0, & |\xi| \geqslant 4/3. \end{cases}$$

Let $\varphi(\xi)$ be the function

$$\varphi(\xi) := \psi(\xi/2) - \psi(\xi).$$

Thus, ψ is supported in the ball $\{\xi \in \mathbb{R}^2\colon\ |\xi| \leqslant 4/3\}$, and φ is also a smooth cut-off function valued in $[0,1]$ and supported in the annulus $\{\xi\colon\ 3/4 \leqslant |\xi| \leqslant 8/3\}$. By construction, we have

$$\sum_{k\in\mathbb{Z}} \varphi\big(2^{-k}\xi\big) = 1, \quad \forall \xi \neq 0.$$

One can define the dyadic blocks as follows. For $k \in \mathbb{Z}$, let

$$\Delta_k f := \mathscr{F}^{-1}\varphi\big(2^{-k}\xi\big)\mathscr{F} f.$$

The formal decomposition

$$f = \sum_{k\in\mathbb{Z}} \Delta_k f \tag{3}$$

is called homogeneous Littlewood–Paley decomposition. Actually, this decomposition works for just about any locally integrable function which yields some decay at infinity, and one usually has all the convergence properties of the summation that one needs. Thus, the r.h.s. of (3) does not necessarily converge in $\mathscr{S}'(\mathbb{R}^2)$. Even if it does, the equality is not always true in $\mathscr{S}'(\mathbb{R}^2)$. For instance, if $f \equiv 1$, then all the projections $\Delta_k f$ vanish. Nevertheless, (3) is true modulo polynomials, in other words (cf. [12,22]), if $f \in \mathscr{S}'(\mathbb{R}^2)$, then $\sum_{k\in\mathbb{Z}} \Delta_k f$ converges modulo $\mathscr{P}[\mathbb{R}^2]$ and (3) holds in $\mathscr{S}'(\mathbb{R}^2)/\mathscr{P}[\mathbb{R}^2]$.

Definition 2.1. Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. For $f \in \mathscr{S}'(\mathbb{R}^2)$, we write

$$\|f\|_{\dot{B}^s_{2,1}} = \sum_{k\in\mathbb{Z}} 2^{ks}\|\Delta_k f\|_{L^2}.$$

A difficulty comes from the choice of homogeneous spaces at this point. Indeed, $\|\cdot\|_{\dot{B}^s_{2,1}}$ cannot be a norm on $\{f \in \mathscr{S}'(\mathbb{R}^2)\colon\ \|f\|_{\dot{B}^s_{2,1}} < \infty\}$ because $\|f\|_{\dot{B}^s_{2,1}} = 0$ means that f is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces (cf. [11]).

Definition 2.2. Let $s \in \mathbb{R}$ and $m = -[2-s]$. If $m < 0$, then we define $\dot{B}^s_{2,1}(\mathbb{R}^2)$ as

$$\dot{B}^s_{2,1} = \left\{ f \in \mathscr{S}'\big(\mathbb{R}^2\big)\colon\ \|f\|_{\dot{B}^s_{2,1}} < \infty \text{ and } f = \sum_{k\in\mathbb{Z}} \Delta_k f \text{ in } \mathscr{S}'\big(\mathbb{R}^2\big) \right\}.$$

If $m \geqslant 0$, we denote by $\mathscr{P}_m$ the set of two variables polynomials of degree less than or equal to m and define

$$\dot{B}^s_{2,1} = \left\{ f \in \mathscr{S}'\big(\mathbb{R}^2\big)/\mathscr{P}_m\colon\ \|f\|_{\dot{B}^s_{2,1}} < \infty \text{ and } f = \sum_{k\in\mathbb{Z}} \Delta_k f \text{ in } \mathscr{S}'\big(\mathbb{R}^2\big)/\mathscr{P}_m \right\}.$$

For the composition of functions, we have the following estimates.

Lemma 2.3. *(See [11, Lemma 2.7].) Let $s > 0$ and $u \in \dot{B}^s_{2,1} \cap L^\infty$.*

(i) *Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^2)$ such that $F(0) = 0$. Then $F(u) \in \dot{B}^s_{2,1}$. Moreover, there exists a function of one variable C_0 depending only on s and F, and such that*

$$\|F(u)\|_{\dot{B}^s_{2,1}} \leqslant C_0\big(\|u\|_{L^\infty}\big)\|u\|_{\dot{B}^s_{2,1}}.$$

(ii) *If $u, v \in \dot{B}^1_{2,1}$, $(v-u) \in \dot{B}^s_{2,1}$ for an $s \in (-1, 1]$ and $G \in W^{4,\infty}_{loc}(\mathbb{R}^2)$ satisfies $G'(0) = 0$, then $G(v) - G(u) \in \dot{B}^s_{2,1}$ and there exists a function of two variables C depending only on s and G, and such that*

$$\|G(v) - G(u)\|_{\dot{B}^s_{2,1}} \leqslant C\big(\|u\|_{L^\infty}, \|v\|_{L^\infty}\big)\big(\|u\|_{\dot{B}^1_{2,1}} + \|v\|_{\dot{B}^1_{2,1}}\big)\|v-u\|_{\dot{B}^s_{2,1}}.$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies [11]. We are going to recall the definition of these new spaces and some of their main properties.

Definition 2.4. Let $s, t \in \mathbb{R}$. We define

$$\|f\|_{\tilde{B}^{s,t}_{2,1}} = \sum_{k\leqslant 0} 2^{ks}\|\Delta_k f\|_{L^2} + \sum_{k>0} 2^{kt}\|\Delta_k f\|_{L^2}.$$

Let $m = -[2-s]$, we then define

$$\tilde{B}^{s,t}_{2,1}\big(\mathbb{R}^2\big) = \big\{f \in \mathscr{S}'\big(\mathbb{R}^2\big)\colon \|f\|_{\tilde{B}^{s,t}_{2,1}} < \infty\big\}, \quad \text{if } m < 0,$$

$$\tilde{B}^{s,t}_{2,1}\big(\mathbb{R}^2\big) = \big\{f \in \mathscr{S}'\big(\mathbb{R}^2\big)/\mathscr{P}_m\colon \|f\|_{\tilde{B}^{s,t}_{2,1}} < \infty\big\}, \quad \text{if } m \geqslant 0.$$

Lemma 2.5. *We have the following inclusions.*

(i) *We have $\tilde{B}^{s,s}_{2,1} = \dot{B}^s_{2,1}$.*
(ii) *If $s \leqslant t$, then $\tilde{B}^{s,t}_{2,1} = \dot{B}^s_{2,1} \cap \dot{B}^t_{2,1}$. Otherwise, $\tilde{B}^{s,t}_{2,1} = \dot{B}^s_{2,1} + \dot{B}^t_{2,1}$.*
(iii) *The space $\tilde{B}^{0,s}_{2,1}$ coincides with the usual inhomogeneous Besov space $B^s_{2,1}$.*
(iv) *If $s_1 \leqslant s_2$ and $t_1 \geqslant t_2$, then $\tilde{B}^{s_1,t_1}_{2,1} \hookrightarrow \tilde{B}^{s_2,t_2}_{2,1}$.*

Let us now recall some useful estimates for the product in hybrid Besov spaces.

Lemma 2.6. *(See [11, Proposition 2.10].) Let $s_1, s_2 > 0$ and $f, g \in L^\infty \cap \tilde{B}^{s_1,s_2}_{2,1}$. Then $fg \in \tilde{B}^{s_1,s_2}_{2,1}$ and*

$$\|fg\|_{\tilde{B}^{s_1,s_2}_{2,1}} \lesssim \|f\|_{L^\infty}\|g\|_{\tilde{B}^{s_1,s_2}_{2,1}} + \|f\|_{\tilde{B}^{s_1,s_2}_{2,1}}\|g\|_{L^\infty}.$$

Let $s_1, s_2, t_1, t_2 \leqslant 1$ such that $\min(s_1+s_2, t_1+t_2) > 0$, $f \in \tilde{B}^{s_1,t_1}_{2,1}$ and $g \in \tilde{B}^{s_2,t_2}_{2,1}$. Then $fg \in \tilde{B}^{s_1+s_2-1,t_1+t_2-1}_{2,1}$ and

$$\|fg\|_{\tilde{B}^{s_1+s_2-1,t_1+t_2-1}_{2,1}} \lesssim \|f\|_{\tilde{B}^{s_1,t_1}_{2,1}}\|g\|_{\tilde{B}^{s_2,t_2}_{2,1}}.$$

In the context of this paper, we also need to use the interpolation spaces of hybrid Besov spaces together with a time space such as $L^p(0,T;\tilde{B}^{s,t}_{2,1})$. Thus, we have to introduce the Chemin–Lerner type space (cf. [5]) which is a refinement of the space $L^p(0,T;\tilde{B}^{s,t}_{2,1})$.

Definition 2.7. Let $p\in[1,\infty]$, $T\in(0,\infty]$ and $s_1,s_2\in\mathbb{R}$. Then we define

$$\|f\|_{\tilde{L}^p_T(\tilde{B}^{s,t}_{2,1})}=\sum_{k\leqslant 0}2^{ks}\|\Delta_k f\|_{L^p(0,T;L^2)}+\sum_{k>0}2^{kt}\|\Delta_k f\|_{L^p(0,T;L^2)}.$$

Noting that Minkowski's inequality yields $\|f\|_{L^p_T(\tilde{B}^{s,t}_{2,1})}\leqslant\|f\|_{\tilde{L}^p_T(\tilde{B}^{s,t}_{2,1})}$, we define spaces $\tilde{L}^p_T(\tilde{B}^{s,t}_{2,1})$ as follows

$$\tilde{L}^p_T\big(\tilde{B}^{s,t}_{2,1}\big)=\big\{f\in L^p_T\big(\tilde{B}^{s,t}_{2,1}\big):\ \|f\|_{\tilde{L}^p_T(\tilde{B}^{s,t}_{2,1})}<\infty\big\}.$$

If $T=\infty$, then we omit the subscript T from the notation $\tilde{L}^p_T(\tilde{B}^{s,t}_{2,1})$, that is, $\tilde{L}^p(\tilde{B}^{s,t}_{2,1})$ for simplicity. We will denote by $\tilde{\mathcal{C}}([0,T];\tilde{B}^{s,t}_{2,1})$ the subset of functions of $\tilde{L}^\infty_T(\tilde{B}^{s,t}_{2,1})$ which are continuous on $[0,T]$ with values in $\tilde{B}^{s,t}_{2,1}$.

Let us observe that $L^1_T(\tilde{B}^{s,t}_{2,1})=\tilde{L}^1_T(\tilde{B}^{s,t}_{2,1})$, but the embedding $\tilde{L}^p_T(\tilde{B}^{s,t}_{2,1})\subset L^p_T(\tilde{B}^{s,t}_{2,1})$ is strict if $p>1$.

We will use the following interpolation property which can be verified easily (cf. [1]).

Lemma 2.8. *Let $s,t,s_1,t_1,s_2,t_2\in\mathbb{R}$ and $p,p_1,p_2\in[1,\infty]$. We have*

$$\|f\|_{\tilde{L}^p_T(\tilde{B}^{s,t}_{2,1})}\leqslant\|f\|^{\theta}_{\tilde{L}^{p_1}_T(\tilde{B}^{s_1,t_1}_{2,1})}\|f\|^{1-\theta}_{\tilde{L}^{p_2}_T(\tilde{B}^{s_2,t_2}_{2,1})},$$

where $\frac{1}{p}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2}$, $s=\theta s_1+(1-\theta)s_2$ and $t=\theta t_1+(1-\theta)t_2$.

Now, we define the following work space.

Definition 2.9. For $T>0$ and $s\in\mathbb{R}$, we denote

$$E^s_T=\big\{(h,\mathbf{u})\in\tilde{\mathcal{C}}\big([0,T];\tilde{B}^{s-1,s}_{2,1}\big)\cap L^1\big(0,T;\tilde{B}^{s+3,s+2}_{2,1}\big)\times\big(\tilde{\mathcal{C}}\big([0,T];\dot{B}^{s-1}_{2,1}\big)\cap L^1\big(0,T;\dot{B}^{s+1}_{2,1}\big)\big)^2\big\}$$

and

$$\big\|(h,\mathbf{u})\big\|_{E^s_T}=\|h\|_{\tilde{L}^\infty_T(\tilde{B}^{s-1,s}_{2,1})}+\|\mathbf{u}\|_{\tilde{L}^\infty_T(\dot{B}^{s-1}_{2,1})}+\|h\|_{L^1_T(\tilde{B}^{s+3,s+2}_{2,1})}+\|\mathbf{u}\|_{L^1_T(\dot{B}^{s+1}_{2,1})}.$$

We use the notation E^s if $T=+\infty$, changing $[0,T]$ into $[0,+\infty)$ in the definition above.

3. A priori estimates

Noticing that $\operatorname{div}D(\mathbf{u})=\frac{1}{2}\nabla\operatorname{div}\mathbf{u}+\frac{1}{2}\Delta\mathbf{u}$ and substituting h by $h+\bar{h}_0$ in (2), we have

$$\begin{cases} h_t+\mathbf{u}\cdot\nabla h+\bar{h}_0\operatorname{div}\mathbf{u}=-h\operatorname{div}\mathbf{u},\\ \mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}-3\mu\nabla\operatorname{div}\mathbf{u}+\mathrm{f}\mathbf{u}^\perp+\mathrm{g}\nabla h-\beta\nabla\Delta h=2\mu\dfrac{\nabla hD(\mathbf{u})+\nabla h\operatorname{div}\mathbf{u}}{h+\bar{h}_0},\\ h(0)=h_0-\bar{h}_0,\quad \mathbf{u}(0)=\mathbf{u}_0. \end{cases}\tag{4}$$

For all $s \in \mathbb{R}$, we denote $\Lambda^s f = \mathscr{F}^{-1}(|\xi|^s \hat{f})$. Let $c = \Lambda^{-1} \operatorname{div} \mathbf{u}$ and $d = \Lambda^{-1} \operatorname{div}^{\perp} \mathbf{u}$ where $\operatorname{div}^{\perp} \mathbf{u} = \nabla^{\perp} \cdot \mathbf{u}$ and $\nabla^{\perp} = (-\partial_2, \partial_1)$. Then, it is easy to check that

$$\mathbf{u} = -\Lambda^{-1}\nabla c - \Lambda^{-1}\nabla^{\perp} d.$$

Now, we can rewrite the system (4) in terms of these notations as the following:

$$\begin{cases} h_t + \mathbf{u} \cdot \nabla h + \bar{h}_0 \Lambda c = F, \\ c_t + \mathbf{u} \cdot \nabla c - 4\mu \Delta c - \mathrm{f} d - \mathrm{g} \Lambda h - \beta \Lambda^3 h = G, \\ d_t - \mu \Delta d + \mathrm{f} c = \Lambda^{-1} \operatorname{div}^{\perp} H, \\ \mathbf{u} = -\Lambda^{-1}\nabla c - \Lambda^{-1}\nabla^{\perp} d, \\ h(0) = h_0 - \bar{h}_0, \quad \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{5}$$

where

$$\begin{aligned} F &= -h \operatorname{div} \mathbf{u}, \\ G &= \mathbf{u} \cdot \nabla c + \Lambda^{-1} \operatorname{div} H, \\ H &= -\mathbf{u} \cdot \nabla \mathbf{u} + 2\mu \frac{\nabla h D(\mathbf{u}) + \nabla h \operatorname{div} \mathbf{u}}{h + \bar{h}_0}. \end{aligned}$$

For these equations, we study the following system:

$$\begin{cases} h_t + \mathbf{v} \cdot \nabla h + \bar{h}_0 \Lambda c = F, \\ c_t + \mathbf{v} \cdot \nabla c - 4\mu \Delta c - \mathrm{f} d - \mathrm{g} \Lambda h - \beta \Lambda^3 h = G, \\ d_t - \mu \Delta d + \mathrm{f} c = P, \end{cases} \tag{6}$$

where $\mathbf{v}$ is a vector function and we will precise its regularity in the following proposition.

Proposition 3.1. *Let (h, c, d) be a solution of (6) on $[0, T)$, $T > 0$, $0 < s \leqslant 2$ and $V(t) = \int_0^t \|\mathbf{v}(\tau)\|_{\dot{B}^2_{2,1}} \, d\tau$. The following estimate holds on $[0, T)$:*

$$\begin{aligned} &\|h\|_{\tilde{L}^\infty_T(\tilde{B}^{s-1,s}_{2,1})} + \|c\|_{\tilde{L}^\infty_T(\dot{B}^{s-1}_{2,1})} + \|d\|_{\tilde{L}^\infty_T(\dot{B}^{s-1}_{2,1})} + \int_0^t \left(\|h(\tau)\|_{\tilde{B}^{s+3,s+2}_{2,1}} + \|c(\tau)\|_{\dot{B}^{s+1}_{2,1}} + \|d(\tau)\|_{\dot{B}^{s+1}_{2,1}} \right) d\tau \\ &\quad \leqslant C e^{CV(t)} \Biggl(\|h(0)\|_{\tilde{B}^{s-1,s}_{2,1}} + \|c(0)\|_{\dot{B}^{s-1}_{2,1}} + \|d(0)\|_{\dot{B}^{s-1}_{2,1}} \\ &\qquad + \int_0^t e^{-CV(\tau)} \left(\|F(\tau)\|_{\tilde{B}^{s-1,s}_{2,1}} + \|G(\tau)\|_{\dot{B}^{s-1}_{2,1}} + \|P(\tau)\|_{\dot{B}^{s-1}_{2,1}} \right) d\tau \Biggr), \end{aligned}$$

where C depends only on s, $\bar{h}_0$ and coefficients μ, f, g and β.

Proof. Let (h, c, d) be a solution of (6) and we set

$$(\tilde{h}, \tilde{c}, \tilde{d}, \tilde{F}, \tilde{G}, \tilde{P}) = e^{-KV(t)}(h, c, d, F, G, P).$$

Thus, (6) can be transformed into

$$\begin{cases} \tilde{h}_t + \mathbf{v}\cdot\nabla\tilde{h} + \bar{h}_0\Lambda\tilde{c} = \tilde{F} - KV'(t)\tilde{h}, \\ \tilde{c}_t + \mathbf{v}\cdot\nabla\tilde{c} - 4\mu\Delta\tilde{c} - \mathrm{f}\tilde{d} - \mathrm{g}\Lambda\tilde{h} - \beta\Lambda^3\tilde{h} = \tilde{G} - KV'(t)\tilde{c}, \\ \tilde{d}_t - \mu\Delta\tilde{d} + \mathrm{f}\tilde{c} = \tilde{P} - KV'(t)\tilde{d}. \end{cases} \tag{7}$$

Applying the operator Δ_k to the system (7), we obtain the following system by noting $(\tilde{h}_k, \tilde{c}_k, \tilde{d}_k, \tilde{F}_k, \tilde{G}_k, \tilde{P}_k) = (\Delta_k\tilde{h}, \Delta_k\tilde{c}, \Delta_k\tilde{d}, \Delta_k\tilde{F}, \Delta_k\tilde{G}, \Delta_k\tilde{P})$:

$$\begin{cases} \partial_t\tilde{h}_k + \Delta_k(\mathbf{v}\cdot\nabla\tilde{h}) + \bar{h}_0\Lambda\tilde{c}_k = \tilde{F}_k - KV'(t)\tilde{h}_k, \\ \partial_t\tilde{c}_k + \Delta_k(\mathbf{v}\cdot\nabla\tilde{c}) - 4\mu\Delta\tilde{c}_k - \mathrm{f}\tilde{d}_k - \mathrm{g}\Lambda\tilde{h}_k - \beta\Lambda^3\tilde{h}_k = \tilde{G}_k - KV'(t)\tilde{c}_k, \\ \partial_t\tilde{d}_k - \mu\Delta\tilde{d}_k + \mathrm{f}\tilde{c}_k = \tilde{P}_k - KV'(t)\tilde{d}_k. \end{cases} \tag{8}$$

To begin with, we consider the case where $\mathbf{v} = \mathbf{0}$, $K = 0$ and $F = G = P = 0$ which implies that (8) takes the form

$$\begin{cases} \partial_t\tilde{h}_k + \bar{h}_0\Lambda\tilde{c}_k = 0, \\ \partial_t\tilde{c}_k - 4\mu\Delta\tilde{c}_k - \mathrm{f}\tilde{d}_k - \mathrm{g}\Lambda\tilde{h}_k - \beta\Lambda^3\tilde{h}_k = 0, \\ \partial_t\tilde{d}_k - \mu\Delta\tilde{d}_k + \mathrm{f}\tilde{c}_k = 0. \end{cases} \tag{9}$$

3.1. The case of high frequencies

Taking the L^2 scalar product of the first equation of (9) with $\tilde{h}_k$, of the second equation with $\tilde{c}_k$, and the third one with $\tilde{d}_k$, we get the following three identities:

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\tilde{h}_k\|_{L^2}^2 + \bar{h}_0(\Lambda\tilde{c}_k, \tilde{h}_k) = 0, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{c}_k\|_{L^2}^2 + 4\mu\|\Lambda\tilde{c}_k\|_{L^2}^2 - \mathrm{f}(\tilde{d}_k, \tilde{c}_k) - \mathrm{g}(\Lambda\tilde{h}_k, \tilde{c}_k) - \beta\big(\Lambda^2\tilde{h}_k, \Lambda\tilde{c}_k\big) = 0, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{d}_k\|_{L^2}^2 + \mu\|\Lambda\tilde{d}_k\|_{L^2}^2 + \mathrm{f}(\tilde{c}_k, \tilde{d}_k) = 0. \end{cases} \tag{10}$$

Now we want to get an equality involving $\Lambda\tilde{h}_k$. To achieve it, we take L^2 scalar product of the first equation of (9) with $\Lambda^2\tilde{h}_k$ and $\Lambda\tilde{c}_k$ respectively, then take the L^2 scalar product of the second equation with $\Lambda\tilde{h}_k$ and sum with both last two equalities. This yields

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\Lambda\tilde{h}_k\|_{L^2}^2 + \bar{h}_0\big(\Lambda^2\tilde{c}_k, \Lambda\tilde{h}_k\big) = 0, \\ \frac{d}{dt}(\Lambda\tilde{h}_k, \tilde{c}_k) + \bar{h}_0\|\Lambda\tilde{c}_k\|_{L^2}^2 - \mathrm{f}(\tilde{d}_k, \Lambda\tilde{h}_k) - \mathrm{g}\|\Lambda\tilde{h}_k\|_{L^2}^2 - \beta\big\|\Lambda^2\tilde{h}_k\big\|_{L^2}^2 + 4\mu\big(\Lambda\tilde{c}_k, \Lambda^2\tilde{h}_k\big) = 0. \end{cases} \tag{11}$$

Let $K_1 > 0$ be a constant to be chosen later and denote for $k > 0$

$$\alpha_k^2 = \frac{\mathrm{g}}{\bar{h}_0}\|\tilde{h}_k\|_{L^2}^2 + \frac{\beta}{\bar{h}_0}\|\Lambda\tilde{h}_k\|_{L^2}^2 + \|\tilde{c}_k\|_{L^2}^2 + \|\tilde{d}_k\|_{L^2}^2 - 2K_1(\Lambda\tilde{h}_k, \tilde{c}_k).$$

By a linear combination of (10) and (11), we can get

$$\frac{1}{2}\frac{d}{dt}\alpha_k^2 + (4\mu - \bar{h}_0 K_1)\|\Lambda\tilde{c}_k\|_{L^2}^2 + \mathrm{g}K_1\|\Lambda\tilde{h}_k\|_{L^2}^2 + \beta K_1\|\Lambda^2\tilde{h}_k\|_{L^2}^2$$
$$+ \mu\|\Lambda\tilde{d}_k\|_{L^2}^2 - 4\mu K_1\big(\Lambda\tilde{c}_k, \Lambda^2\tilde{h}_k\big) + \mathrm{f}K_1(\tilde{d}_k, \Lambda\tilde{h}_k) = 0. \tag{12}$$

Using Schwartz' inequality, Young's inequality and Bernstein's inequality

$$\|\tilde{d}_k\|_{L^2} \leqslant \frac{4}{3}2^{-k}\|\Lambda\tilde{d}_k\|_{L^2},$$

we find, for any positive numbers M_1, M_2, M_3, that

$$\big|\big(\Lambda\tilde{c}_k, \Lambda^2\tilde{h}_k\big)\big| \leqslant \frac{M_1}{2}\|\Lambda\tilde{c}_k\|_{L^2}^2 + \frac{1}{2M_1}\|\Lambda^2\tilde{h}_k\|_{L^2}^2,$$
$$\big|(\tilde{d}_k, \Lambda\tilde{h}_k)\big| \leqslant \frac{8M_2}{9}\|\Lambda\tilde{d}_k\|_{L^2}^2 + \frac{1}{2M_2}\|\Lambda\tilde{h}_k\|_{L^2}^2,$$
$$\big|(\Lambda\tilde{h}_k, \tilde{c}_k)\big| \leqslant \frac{M_3}{2}\|\Lambda\tilde{h}_k\|_{L^2}^2 + \frac{1}{2M_3}\|\tilde{c}_k\|_{L^2}^2.$$

Thus, we need to determine the values of K_1, M_1, M_2 and M_3 such that

$$4\mu - \bar{h}_0K_1 - 4\mu K_1\frac{M_1}{2} > 0, \qquad \beta - \frac{4\mu}{2M_1} > 0, \qquad \mathrm{g} - \frac{\mathrm{f}}{2M_2} > 0,$$
$$\mu - \frac{8\mathrm{f}K_1M_2}{9} > 0, \qquad \frac{\beta}{\bar{h}_0} - K_1M_3 > 0, \qquad 1 - \frac{K_1}{M_3} > 0.$$

One can verify that the above inequalities will hold if one has

$$0 < K_1 < \min\left(\frac{4\mu\beta}{\bar{h}_0\beta + 5\mu^2}, \frac{2}{3}\sqrt{\frac{\beta}{\bar{h}_0}}, \frac{9\mu\mathrm{g}}{4\mathrm{f}^2}\right),$$
$$M_1 = \frac{5\mu}{2\beta}, \qquad M_2 = \frac{\mathrm{f}}{4\mathrm{g}} + \frac{9\mu}{16\mathrm{f}K_1}, \qquad M_3 = \frac{2}{3}\sqrt{\frac{\beta}{\bar{h}_0}}.$$

Hence, we obtain

$$c_1\alpha_k^2 \leqslant \|\tilde{h}_k\|_{L^2}^2 + \|\Lambda\tilde{h}_k\|_{L^2}^2 + \|\tilde{c}_k\|_{L^2}^2 + \|\tilde{d}_k\|_{L^2}^2 \leqslant c_2\alpha_k^2. \tag{13}$$

Therefore, there exists a constant $\dot{c} > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\alpha_k^2 + \dot{c}2^{2k}\alpha_k^2 \leqslant 0.$$

In the general case where F, G, P, K and $\mathbf{v}$ are not zero, we have, with the help of Lemma 6.2 in [11], that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\alpha_k^2 + \big(\dot{c}2^{2k} + KV'\big)\alpha_k^2 \\
&\leqslant \frac{g}{\bar{h}_0}(\tilde{F}_k, \tilde{h}_k) + (\tilde{G}_k, \tilde{c}_k) + (\tilde{P}_k, \tilde{c}_k) + \frac{\beta}{\bar{h}_0}(\Lambda\tilde{F}_k, \Lambda\tilde{h}_k) - K_1(\Lambda\tilde{F}_k, \tilde{c}_k) - K_1(\tilde{G}_k, \Lambda\tilde{h}_k) \\
&\quad - \frac{g}{\bar{h}_0}\big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{h}), \tilde{h}_k\big) - \big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{c}), \tilde{c}_k\big) - \frac{\beta}{\bar{h}_0}\big(\Lambda\Delta_k(\mathbf{v}\cdot\nabla\tilde{h}), \Lambda\tilde{h}_k\big) \\
&\quad + K_1\big(\Lambda\Delta_k(\mathbf{v}\cdot\nabla\tilde{h}), \tilde{c}_k\big) + K_1\big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{c}), \Lambda\tilde{h}_k\big) \\
&\lesssim \alpha_k\big(\|\tilde{F}_k\|_{L^2} + \|\tilde{G}_k\|_{L^2} + \|\tilde{P}_k\|_{L^2} + \|\Lambda\tilde{F}_k\|_{L^2} \\
&\quad + \gamma_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{h}\|_{\dot{B}^{s-1}_{2,1}} + \gamma_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{c}\|_{\dot{B}^{s-1}_{2,1}} \\
&\quad + \gamma_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{h}\|_{\dot{B}^s_{2,1}} + \gamma_k\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\big(2^{-k(s-1)}\|\tilde{c}\|_{\dot{B}^{s-1}_{2,1}} + 2^{-k(s-1)}\|\tilde{h}\|_{\dot{B}^s_{2,1}}\big)\big),
\end{aligned}
$$

where $\sum_k \gamma_k \leqslant 1$ and $s \in (0, 2]$.

3.2. The case of low frequencies

We replace the second equation of (11) by the following equation

$$
\begin{aligned}
&\frac{d}{dt}\big(\Lambda^3\tilde{h}_k, \tilde{c}_k\big) + \bar{h}_0\big\|\Lambda^2\tilde{c}_k\big\|^2_{L^2} - \mathrm{f}\big(\Lambda\tilde{d}_k, \Lambda^2\tilde{h}_k\big) - \mathrm{g}\big\|\Lambda^2\tilde{h}_k\big\|^2_{L^2} \\
&\quad - \beta\big\|\Lambda^3\tilde{h}_k\big\|^2_{L^2} + 4\mu\big(\Lambda^2\tilde{c}_k, \Lambda^3\tilde{h}_k\big) = 0.
\end{aligned} \tag{14}
$$

Let $K_2 > 0$ be a constant to be chosen later and denote for $k \leqslant 0$

$$
\alpha_k^2 = \frac{g}{\bar{h}_0}\|\tilde{h}_k\|^2_{L^2} + \frac{\beta}{\bar{h}_0}\|\Lambda\tilde{h}_k\|^2_{L^2} + \|\tilde{c}_k\|^2_{L^2} + \|\tilde{d}_k\|^2_{L^2} - 2K_2\big(\Lambda^3\tilde{h}_k, \tilde{c}_k\big).
$$

A linear combination of (10), the first equation of (11) and (14) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\alpha_k^2 + 4\mu\|\Lambda\tilde{c}_k\|^2_{L^2} - \bar{h}_0K_2\big\|\Lambda^2\tilde{c}_k\big\|^2_{L^2} + \mathrm{g}K_2\big\|\Lambda^2\tilde{h}_k\big\|^2_{L^2} \\
&\quad + \beta K_2\big\|\Lambda^3\tilde{h}_k\big\|^2_{L^2} + \mu\|\Lambda\tilde{d}_k\|^2_{L^2} - 4\mu K_2\big(\Lambda^2\tilde{c}_k, \Lambda^3\tilde{h}_k\big) + \mathrm{f}K_2\big(\Lambda\tilde{d}_k, \Lambda^2\tilde{h}_k\big) = 0.
\end{aligned} \tag{15}
$$

Using Schwartz' inequality, Young's inequality and Bernstein's inequality

$$
\|\Lambda\tilde{h}_k\|_{L^2} \leqslant \frac{8}{3}2^k\|\tilde{h}_k\|_{L^2},
$$

we find, for any positive numbers M_4, M_5, M_6, that

$$
\begin{aligned}
\big|\big(\Lambda^2\tilde{c}_k, \Lambda^3\tilde{h}_k\big)\big| &\leqslant \frac{8^2M_4}{2\cdot 3^2}\|\Lambda\tilde{c}_k\|^2_{L^2} + \frac{1}{2M_4}\big\|\Lambda^3\tilde{h}_k\big\|^2_{L^2}, \\
\big|\big(\Lambda\tilde{d}_k, \Lambda^2\tilde{h}_k\big)\big| &\leqslant \frac{M_5}{2}\|\Lambda\tilde{d}_k\|^2_{L^2} + \frac{1}{2M_5}\big\|\Lambda^2\tilde{h}_k\big\|^2_{L^2}, \\
\big|\big(\Lambda^3\tilde{h}_k, \tilde{c}_k\big)\big| &\leqslant \frac{8^4M_6}{2\cdot 3^4}\|\Lambda\tilde{h}_k\|^2_{L^2} + \frac{1}{2M_6}\|\tilde{c}_k\|^2_{L^2}.
\end{aligned}
$$

Thus, we need to determine the values of K_2, M_4, M_5 and M_6 such that

$$4\mu - \frac{8^2}{3^2}\bar{h}_0 K_2 - 4\mu K_2 \frac{8^2 M_4}{2\cdot 3^2} > 0, \qquad \beta - \frac{4\mu}{2M_4} > 0, \qquad \mathrm{g} - \frac{\mathrm{f}}{2M_5} > 0,$$
$$\mu - \frac{\mathrm{f}K_2 M_5}{2} > 0, \qquad \frac{\beta}{\bar{h}_0} - \frac{8^4}{2\cdot 3^4}K_2 M_6 > 0, \qquad 1 - \frac{K_2}{M_6} > 0.$$

One can verify that the above inequalities will hold if one chooses

$$0 < K_2 < \min\left(\frac{3^2\cdot 4\mu\beta}{8^2(\bar{h}_0\beta + 5\mu^2)}, \frac{3^2}{8^2}\sqrt{\frac{\beta}{\bar{h}_0}}, \frac{4\mu \mathrm{g}}{\mathrm{f}^2}\right),$$
$$M_4 = \frac{5\mu}{2\beta}, \qquad M_5 = \frac{\mathrm{f}}{4\mathrm{g}} + \frac{\mu}{\mathrm{f}K_2}, \qquad M_6 = \frac{3^2}{8^2}\sqrt{\frac{\beta}{\bar{h}_0}}.$$

Hence, we obtain

$$c_3\alpha_k^2 \leqslant \|\tilde{h}_k\|_{L^2}^2 + \|\Lambda\tilde{h}_k\|_{L^2}^2 + \|\tilde{c}_k\|_{L^2}^2 + \|\tilde{d}_k\|_{L^2}^2 \leqslant c_4\alpha_k^2. \tag{16}$$

Therefore, there exists a constant $\ddot{c} > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\alpha_k^2 + \ddot{c}2^{4k}\alpha_k^2 \leqslant 0.$$

In the general case where F, G, P, K and $\mathbf{v}$ are not zero, we have, with the help of Lemma 6.2 in [11], that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\alpha_k^2 + \big(\ddot{c}2^{4k} + KV'\big)\alpha_k^2 \\
&\quad\leqslant \frac{\mathrm{g}}{\bar{h}_0}(\tilde{F}_k, \tilde{h}_k) + (\tilde{G}_k, \tilde{c}_k) + (\tilde{P}_k, \tilde{c}_k) + \frac{\beta}{\bar{h}_0}(\Lambda\tilde{F}_k, \Lambda\tilde{h}_k) - K_1\big(\Lambda^3\tilde{F}_k, \tilde{c}_k\big) - K_1\big(\tilde{G}_k, \Lambda^3\tilde{h}_k\big) \\
&\qquad - \frac{\mathrm{g}}{\bar{h}_0}\big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{h}), \tilde{h}_k\big) - \big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{c}), \tilde{c}_k\big) - \frac{\beta}{\bar{h}_0}\big(\Lambda\Delta_k(\mathbf{v}\cdot\nabla\tilde{h}), \Lambda\tilde{h}_k\big) \\
&\qquad + K_1\big(\Lambda^3\Delta_k(\mathbf{v}\cdot\nabla\tilde{h}), \tilde{c}_k\big) + K_1\big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{c}), \Lambda^3\tilde{h}_k\big) \\
&\quad\lesssim \alpha_k\big(\|\tilde{F}_k\|_{L^2} + \|\tilde{G}_k\|_{L^2} + \|\tilde{P}_k\|_{L^2} + \gamma_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{h}\|_{\dot{B}^{s-1}_{2,1}} + \gamma_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{c}\|_{\dot{B}^{s-1}_{2,1}} \\
&\qquad + \gamma_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{h}\|_{\dot{B}^s_{2,1}} + \gamma_k\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\big(2^{-k(s-1)}\|\tilde{c}\|_{\dot{B}^{s-1}_{2,1}} + 2^{-k(s-1)}\|\tilde{h}\|_{\dot{B}^{s-1}_{2,1}}\big)\big),
\end{aligned}$$

where $\sum_k \gamma_k \leqslant 1$ and $s \in (0, 3]$.

Thus, combining two cases of high and low frequencies, we obtain for any $k \in \mathbb{Z}$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\alpha_k^2 + \big(\check{c}2^{2k}\min\big(1, 2^{2k}\big) + KV'\big)\alpha_k^2 \\
&\quad\lesssim \alpha_k\big(\|\tilde{F}_k\|_{L^2} + \|\tilde{G}_k\|_{L^2} + \|\tilde{P}_k\|_{L^2} + \|\Lambda\tilde{F}_k\|_{L^2} + \gamma_k 2^{-k(s-1)}V'\big\|(\tilde{h}, \tilde{c})\big\|_{\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}}\big),
\end{aligned} \tag{17}$$

where we choose $\check{c} = \min(\dot{c}, \ddot{c})$ and $V(t) = \int_0^t \|\mathbf{v}\|_{\dot{B}^2_{2,1}}$.

We are now going to show that the inequality (17) implies a decay for h, c and d.

3.3. The damping effect for h

Dividing (17) by α_k, we get

$$\frac{d}{dt}\alpha_k(t)+\big(\check{c}2^{2k}\min\big(1,2^{2k}\big)+KV'\big)\alpha_k \lesssim \|\tilde{F}_k\|_{L^2}+\|\tilde{G}_k\|_{L^2}+\|\tilde{P}_k\|_{L^2}+\|\Lambda\tilde{F}_k\|_{L^2}+\gamma_k 2^{-k(s-1)}V'\big\|(\tilde{h},\tilde{c})\big\|_{\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}}. \tag{18}$$

Integrating over $[0,t]$, we have

$$\begin{aligned}&\alpha_k(t)+\check{c}2^{2k}\min\big(1,2^{2k}\big)\int_0^t\alpha_k(\tau)\,d\tau\\ &\quad\lesssim\alpha_k(0)+\int_0^t\big(\big\|\tilde{F}_k(\tau)\big\|_{L^2}+\big\|\tilde{G}_k(\tau)\big\|_{L^2}+\big\|\tilde{P}_k(\tau)\big\|_{L^2}+\big\|\Lambda\tilde{F}_k(\tau)\big\|_{L^2}\big)\,d\tau\\ &\qquad+\int_0^t V'(\tau)\big[\gamma_k(\tau)2^{-k(s-1)}\big\|(\tilde{h},\tilde{c})\big\|_{\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}}-K\alpha_k(\tau)\big]\,d\tau.\end{aligned} \tag{19}$$

By the definition of α_k^2, we have

$$2^{k(s-1)}\alpha_k\approx 2^{ks}\max\big(1,2^{-k}\big)\|\tilde{h}_k\|_{L^2}+2^{k(s-1)}\|\tilde{c}_k\|_{L^2}+2^{k(s-1)}\|\tilde{d}_k\|_{L^2},\quad\forall k\in\mathbb{Z}. \tag{20}$$

Thus, we have, by taking K large enough, that

$$\sum_{k\in\mathbb{Z}}\big[\gamma_k(\tau)\big\|(\tilde{h},\tilde{c})\big\|_{\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}}-K2^{k(s-1)}\alpha_k(\tau)\big]\leqslant 0.$$

Changing the functions $(\tilde{h},\tilde{c},\tilde{d},\tilde{F},\tilde{G},\tilde{P})$ into the original ones (h,c,d,F,G,P) and multiplying both sides of (19) by $2^{k(s-1)}$. According to the last inequality, and due to (19) and (20), we conclude after summation on k in $\mathbb{Z}$, that

$$\begin{aligned}&\|h\|_{\tilde{L}^\infty_T(\tilde{B}^{s-1,s}_{2,1})}+\|c\|_{\tilde{L}^\infty_T(\dot{B}^{s-1}_{2,1})}+\|d\|_{\tilde{L}^\infty_T(\dot{B}^{s-1}_{2,1})}+\check{c}\int_0^t\big\|h(\tau)\big\|_{\tilde{B}^{s+3,s+2}_{2,1}}\,d\tau\\ &\qquad+\check{c}\sum_{k\in\mathbb{Z}}\int_0^t 2^{k(s+1)}\min\big(1,2^{2k}\big)\big\|c_k(\tau)\big\|_{L^2}\,d\tau+\check{c}\sum_{k\in\mathbb{Z}}\int_0^t 2^{k(s+1)}\min\big(1,2^{2k}\big)\big\|d_k(\tau)\big\|_{L^2}\,d\tau\\ &\quad\lesssim e^{CV(t)}\big\|\big(h(0),c(0),d(0)\big)\big\|_{\tilde{B}^{s-1,s}_{2,1}\times(\dot{B}^{s-1}_{2,1})^2}\\ &\qquad+e^{CV(t)}\int_0^t e^{-CV(\tau)}\big\|(F,G,P)(\tau)\big\|_{\tilde{B}^{s-1,s}_{2,1}\times(\dot{B}^{s-1}_{2,1})^2}\,d\tau.\end{aligned} \tag{21}$$

3.4. The smoothing effects of c and d

Once the damping effect for h is established, it is easy to get the smoothing effect on c and d. Since (21) implies the desired estimate for high frequencies, it suffices to prove it for low frequencies only. We therefore suppose in this subsection that $k \leqslant 0$.

Taking the L^2 scalar product of the last two equations of (8) with $\tilde{c}_k$ and $\tilde{d}_k$ respectively, we have

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|\tilde{h}_k\|_{L^2}^2 + \bar{h}_0(\tilde{c}_k, \Lambda\tilde{h}_k) = (\tilde{F}_k, \tilde{h}_k) - KV'(t)\|\tilde{h}_k\|_{L^2}^2 - \big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{h}), \tilde{h}_k\big), \\ \dfrac{1}{2}\dfrac{d}{dt}\|\tilde{c}_k\|_{L^2}^2 + 4\mu\|\Lambda\tilde{c}_k\|_{L^2}^2 - \mathrm{f}(\tilde{d}_k, \tilde{c}_k) - \mathrm{g}(\Lambda\tilde{h}_k, \tilde{c}_k) - \beta\big(\Lambda^3\tilde{h}_k, \tilde{c}_k\big) \\ \quad = (\tilde{G}_k, \tilde{c}_k) - KV'(t)\|\tilde{c}_k\|_{L^2}^2 - \big(\Delta_k(\mathbf{v}\cdot\nabla\tilde{c}), \tilde{c}_k\big), \\ \dfrac{1}{2}\dfrac{d}{dt}\|\tilde{d}_k\|_{L^2}^2 + \mu\|\Lambda\tilde{d}_k\|_{L^2}^2 + \mathrm{f}(\tilde{d}_k, \tilde{c}_k) = (\tilde{P}_k, \tilde{d}_k) - KV'(t)\|\tilde{d}_k\|_{L^2}^2. \end{cases} \tag{22}$$

Define $\theta_k^2 = \frac{\mathrm{g}}{\bar{h}_0}\|\tilde{h}_k\|_{L^2}^2 + \|\tilde{c}_k\|_{L^2}^2 + \|\tilde{d}_k\|_{L^2}^2$. By using Lemma 6.2 in [11], (22) yields, for a constant $c > 0$, that

$$\frac{1}{2}\frac{d}{dt}\theta_k^2 + C2^{2k}\theta_k^2 \lesssim \theta_k\big(\big\|\Lambda^3\tilde{h}_k\big\|_{L^2} + \|\tilde{G}_k\|_{L^2} + \|\tilde{P}_k\|_{L^2}\big)$$
$$+ \theta_k V'(t)\big(C\gamma_k 2^{-k(s-1)}\big(\|\tilde{c}\|_{\dot{B}_{2,1}^{s-1}} + \|\tilde{h}\|_{\dot{B}_{2,1}^{s-1}}\big) - K\theta_k\big).$$

Dividing by θ_k and integrating over $[0, t]$, we infer

$$\theta_k(t) + C\int_0^t 2^{2k}\theta_k(\tau)\,d\tau \leqslant \theta_k(0) + C\int_0^t \big[\big\|\tilde{G}_k(\tau)\big\|_{L^2} + \big\|\tilde{P}_k(\tau)\big\|_{L^2}\big]\,d\tau$$
$$+ C\int_0^t V'(\tau)\gamma_k(\tau)2^{-k(s-1)}\big(\big\|\tilde{c}(\tau)\big\|_{\dot{B}_{2,1}^{s-1}} + \big\|\tilde{h}(\tau)\big\|_{\dot{B}_{2,1}^{s-1}}\big)\,d\tau.$$

Therefore, changing the functions $(\tilde{h}, \tilde{c}, \tilde{d}, \tilde{F}, \tilde{G}, \tilde{P})$ into the original ones, we get

$$\sum_{k\leqslant 0} 2^{k(s-1)}\big\|h_k(t)\big\|_{\tilde{L}_T^\infty(L^2)} + \sum_{k\leqslant 0} 2^{k(s-1)}\big\|c_k(t)\big\|_{\tilde{L}_T^\infty(L^2)}$$
$$+ \sum_{k\leqslant 0} 2^{k(s-1)}\big\|d_k(t)\big\|_{\tilde{L}_T^\infty(L^2)} + C\int_0^t \sum_{k\leqslant 0} 2^{k(s+1)}\big\|c_k(\tau)\big\|_{L^2}\,d\tau + C\int_0^t \sum_{k\leqslant 0} 2^{k(s+1)}\big\|d_k(\tau)\big\|_{L^2}\,d\tau$$
$$\lesssim e^{CV(t)}\big\|\big(h(0), c(0), d(0)\big)\big\|_{\tilde{B}_{2,1}^{s-1,s}\times(\dot{B}_{2,1}^{s-1})^2} + \int_0^t e^{CV(t-\tau)}\big\|\big(G(\tau), P(\tau)\big)\big\|_{(\dot{B}_{2,1}^{s-1})^2}\,d\tau$$
$$+ e^{CV(t)}\big(\|c\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{s-1})} + \big\|h(\tau)\big\|_{\tilde{L}_T^\infty(\tilde{B}_{2,1}^{s-1,s})}\big).$$

Using (21), we eventually conclude that

$$C\int_0^t \sum_{k\leqslant 0} 2^{k(s+1)} \|c_k(\tau)\|_{L^2}\,d\tau + C\int_0^t \sum_{k\leqslant 0} 2^{k(s+1)} \|d_k(\tau)\|_{L^2}\,d\tau$$
$$\lesssim e^{CV(t)}\left(\big\|\big(h(0),c(0),d(0)\big)\big\|_{\tilde{B}^{s-1,s}_{2,1}\times(\dot{B}^{s-1}_{2,1})^2} + \int_0^t e^{-CV(\tau)}\big\|\big(F(\tau),G(\tau),P(\tau)\big)\big\|_{\tilde{B}^{s-1,s}_{2,1}\times(\dot{B}^{s-1}_{2,1})^2}\,d\tau\right).$$

Combining the last inequality with (21), we complete the proof. □

4. Existence and uniqueness

This section is devoted to the proof of Theorem 1.1. The principle of the proof is a very classical one. We shall use the classical Friedrichs' regularization method, which was used in [6–8,16] for examples, to construct the approximate solutions $(h^n, \mathbf{u}^n)_{n\in\mathbb{N}}$ to (4), and then we will use Proposition 3.1 to get some uniform bounds on $(h^n, \mathbf{u}^n)_{n\in\mathbb{N}}$.

4.1. Construction of the approximate sequence

To this end, let us define the sequence of operators $(J_n)_{n\in\mathbb{N}}$ by

$$J_n f := \mathscr{F}^{-1}\mathbf{1}_{B(\frac{1}{n},n)}(\xi)\mathscr{F}f,$$

and consider the following approximate system:

$$\begin{cases} h^n_t + J_n\big(J_n\mathbf{u}^n\cdot\nabla J_n h^n\big) + \bar{h}_0\Lambda J_n c^n = F^n,\\ c^n_t + J_n\big(J_n\mathbf{u}^n\cdot\nabla J_n c^n\big) - 4\mu\Delta J_n c^n - \mathrm{f}J_n d^n - \mathrm{g}\Lambda J_n h^n - \beta\Lambda^3 J_n h^n = G^n,\\ d^n_t - \mu\Delta J_n d^n + \mathrm{f}J_n c^n = J_n\Lambda^{-1}\operatorname{div}^{\perp} H^n,\\ \mathbf{u}^n = -\Lambda^{-1}\nabla c^n - \Lambda^{-1}\nabla^{\perp} d^n,\\ \big(h^n, c^n, d^n\big)(0) = \big(h_n, \Lambda^{-1}\operatorname{div}\mathbf{u}_n, \Lambda^{-1}\operatorname{div}^{\perp}\mathbf{u}_n\big), \end{cases} \tag{23}$$

where

$$h_n = J_n(h_0 - \bar{h}_0), \qquad \mathbf{u}_n = J_n\mathbf{u}_0,$$
$$F^n = -J_n\big(J_n h^n \operatorname{div} J_n\mathbf{u}^n\big),$$
$$G^n = J_n\big(J_n\mathbf{u}^n\cdot\nabla J_n c^n\big) + J_n\Lambda^{-1}\operatorname{div} H^n,$$
$$H^n = -J_n\mathbf{u}^n\cdot\nabla J_n\mathbf{u}^n + 2\mu\frac{\nabla J_n h^n D(J_n\mathbf{u}^n) + \nabla J_n h^n \operatorname{div} J_n\mathbf{u}^n}{\zeta(J_n h^n + \bar{h}_0)},$$

with ζ a smooth function satisfying

$$\zeta(s) = \begin{cases} \bar{h}_0/4, & |s| \leqslant \bar{h}_0/4,\\ s, & \bar{h}_0/2 \leqslant |s| \leqslant 3\bar{h}_0/2,\\ 7\bar{h}_0/4, & |s| \geqslant 7\bar{h}_0/4,\\ \text{smooth}, & \text{otherwise}. \end{cases}$$

We want to show that (23) is only an ordinary differential equation in $L^2 \times L^2 \times L^2$. We can observe easily that all the source term in (23) turn out to be continuous in $L^2 \times L^2 \times L^2$. For example, we consider the term $J_n\Lambda^{-1}\operatorname{div}\frac{\nabla J_n h^n \operatorname{div} J_n\mathbf{u}^n}{\zeta(J_n h^n+\bar{h}_0)}$. By Plancherel's theorem, Hausdorff–Young's inequality and Hölder's inequality, we have

$$
\begin{aligned}
\left\| J_n\Lambda^{-1}\operatorname{div}\frac{\nabla J_n h^n \operatorname{div} J_n\mathbf{u}^n}{\zeta(J_n h^n+\bar{h}_0)}\right\|_{L^2} &= \left\| \mathbf{1}_{B(\frac{1}{n},n)}|\xi|^{-1}(\xi_1,\xi_2)\cdot\mathscr{F}\frac{\nabla J_n h^n \operatorname{div} J_n\mathbf{u}^n}{\zeta(J_n h^n+\bar{h}_0)}\right\|_{L^2} \\
&\leqslant \left\| \frac{\nabla J_n h^n \operatorname{div} J_n\mathbf{u}^n}{\zeta(J_n h^n+\bar{h}_0)}\right\|_{L^2} \leqslant \left\|\nabla J_n h^n \operatorname{div} J_n\mathbf{u}^n\right\|_{L^2}\left\|\frac{1}{\zeta(J_n h^n+\bar{h}_0)}\right\|_{L^\infty} \\
&\leqslant \frac{4}{\bar{h}_0}\left\|\nabla J_n h^n\right\|_{L^\infty}\left\|\operatorname{div} J_n\mathbf{u}^n\right\|_{L^2} \leqslant \frac{4n}{\bar{h}_0}\left\||\xi|\mathbf{1}_{B(\frac{1}{n},n)}\mathscr{F}h^n\right\|_{L^1}\left\|\mathbf{u}^n\right\|_{L^2} \\
&\leqslant \frac{4n^3}{\bar{h}_0}\left\|h^n\right\|_{L^2}\left\|\mathbf{u}^n\right\|_{L^2}.
\end{aligned}
$$

Thus, the usual Cauchy–Lipschitz theorem implies the existence of a strictly positive maximal time T_n such that a unique solution exists which is continuous in time with value in $L^2 \times L^2 \times L^2$. However, as $J_n^2 = J_n$, we claim that $J_n(h^n, c^n, d^n)$ is also a solution, so uniqueness implies that $J_n(h^n, c^n, d^n) = (h^n, c^n, d^n)$. So (h^n, c^n, d^n) is also a solution of the following system:

$$
\begin{cases}
h_t^n + J_n(\mathbf{u}^n\cdot\nabla h^n) + \bar{h}_0\Lambda c^n = F_1^n, \\
c_t^n + J_n(\mathbf{u}^n\cdot\nabla c^n) - 4\mu\Delta c^n - \mathrm{f}d^n - \mathrm{g}\Lambda h^n - \beta\Lambda^3 h^n = G_1^n, \\
d_t^n - \mu\Delta d^n + \mathrm{f}c^n = \Lambda^{-1}\operatorname{div}^{\perp} H_1^n, \\
\mathbf{u}^n = -\Lambda^{-1}\nabla c^n - \Lambda^{-1}\nabla^{\perp} d^n, \\
(h^n, c^n, d^n)(0) = (h_n, \Lambda^{-1}\operatorname{div}\mathbf{u}_n, \Lambda^{-1}\operatorname{div}^{\perp}\mathbf{u}_n),
\end{cases}
\tag{24}
$$

with

$$
\begin{aligned}
&h_n = J_n(h_0 - \bar{h}_0), \qquad \mathbf{u}_n = J_n\mathbf{u}_0, \\
&F_1^n = -J_n(h^n \operatorname{div}\mathbf{u}^n), \\
&G_1^n = J_n(\mathbf{u}^n\cdot\nabla c^n) + J_n\Lambda^{-1}\operatorname{div} H_1^n, \\
&H_1^n = -\mathbf{u}^n\cdot\nabla\mathbf{u}^n + 2\mu\frac{\nabla h^n D(\mathbf{u}^n) + \nabla h^n \operatorname{div}\mathbf{u}^n}{\zeta(h^n+\bar{h}_0)}.
\end{aligned}
$$

The system (24) appears to be an ordinary differential equation in the space

$$
L_n^2 := \left\{ a \in L^2(\mathbb{R}^2)\colon \operatorname{supp}\mathscr{F}a \subset B\left(\frac{1}{n}, n\right)\right\}.
$$

Due to the Cauchy–Lipschitz theorem again, a unique maximal solution exists on an interval $[0, T_n^*)$ which is continuous in time with value in $L_n^2 \times L_n^2 \times L_n^2$.

4.2. Uniform bounds

In this subsection, we prove uniform estimates independent of $T < T_n^*$ in $E_T^1 \cap E_T^{1+\varepsilon}$ for $(h^n, \mathbf{u}^n)$. We shall show that $T_n^* = +\infty$ by the Cauchy–Lipschitz theorem. Denote

$$E(0) := \|h_0 - \bar{h}_0\|_{\tilde{B}^{0,1+\varepsilon}_{2,1}} + \|\mathbf{u}_0\|_{\tilde{B}^{0,\varepsilon}_{2,1}},$$
$$E(h, \mathbf{u}, t) := \big\|(h, \mathbf{u})\big\|_{E_t^1} + \big\|(h, \mathbf{u})\big\|_{E_t^{1+\varepsilon}},$$
$$\tilde{T}_n := \sup\big\{t \in \big[0, T_n^*\big):\ E\big(h^n, \mathbf{u}^n, t\big) \leqslant A\tilde{C}E(0)\big\},$$

where $\tilde{C}$ corresponds to the constant in Proposition 3.1 and $A > \max(2, \tilde{C}^{-1})$ is a constant. Thus, by the continuity we have $\tilde{T}_n > 0$.

We are going to prove that $\tilde{T}_n = T_n^*$ for all $n \in \mathbb{N}$ and we will conclude that $T_n^* = +\infty$ for any $n \in \mathbb{N}$.

According to Proposition 3.1 and the definition of $(h_n, \mathbf{u}_n)$, the following inequality holds

$$\begin{aligned}\big\|\big(h^n, \mathbf{u}^n\big)\big\|_{E_T^1} \leqslant \tilde{C}e^{\tilde{C}\|\mathbf{u}^n\|_{L^1_T(\dot{B}^2_{2,1})}}\big(&\|h_0 - \bar{h}_0\|_{\tilde{B}^{0,1}_{2,1}} + \|\mathbf{u}_0\|_{\dot{B}^0_{2,1}} \\ &+ \big\|F_1^n\big\|_{L^1_T(\tilde{B}^{0,1}_{2,1})} + \big\|\mathbf{u}^n \cdot \nabla c^n\big\|_{L^1_T(\dot{B}^0_{2,1})} + \big\|H_1^n\big\|_{L^1_T(\dot{B}^0_{2,1})}\big).\end{aligned}$$

Therefore, it is only a matter to prove appropriate estimates for F_1^n, H_1^n and $\mathbf{u}^n \cdot \nabla c^n$. The estimate of F_1^n is straightforward. From Lemma 2.6, we have

$$\big\|F_1^n\big\|_{L^1_T(\tilde{B}^{0,1}_{2,1})} \leqslant C\big\|h^n\big\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}\big\|\mathbf{u}^n\big\|_{L^1_T(\dot{B}^2_{2,1})} \leqslant CE^2\big(h^n, \mathbf{u}^n, T\big). \tag{25}$$

With the help of Lemma 2.6 and interpolation arguments, we have

$$\begin{aligned}\big\|\mathbf{u}^n \cdot \nabla c^n\big\|_{L^1_T(\dot{B}^0_{2,1})} &\leqslant C\big\|\mathbf{u}^n\big\|_{L^2_T(\dot{B}^1_{2,1})}\big\|\nabla c^n\big\|_{L^2_T(\dot{B}^0_{2,1})} \leqslant C\big\|\mathbf{u}^n\big\|^2_{L^2_T(\dot{B}^1_{2,1})} \\ &\leqslant C\big\|\mathbf{u}^n\big\|_{L^\infty_T(\dot{B}^0_{2,1})}\big\|\mathbf{u}^n\big\|_{L^1_T(\dot{B}^2_{2,1})} \leqslant CE^2\big(h^n, \mathbf{u}^n, T\big).\end{aligned} \tag{26}$$

In the same way, we can get

$$\big\|\mathbf{u}^n \cdot \nabla \mathbf{u}^n\big\|_{L^1_T(\dot{B}^0_{2,1})} \leqslant CE^2\big(h^n, \mathbf{u}^n, T\big). \tag{27}$$

To estimate other terms of H_1^n, we make the following assumption on $E(0)$:

$$2C_1A\tilde{C}E(0) \leqslant \bar{h}_0,$$

where C_1 is the continuity modulus of $\dot{B}^1_{2,1} \subset L^\infty$. If $T < \tilde{T}_n$, it implies

$$\big\|h^n\big\|_{L^\infty} \leqslant C_1\big\|h^n\big\|_{\dot{B}^1_{2,1}} \leqslant C_1\big\|h^n\big\|_{\tilde{B}^{0,1}_{2,1}} \leqslant C_1A\tilde{C}E(0) \leqslant \frac{1}{2}\bar{h}_0.$$

Thus, we have

$$\big\|h^n\big\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant \frac{1}{2}\bar{h}_0,$$

which yields

$$h^n + \bar{h}_0 \in \left[\frac{1}{2}\bar{h}_0, \frac{3}{2}\bar{h}_0\right] \quad \text{and} \quad \zeta\big(h^n + \bar{h}_0\big) = h^n + \bar{h}_0.$$

From Lemmas 2.6 and 2.3, and interpolation arguments, we have

$$\begin{aligned}
\bar{h}_0 \left\| \frac{\nabla h^n \cdot \nabla \mathbf{u}^n}{\bar{h}_0 + h^n} \right\|_{L^1_T(\dot{B}^0_{2,1})} &\leqslant \|\nabla h^n \cdot \nabla \mathbf{u}^n\|_{L^1_T(\dot{B}^0_{2,1})} + \left\| \frac{h^n \nabla h^n \cdot \nabla \mathbf{u}^n}{\bar{h}_0 + h^n} \right\|_{L^1_T(\dot{B}^0_{2,1})} \\
&\leqslant C\|\nabla h^n\|_{L^\infty_T(\dot{B}^0_{2,1})} \|\nabla \mathbf{u}^n\|_{L^1_T(\dot{B}^1_{2,1})} + C\left\| \frac{h^n \nabla h^n}{\bar{h}_0 + h^n} \right\|_{L^\infty_T(\dot{B}^0_{2,1})} \|\nabla \mathbf{u}^n\|_{L^1_T(\dot{B}^1_{2,1})} \\
&\leqslant C\|h^n\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})} \|\mathbf{u}^n\|_{L^1_T(\dot{B}^2_{2,1})} \left(1 + \left\| \frac{h^n}{\bar{h}_0 + h^n} \right\|_{L^\infty_T(\dot{B}^1_{2,1})}\right) \\
&\leqslant C\|h^n\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})} \|\mathbf{u}^n\|_{L^1_T(\dot{B}^2_{2,1})} \big(1 + \|h^n\|_{L^\infty_T(\dot{B}^1_{2,1})}\big) \\
&\leqslant CE^2\big(h^n, \mathbf{u}^n, T\big)\big(1 + E\big(h^n, \mathbf{u}^n, T\big)\big). \qquad (28)
\end{aligned}$$

Similarly, we can get

$$\bar{h}_0 \left\| \frac{\nabla h^n \cdot D(\mathbf{u}^n)}{\bar{h}_0 + h^n} \right\|_{L^1_T(\dot{B}^0_{2,1})} \leqslant CE^2\big(h^n, \mathbf{u}^n, T\big)\big(1 + E\big(h^n, \mathbf{u}^n, T\big)\big). \qquad (29)$$

Hence, from (26)–(28), we gather

$$\|\mathbf{u}^n \cdot \nabla c^n\|_{L^1_T(\dot{B}^0_{2,1})} + \|H_1^n\|_{L^1_T(\dot{B}^0_{2,1})} \leqslant C\big(1 + 4\mu\bar{h}_0^{-1}\big(1 + E\big(h^n, \mathbf{u}^n, T\big)\big)\big)E^2\big(h^n, \mathbf{u}^n, T\big). \qquad (30)$$

Similarly, according to Proposition 3.1 and the definition of $(h_n, \mathbf{u}_n)$, the following inequality holds

$$\begin{aligned}
\big\|\big(h^n, \mathbf{u}^n\big)\big\|_{E_T^{1+\varepsilon}} \leqslant \tilde{C} e^{\tilde{C}\|\mathbf{u}^n\|_{L^1_T(\dot{B}^2_{2,1})}} \big(&\|h_0 - \bar{h}_0\|_{\tilde{B}^{\varepsilon,1+\varepsilon}_{2,1}} + \|\mathbf{u}_0\|_{\dot{B}^\varepsilon_{2,1}} \\
&+ \|F_1^n\|_{L^1_T(\tilde{B}^{\varepsilon,1+\varepsilon}_{2,1})} + \|\mathbf{u}^n \cdot \nabla c^n\|_{L^1_T(\dot{B}^\varepsilon_{2,1})} + \|H_1^n\|_{L^1_T(\dot{B}^\varepsilon_{2,1})}\big).
\end{aligned}$$

The estimate of F_1^n is straightforward. From Lemma 2.6, we have

$$\|F_1^n\|_{L^1_T(\tilde{B}^{\varepsilon,1+\varepsilon}_{2,1})} \leqslant C\|h^n\|_{L^\infty_T(\tilde{B}^{\varepsilon,1+\varepsilon}_{2,1})} \|\mathbf{u}^n\|_{L^1_T(\dot{B}^2_{2,1})} \leqslant CE^2\big(h^n, \mathbf{u}^n, T\big). \qquad (31)$$

With the help of Lemma 2.6 and interpolation arguments, we have

$$\begin{aligned}
\|\mathbf{u}^n \cdot \nabla c^n\|_{L^1_T(\dot{B}^\varepsilon_{2,1})} &\leqslant C\|\mathbf{u}^n\|_{L^{2+\varepsilon}_T(\dot{B}^1_{2,1})} \|\nabla c^n\|_{L_T^{\frac{2+\varepsilon}{1+\varepsilon}}(\dot{B}^\varepsilon_{2,1})} \\
&\leqslant C\|\mathbf{u}^n\|_{L^\infty_T(\dot{B}^0_{2,1})} \|\mathbf{u}^n\|_{L^1_T(\dot{B}^{2+\varepsilon}_{2,1})} \leqslant CE^2\big(h^n, \mathbf{u}^n, T\big). \qquad (32)
\end{aligned}$$

In the same way, we can get

$$\|\mathbf{u}^n \cdot \nabla \mathbf{u}^n\|_{L^1_T(\dot{B}^\varepsilon_{2,1})} \leqslant CE^2\big(h^n, \mathbf{u}^n, T\big). \qquad (33)$$

From Lemmas 2.6 and 2.3, and interpolation arguments, we have

$$\begin{aligned}
\bar{h}_0\left\|\frac{\nabla h^n\cdot\nabla\mathbf{u}^n}{\bar{h}_0+h^n}\right\|_{L^1_T(\dot{B}^{\varepsilon}_{2,1})} &\leqslant \left\|\nabla h^n\cdot\nabla\mathbf{u}^n\right\|_{L^1_T(\dot{B}^{\varepsilon}_{2,1})}+\left\|\frac{h^n\nabla h^n\cdot\nabla\mathbf{u}^n}{\bar{h}_0+h^n}\right\|_{L^1_T(\dot{B}^{\varepsilon}_{2,1})}\\
&\leqslant C\left\|\nabla h^n\right\|_{L^\infty_T(\dot{B}^{\varepsilon}_{2,1})}\left\|\nabla\mathbf{u}^n\right\|_{L^1_T(\dot{B}^{1}_{2,1})}+C\left\|\frac{h^n\nabla h^n}{\bar{h}_0+h^n}\right\|_{L^\infty_T(\dot{B}^{\varepsilon}_{2,1})}\left\|\nabla\mathbf{u}^n\right\|_{L^1_T(\dot{B}^{1}_{2,1})}\\
&\leqslant C\left\|h^n\right\|_{L^\infty_T(\tilde{B}^{0,1+\varepsilon}_{2,1})}\left\|\mathbf{u}^n\right\|_{L^1_T(\dot{B}^{2}_{2,1})}\left(1+\left\|\frac{h^n}{\bar{h}_0+h^n}\right\|_{L^\infty_T(\dot{B}^{1}_{2,1})}\right)\\
&\leqslant C\left\|h^n\right\|_{L^\infty_T(\tilde{B}^{0,1+\varepsilon}_{2,1})}\left\|\mathbf{u}^n\right\|_{L^1_T(\dot{B}^{2}_{2,1})}\left(1+\left\|h^n\right\|_{L^\infty_T(\dot{B}^{1}_{2,1})}\right)\\
&\leqslant CE^2\left(h^n,\mathbf{u}^n,T\right)\left(1+E\left(h^n,\mathbf{u}^n,T\right)\right).
\end{aligned}\tag{34}$$

Similarly, we can get

$$\bar{h}_0\left\|\frac{\nabla h^n\cdot D(\mathbf{u}^n)}{\bar{h}_0+h^n}\right\|_{L^1(\dot{B}^{\varepsilon}_{2,1})}\leqslant CE^2\left(h^n,\mathbf{u}^n,T\right)\left(1+E\left(h^n,\mathbf{u}^n,T\right)\right).\tag{35}$$

Hence, from (32)–(35), we gather

$$\left\|\mathbf{u}^n\cdot\nabla c^n\right\|_{L^1_T(\dot{B}^{\varepsilon}_{2,1})}+\left\|H^n_1\right\|_{L^1_T(\dot{B}^{\varepsilon}_{2,1})}\leqslant C\left(1+4\mu\bar{h}_0^{-1}\left(1+E\left(h^n,\mathbf{u}^n,T\right)\right)\right)E^2\left(h^n,\mathbf{u}^n,T\right).\tag{36}$$

From (25), (30), (31) and (36), it follows

$$\left\|\left(h^n,\mathbf{u}^n\right)\right\|_{E^1_T\cap E^{1+\varepsilon}_T}\leqslant\tilde{C}e^{A\tilde{C}^2E(0)}\left[1+CA^2\tilde{C}^2\left(1+4\mu\bar{h}_0^{-1}\left(1+A\tilde{C}E(0)\right)\right)E(0)\right]E(0).$$

So we can choose $E(0)$ so small that

$$\begin{aligned}
&1+CA^2\tilde{C}^2\left(1+4\mu\bar{h}_0^{-1}\left(1+A\tilde{C}E(0)\right)\right)E(0)\leqslant\frac{A^2}{A+2},\\
&e^{A\tilde{C}^2E(0)}\leqslant\frac{A+1}{A}\quad\text{and}\quad 2C_1A\tilde{C}E(0)\leqslant\bar{h}_0,
\end{aligned}\tag{37}$$

which yields $\|(h^n,\mathbf{u}^n)\|_{E^1_T}\leqslant\frac{A+1}{A+2}A\tilde{C}E(0)$ for any $T<\tilde{T}_n$. It follows that $\tilde{T}_n=T^*_n$. In fact, if $\tilde{T}_n<T^*_n$, we have seen that $E(h^n,\mathbf{u}^n,\tilde{T}_n)\leqslant\frac{A+1}{A+2}A\tilde{C}E(0)$. So by continuity, for a sufficiently small constant $\sigma>0$ we can obtain $E(h^n,\mathbf{u}^n,\tilde{T}_n+\sigma)\leqslant A\tilde{C}E(0)$. This yields a contradiction with the definition of $\tilde{T}_n$.

Now, if $\tilde{T}_n=T^*_n<\infty$, then we have obtained $F(h^n,\mathbf{u}^n,T^*_n)\leqslant A\tilde{C}E(0)$. As $\|h^n\|_{L_{T^*_n}(\tilde{B}^{0,1+\varepsilon}_{2,1})}<\infty$ and $\|\mathbf{u}^n\|_{L_{T^*_n}(\tilde{B}^{0,\varepsilon}_{2,1})}<\infty$, it implies that $\|h^n\|_{L_{T^*_n}(L^2_n)}<\infty$ and $\|\mathbf{u}^n\|_{L_{T^*_n}(L^2_n)}<\infty$. Thus, we may continue the solution beyond T^*_n by the Cauchy–Lipschitz theorem. This contradicts the definition of T^*_n. Therefore, the approximate solution $(h^n,\mathbf{u}^n)_{n\in\mathbb{N}}$ is global in time.

4.3. Existence of a solution

In this subsection, we shall show that, up to an extraction, the sequence $(h^n, \mathbf{u}^n)_{n\in\mathbb{N}}$ converges in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^2)$ to a solution $(h, \mathbf{u})$ of (4) which has the desired regularity properties. The proof lies on compactness arguments. To start with, we show that the time first derivative of $(h^n, \mathbf{u}^n)$ is uniformly bounded in appropriate spaces. This enables us to apply Ascoli's theorem and get the existence of a limit $(h, \mathbf{u})$ for a subsequence. Now, the uniform bounds of the previous subsection provide us with additional regularity and convergence properties so that we may pass to the limit in the system.

It is convenient to split $(h^n, \mathbf{u}^n)$ into the solution of a linear system with initial data $(h_n, \mathbf{u}_n)$ and the discrepancy to that solution. More precisely, we denote by $(h_L^n, \mathbf{u}_L^n)$ the solution to the linear system

$$\begin{cases} \partial_t h_L^n + \operatorname{div} \mathbf{u}_L^n = 0, \\ \partial_t \mathbf{u}_L^n - \mu \Delta \mathbf{u}_L^n - 3\mu \nabla \operatorname{div} \mathbf{u}_L^n + \mathrm{f}\big(\mathbf{u}_L^n\big)^{\perp} + \mathrm{g} \nabla h_L^n - \beta \nabla \Delta h_L^n = 0, \\ \big(h_L^n, \mathbf{u}_L^n\big)_{t=0} = (h_n, \mathbf{u}_n), \end{cases} \tag{38}$$

and $(\bar{h}^n, \bar{\mathbf{u}}^n) = (h^n - h_L^n, \mathbf{u}^n - \mathbf{u}_L^n)$.

Obviously, the definition of $(h_n, \mathbf{u}_n)$ entails

$$h_n \to h_0 - \bar{h}_0 \quad \text{in } \tilde{B}_{2,1}^{0,1+\varepsilon}, \qquad \mathbf{u}_n \to \mathbf{u}_0 \quad \text{in } \tilde{B}_{2,1}^{0,\varepsilon}, \text{ as } n \to +\infty.$$

Proposition 3.1 insures us that

$$\big(h_L^n, \mathbf{u}_L^n\big) \to (h_L, \mathbf{u}_L) \quad \text{in } E^1 \cap E^{1+\varepsilon}, \tag{39}$$

where $(h_L, \mathbf{u}_L)$ is the solution of the linear system

$$\begin{cases} \partial_t h_L + \operatorname{div} \mathbf{u}_L = 0, \\ \partial_t \mathbf{u}_L - \mu \Delta \mathbf{u}_L - 3\mu \nabla \operatorname{div} \mathbf{u}_L + \mathrm{f} \mathbf{u}_L^{\perp} + \mathrm{g} \nabla h_L - \beta \nabla \Delta h_L = 0, \\ (h_L, \mathbf{u}_L)_{t=0} = (h_0 - \bar{h}_0, \mathbf{u}_0). \end{cases} \tag{40}$$

Now, we have to prove the convergence of $(\bar{h}^n, \bar{\mathbf{u}}^n)$. This is of course a trifle more difficult and requires compactness results. Let us first state the following lemma.

Lemma 4.1. *$((\bar{h}^n, \bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}(\mathbb{R}^+; \dot{B}_{2,1}^0) \times (C^{\frac{\varepsilon}{3+\varepsilon}}(\mathbb{R}^+; \dot{B}_{2,1}^0))^2$.*

Proof. Throughout the proof, we will note u.b. for uniformly bounded. We first prove that $\partial_t \bar{h}^n$ is u.b. in $L^2(\mathbb{R}^+, \dot{B}_{2,1}^0)$, which yields the desired result for $\bar{h}$. Let us observe that $\bar{h}^n$ verifies the following equation

$$\partial_t \bar{h}^n = -J_n\big(h^n \operatorname{div} \mathbf{u}^n\big) - J_n\big(\mathbf{u}^n \cdot \nabla h^n\big) - \bar{h}_0 \operatorname{div} \mathbf{u}^n + \bar{h}_0 \operatorname{div} \mathbf{u}_L^n.$$

According to the previous subsection, $(h^n)_{n\in\mathbb{N}}$ is u.b. in $\tilde{L}^\infty(\dot{B}_{2,1}^1)$ and $(\mathbf{u}^n)_{n\in\mathbb{N}}$ is u.b. in $\tilde{L}^2(\dot{B}_{2,1}^1)$ in view of interpolation arguments. Thus, $-J_n(h^n \operatorname{div} \mathbf{u}^n) - J_n(\mathbf{u}^n \cdot \nabla h^n) - \operatorname{div} \mathbf{u}^n$ is u.b. in $\tilde{L}^2(\dot{B}_{2,1}^0)$. The definition of $\mathbf{u}_L^n$ obviously provides us with uniform bounds for $\operatorname{div} \mathbf{u}_L^n$ in $\tilde{L}^2(\dot{B}_{2,1}^0)$, so we can conclude that $\partial_t \bar{h}^n$ is u.b. in $L^2(\dot{B}_{2,1}^0)$.

Denote $c_L^n = \Lambda^{-1} \operatorname{div} \mathbf{u}_L^n$, $\bar{c}^n = \Lambda^{-1} \operatorname{div} \bar{\mathbf{u}}^n$, $d_L^n = \Lambda^{-1} \operatorname{div}^\perp \mathbf{u}_L^n$ and $\bar{d}^n = \Lambda^{-1} \operatorname{div}^\perp \bar{\mathbf{u}}^n$. Let us prove now that $\partial_t \bar{c}^n$ is u.b. in $(L^{\frac{3+\varepsilon}{3}} + L^\infty)(\dot{B}^0_{2,1})$ and that $\partial_t \bar{d}^n$ is u.b. in $(L^{\frac{2+\varepsilon}{2}} + L^\infty)(\dot{B}^0_{2,1})$ which gives the required result for $\bar{\mathbf{u}}^n$ by using the relation $\mathbf{u}^n = -\Lambda^{-1}\nabla c^n - \Lambda^{-1}\nabla^\perp d^n$.

Let us recall that

$$\begin{aligned}
\partial_t \bar{c}^n &= 4\mu\Delta\big(c^n - c_L^n\big) + \mathrm{f}\big(d^n - d_L^n\big) + \mathrm{g}\Lambda\big(h^n - h_L^n\big) + \beta\Lambda^3\big(h^n - h_L^n\big) \\
&\quad - J_n\Lambda^{-1} \operatorname{div}\left(\mathbf{u}^n \cdot \nabla \mathbf{u}^n - 2\mu \frac{\nabla h^n D(\mathbf{u}^n) + \nabla h^n \operatorname{div} \mathbf{u}^n}{h^n + \bar{h}_0}\right), \\
\partial_t \bar{d}^n &= \mu\Delta\big(d^n - d_L^n\big) - \mathrm{f}\big(c^n - c_L^n\big) \\
&\quad - J_n\Lambda^{-1} \operatorname{div}^\perp\left(\mathbf{u}^n \cdot \nabla \mathbf{u}^n - 2\mu \frac{\nabla h^n D(\mathbf{u}^n) + \nabla h^n \operatorname{div} \mathbf{u}^n}{h^n + \bar{h}_0}\right).
\end{aligned}$$

Results of the previous subsection and an interpolation argument yield uniform bounds for $\mathbf{u}^n$ in $\tilde{L}^{\frac{2+\varepsilon}{\varepsilon}}(\dot{B}^{\varepsilon}_{2,1}) \cap \tilde{L}^{\frac{2+\varepsilon}{2-\varepsilon}}(\dot{B}^{2-\varepsilon}_{2,1})$. Since h^n is u.b. in $\tilde{L}^\infty(\dot{B}^1_{2,1})$, c_L^n and c^n are u.b. in $\tilde{L}^{\frac{2+\varepsilon}{2}}(\dot{B}^2_{2,1})$, we easily verify that $\Delta(c^n - c_L^n)$ and $J_n\Lambda^{-1} \operatorname{div}(\mathbf{u}^n \cdot \nabla \mathbf{u}^n - 2\mu \frac{\nabla h^n D(\mathbf{u}^n) + \nabla h^n \operatorname{div} \mathbf{u}^n}{h^n + \bar{h}_0})$ are u.b. in $L^{\frac{2+\varepsilon}{2}}(\dot{B}^0_{2,1})$. Obviously, we have d^{n+1} and d_L^{n+1} u.b. in $\tilde{L}^\infty(\dot{B}^0_{2,1})$. Because h^n and h_L^n are u.b. in $\tilde{L}^\infty(\dot{B}^1_{2,1})$, we have $\Lambda(h^{n+1} - h_L^{n+1})$ u.b. in $L^\infty(\dot{B}^0_{2,1})$. We also have h^n and h_L^n are u.b. in $(\tilde{L}^{\frac{4}{3}} + \tilde{L}^{\frac{3+\varepsilon}{3}})(\dot{B}^3_{2,1})$ in view of Lemma 2.8. Thus, $\Lambda^3 h^n$ is u.b. in $(L^{\frac{4}{3}} + L^{\frac{3+\varepsilon}{3}})(\dot{B}^0_{2,1})$. So we finally get $\partial_t \bar{c}^n$ u.b. in $(L^{\frac{3+\varepsilon}{3}} + L^\infty)(\dot{B}^0_{2,1})$. The case of $\partial_t \bar{d}^n$ goes along the same lines. As the terms corresponding to $\Lambda^3(h^n - h_L^n)$ do not appear, we simply get $\partial_t \bar{d}^n$ u.b. in $(L^{\frac{2+\varepsilon}{2}} + L^\infty)(\dot{B}^0_{2,1})$. $\square$

Now, we can turn to the proof of the existence of a solution and use Ascoli's theorem to get strong convergence. We need to localize the spatial space because we have some results of compactness for the local Sobolev spaces. Let $(\chi_p)_{p\in\mathbb{N}}$ be a sequence of $\mathcal{C}_0^\infty(\mathbb{R}^2)$ cut-off functions supported in the ball $B(0, p+1)$ of $\mathbb{R}^2$ and equal to 1 in a neighborhood of $B(0, p)$.

For any $p \in \mathbb{N}$, Lemma 4.1 tells us that $((\chi_p \bar{h}^n, \chi_p \bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ is uniformly equicontinuous in $\mathcal{C}(\mathbb{R}^+; (\dot{B}^0_{2,1})^{1+2})$ and bounded in $E^{1+\varepsilon}$.

Let us observe that the application $f \mapsto \chi_p f$ is compact from $\tilde{B}^{0,1}_{2,1}$ into $\dot{B}^0_{2,1}$, and from $\dot{B}^\varepsilon_{2,1}$ into $\tilde{B}^{\varepsilon,0}_{2,1}$. After we apply Ascoli's theorem to the family $((\chi_p \bar{h}^n, \chi_p \bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ on the time interval $[0, p]$, we use Cantor's diagonal process. This finally provides us with a distribution $(\bar{h}, \bar{\mathbf{u}})$ belonging to $\mathcal{C}(\mathbb{R}^+; \dot{B}^0_{2,1} \times (\tilde{B}^{\varepsilon,0}_{2,1})^2)$ and a subsequence (which we still denote by $(\bar{h}^n, \bar{\mathbf{u}}^n)_{n\in\mathbb{N}}$) such that, for all $p \in \mathbb{N}$, we have

$$\big(\chi_p \bar{h}^n, \chi_p \bar{\mathbf{u}}^n\big) \to (\chi_p \bar{h}, \chi_p \bar{\mathbf{u}}) \quad \text{as } n \to +\infty,$$

in $\mathcal{C}([0, p]; \dot{B}^0_{2,1} \times (\tilde{B}^{\varepsilon,0}_{2,1})^2)$. This obviously infers that $(\bar{h}^n, \bar{\mathbf{u}}^n)$ tends to $(\bar{h}, \bar{\mathbf{u}})$ in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^2)$.

Coming back to the uniform estimates of the previous subsection, we moreover get that $(\bar{h}, \bar{\mathbf{u}})$ belongs to

$$\tilde{L}^\infty\big(\mathbb{R}^+; \tilde{B}^{0,1+\varepsilon}_{2,1} \times \big(\tilde{B}^{0,\varepsilon}_{2,1}\big)^2\big) \cap L^1\big(\mathbb{R}^+; \big(\tilde{B}^{4,3}_{2,1} \cap \tilde{B}^{4+\varepsilon,3+\varepsilon}_{2,1}\big) \times \big(\tilde{B}^{2,2+\varepsilon}_{2,1}\big)^2\big) \tag{41}$$

and to $\mathcal{C}^{1/2}(\mathbb{R}^+; \dot{B}^0_{2,1}) \times (\mathcal{C}^{\frac{\varepsilon}{3+\varepsilon}}(\mathbb{R}^+; \dot{B}^0_{2,1}))^2$.

Let us now prove that $(h,\mathbf{u}) := (h_L,\mathbf{u}_L) + (\bar{h},\bar{\mathbf{u}})$ solves (4). We first observe that, according to (23),

$$\begin{cases} h_t^n + J_n\big(\mathbf{u}^n\cdot\nabla h^n\big) + \bar{h}_0\Lambda c^n = -J_n\big(h^n \operatorname{div}\mathbf{u}^n\big), \\ \mathbf{u}_t^n + J_n\big(\mathbf{u}^n\cdot\nabla\mathbf{u}^n\big) - \mu\Delta\mathbf{u}^n - 3\mu\nabla\operatorname{div}\mathbf{u}^n + \mathrm{f}\big(\mathbf{u}^n\big)^{\perp} + \mathrm{g}\nabla h^n - \beta\nabla\Delta h^n \\ \quad = 2\mu J_n \dfrac{\nabla h^n D(\mathbf{u}^n) + \nabla h^n \operatorname{div}\mathbf{u}^n}{h^n+\bar{h}_0}. \end{cases}$$

The only problem is to pass to the limit in $\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^2)$ in the nonlinear terms. This can be done by using the convergence results stemming from the uniform estimates and the convergence results (39) and (41).

As it is just a matter of doing tedious verifications, we show, as an example, the case of the term $\frac{\nabla h^n \operatorname{div}\mathbf{u}^n}{\bar{h}_0+h^n}$. Denote $L(z) = z/(z+\bar{h}_0)$. Let $\theta\in\mathcal{C}_0^\infty(\mathbb{R}^+\times\mathbb{R}^2)$ and $p\in\mathbb{N}$ be such that $\operatorname{supp}\theta\subset[0,p]\times B(0,p)$. We consider the decomposition

$$\begin{aligned} &J_n\frac{\bar{h}_0\theta\nabla h^n \operatorname{div}\mathbf{u}^n}{\bar{h}_0+h^n} - \frac{\bar{h}_0\theta\nabla h \operatorname{div}\mathbf{u}}{\bar{h}_0+h} \\ &\quad = J_n\big[\theta\big(1-L\big(h^n\big)\big)\chi_p\nabla h^n\chi_p\operatorname{div}\big(\mathbf{u}_L^n-\mathbf{u}_L\big) + \theta\big(1-L\big(h^n\big)\big)\chi_p\nabla h^n\chi_p\operatorname{div}\big(\chi_p\big(\bar{\mathbf{u}}^n-\bar{\mathbf{u}}\big)\big) \\ &\qquad + \theta\big(1-L\big(h^n\big)\big)\chi_p\nabla\big(\chi_p\big(h^n-h\big)\big)\operatorname{div}\mathbf{u} + \theta\nabla h\chi_p\operatorname{div}\mathbf{u}\big(L(\chi_p h) - L\big(\chi_p h^n\big)\big)\big] \\ &\qquad + (J_n - I)\frac{\bar{h}_0\theta\nabla h \operatorname{div}\mathbf{u}}{\bar{h}_0+h}. \end{aligned}$$

The last term tends to zero as $n\to+\infty$ due to the property of J_n. As $\theta L(h^n)$ and h^n are u.b. in $L^\infty(\dot{B}^1_{2,1})$ and $\mathbf{u}_L^n$ tends to $\mathbf{u}_L$ in $L^1(\dot{B}^2_{2,1})$, the first term tends to 0 in $L^1(\dot{B}^0_{2,1})$. According to (41), $\chi_p(\bar{\mathbf{u}}^n-\bar{\mathbf{u}})$ tends to zero in $L^1([0,p];\dot{B}^2_{2,1})$ so that the second term tends to 0 in $L^1([0,p];\dot{B}^0_{2,1})$. Clearly, $\chi_p h^n\to\chi_p h$ in $L^\infty(\dot{B}^1_{2,1})$ and $L(\chi_p h^n)\to L(\chi_p h)$ in $L^\infty(L^\infty\cap\dot{B}^1_{2,1})$, so that the third and the last terms also tend to 0 in $L^1(\dot{B}^0_{2,1})$. The other nonlinear terms can be treated in the same way.

We still have to prove that h is continuous in $\tilde{B}^{0,1+\varepsilon}_{2,1}$ and that $\mathbf{u}$ belongs to $\mathcal{C}(\mathbb{R}^+;(\tilde{B}^{0,\varepsilon}_{2,1})^2)$. The continuity of $\mathbf{u}$ is straightforward. Indeed, $\mathbf{u}$ satisfies

$$\partial_t\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} + \mu\Delta\mathbf{u} + 3\mu\nabla\operatorname{div}\mathbf{u} - \mathrm{f}\mathbf{u}^{\perp} - \mathrm{g}\nabla h + \beta\nabla\Delta h + 2\mu\frac{\nabla h D(\mathbf{u}) + \nabla h\operatorname{div}\mathbf{u}}{h+\bar{h}_0}$$

and the r.h.s. belongs to $(L^1+L^\infty)(\dot{B}^0_{2,1})$ by noting that we also have $h\in L^\infty(\dot{B}^1_{2,1})\cap(L^{\frac{4}{3}}+L^1)(\dot{B}^3_{2,1})$ in view of the interpolation argument. In a similar argument, one can obtain $\mathbf{u}\in\mathcal{C}(\mathbb{R}^+;(\dot{B}^{\varepsilon}_{2,1})^2)$. We have already got that $h\in\mathcal{C}(\mathbb{R}^+;\dot{B}^0_{2,1})$. Indeed, $h_0-\bar{h}_0\in\dot{B}^0_{2,1}$, $\mathbf{u}\in L^2(\mathbb{R}^+;\dot{B}^1_{2,1})$, $h\in L^\infty(\mathbb{R}^+;\dot{B}^1_{2,1})$ and then $\partial_t h\in L^2(\mathbb{R}^+;\dot{B}^0_{2,1})$ from the equation $\partial_t h = -\bar{h}_0\operatorname{div}\mathbf{u} - \operatorname{div}(h\mathbf{u})$. Thus, there remains to prove the continuity of h in $\dot{B}^{1+\varepsilon}_{2,1}$.

Let us apply the operator Δ_k to the first equation of (4) to get

$$\partial_t\Delta_k h = -\Delta_k(\mathbf{u}\cdot\nabla h) - \bar{h}_0\Delta_k\operatorname{div}\mathbf{u} - \Delta_k(h\operatorname{div}\mathbf{u}). \tag{42}$$

Obviously, for fixed k the r.h.s. belongs to $L^1_{loc}(\mathbb{R}^+;L^2)$ so that each $\Delta_k h$ is continuous in time with values in L^2.

Now, we apply an energy method to (42) to obtain, with the help of Lemma 6.2 in [11], that

$$\frac{1}{2}\frac{d}{dt}\|\Delta_k h\|^2_{L^2} \leqslant C\|\Delta_k h\|_{L^2}\big(\alpha_k 2^{-k(1+\varepsilon)}\|h\|_{\dot{B}^{1+\varepsilon}_{2,1}}\|\mathbf{u}\|_{\dot{B}^2_{2,1}} + \|\Delta_k\operatorname{div}\mathbf{u}\|_{L^2} + \big\|\Delta_k(h\operatorname{div}\mathbf{u})\big\|_{L^2}\big),$$

where $\sum_k \alpha_k \leqslant 1$. Integrating in time and multiplying $2^{k(1+\varepsilon)}$, we get

$$2^{k(1+\varepsilon)}\|\Delta_k h(t)\|_{L^2} \leqslant 2^{k(1+\varepsilon)}\|\Delta_k(h_0-\bar{h}_0)\|_{L^2} + C\int_0^t \big(\alpha_k \|h(\tau)\|_{\dot{B}^{1+\varepsilon}_{2,1}}\|\mathbf{u}(\tau)\|_{\dot{B}^2_{2,1}}$$
$$+ 2^{k(2+\varepsilon)}\|\Delta_k \mathbf{u}(\tau)\|_{L^2} + 2^{k(1+\varepsilon)}\|\Delta_k(h\operatorname{div}\mathbf{u})(\tau)\|_{L^2}\big)\,d\tau.$$

Since $h \in L^\infty(\dot{B}^{1+\varepsilon}_{2,1})$, $\mathbf{u} \in L^1(\tilde{B}^{2,2+\varepsilon}_{2,1})$ and $h\operatorname{div}\mathbf{u} \in L^1(\dot{B}^{1+\varepsilon}_{2,1})$, we can get

$$\sum_{k\in\mathbb{Z}} \sup_{t\geqslant 0} 2^{k(1+\varepsilon)}\|\Delta_k h(t)\|_{L^2} \lesssim \|h_0-\bar{h}_0\|_{\dot{B}^{1+\varepsilon}_{2,1}} + \big(1+\|h\|_{L^\infty(\dot{B}^{1+\varepsilon}_{2,1})}\big)\|\mathbf{u}\|_{L^1(\tilde{B}^{2,2+\varepsilon}_{2,1})} + \|h\operatorname{div}\mathbf{u}\|_{L^1(\dot{B}^{1+\varepsilon}_{2,1})}$$
$$< \infty.$$

Thus, $\sum_{|k|\leqslant N} \Delta_k h$ converges uniformly in $L^\infty(\mathbb{R}^+; \dot{B}^{1+\varepsilon}_{2,1})$ and we can conclude that $h \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{1+\varepsilon}_{2,1})$.

4.4. Uniqueness

Let $(h_1, \mathbf{u}_1)$ and $(h_2, \mathbf{u}_2)$ be solutions of (4) in $E^1_T \cap E^{1+\varepsilon}_T$ with the same data $(h_0-\bar{h}_0, \mathbf{u}_0)$ constructed in the previous subsections on the time interval $[0, T]$. Denote $(\delta h, \delta\mathbf{u}) = (h_2-h_1, \mathbf{u}_2-\mathbf{u}_1)$. From (4), we can get

$$\begin{cases} \partial_t \delta h + \mathbf{u}_2\cdot\nabla\delta h + \bar{h}_0 \operatorname{div}\delta\mathbf{u} = F_2, \\ \partial_t \delta\mathbf{u} + \mathbf{u}_2\cdot\nabla\delta\mathbf{u} - \mu\Delta\delta\mathbf{u} - 3\mu\nabla\operatorname{div}\delta\mathbf{u} + \mathrm{f}(\delta\mathbf{u})^\perp + \mathrm{g}\nabla\delta h - \beta\nabla\Delta\delta h = G_2, \\ (\delta h, \delta\mathbf{u}) = (0, \mathbf{0}), \end{cases} \tag{43}$$

where

$$F_2 = -\delta\mathbf{u}\cdot\nabla h_1 - \delta h \operatorname{div}\mathbf{u}_2 - h_1\operatorname{div}\delta\mathbf{u},$$
$$G_2 = -\delta\mathbf{u}\cdot\nabla\mathbf{u}_1 + 2\mu\frac{\nabla\delta h\operatorname{div}\mathbf{u}_2}{\bar{h}_0+h_2} + 2\mu\frac{\nabla h_1\operatorname{div}\delta\mathbf{u}}{\bar{h}_0+h_2} + 2\mu\left(\frac{1}{\bar{h}_0+h_2} - \frac{1}{\bar{h}_0+h_1}\right)\nabla h_1\operatorname{div}\mathbf{u}_1$$
$$+ 2\mu\frac{\nabla\delta h D(\mathbf{u}_2)}{\bar{h}_0+h_2} + 2\mu\frac{\nabla h_1 D(\delta\mathbf{u})}{\bar{h}_0+h_2} + 2\mu\left(\frac{1}{\bar{h}_0+h_2} - \frac{1}{\bar{h}_0+h_1}\right)\nabla h_1 D(\mathbf{u}_1).$$

Similar to (4), we can get

$$\|(\delta h, \delta\mathbf{u})\|_{E^1_T} \leqslant Ce^{C\|\mathbf{u}_2\|_{L^1_T(\dot{B}^2_{2,1})}}\big(\|F_2\|_{L^1_T(\tilde{B}^{0,1}_{2,1})} + \|G_2\|_{L^1_T(\dot{B}^0_{2,1})}\big).$$

Noticing that $h_1 \in L^\infty_T(\tilde{B}^{0,1}_{2,1}) \cap L^1_T(\tilde{B}^{4,3}_{2,1})$ and $\mathbf{u}_2 \in L^1_T(\dot{B}^2_{2,1})$, we can get

$$\|F_2\|_{L^1_T(\dot{B}^0_{2,1})} \lesssim T^{\frac{1}{2}}\|\delta\mathbf{u}\|_{L^\infty_T(\dot{B}^0_{2,1})}\|h_1\|_{L^2_T(\dot{B}^2_{2,1})} + \|\delta h\|_{L^\infty_T(\dot{B}^0_{2,1})}\|\mathbf{u}_2\|_{L^1_T(\dot{B}^2_{2,1})} + \|h_1\|_{L^\infty_T(\dot{B}^0_{2,1})}\|\delta\mathbf{u}\|_{L^1_T(\dot{B}^2_{2,1})}.$$

Moreover, from $h_1 \in L^2_T(\dot{B}^2_{2,1}) \cap L^\infty_T(\dot{B}^1_{2,1})$ by Lemma 2.8, we have

$$\|F_2\|_{L^1_T(\dot{B}^1_{2,1})} \lesssim \|\delta\mathbf{u}\|_{L^2_T(\dot{B}^1_{2,1})}\|h_1\|_{L^2_T(\dot{B}^2_{2,1})} + \|\delta h\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\mathbf{u}_2\|_{L^1_T(\dot{B}^2_{2,1})} + \|h_1\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\delta\mathbf{u}\|_{L^1_T(\dot{B}^2_{2,1})}.$$

Noting that $h_1, h_2 \in L^\infty_T(\dot{B}^1_{2,1})$, $\mathbf{u}_1, \mathbf{u}_2 \in L^1_T(\dot{B}^2_{2,1})$, and

$$\|h_1\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant \frac{1}{2}\bar{h}_0, \qquad \|h_2\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant \frac{1}{2}\bar{h}_0,$$

by the construction of solutions, we have

$$\begin{aligned}\|G_2\|_{L^1_T(\dot{B}^0_{2,1})} \lesssim{}& \|\delta\mathbf{u}\|_{L^\infty_T(\dot{B}^0_{2,1})}\|\mathbf{u}_1\|_{L^1_T(\dot{B}^2_{2,1})} + 4\mu\big(1+\|h_2\|_{L^\infty_T(\dot{B}^1_{2,1})}\big)\|\delta h\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\mathbf{u}_2\|_{L^1_T(\dot{B}^2_{2,1})}\\ &+ 4\mu\big(1+\|h_2\|_{L^\infty_T(\dot{B}^1_{2,1})}\big)\|h_1\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\delta\mathbf{u}\|_{L^1_T(\dot{B}^2_{2,1})}\\ &+ 4\mu\|\delta h\|_{L^\infty_T(\dot{B}^1_{2,1})}\|h_1\|_{L^\infty_T(\dot{B}^1_{2,1})}\|\mathbf{u}_1\|_{L^1_T(\dot{B}^2_{2,1})}\\ &\times\big(1+\|h_1\|_{L^\infty_T(\dot{B}^1_{2,1})}+\|h_2\|_{L^\infty_T(\dot{B}^1_{2,1})}+\|h_1\|_{L^\infty_T(\dot{B}^1_{2,1})}\|h_2\|_{L^\infty_T(\dot{B}^1_{2,1})}\big).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\big\|(\delta h,\delta\mathbf{u})\big\|_{E^1_T} \leqslant{}& Ce^{C\|\mathbf{u}_2\|_{L^1_T(\dot{B}^2_{2,1})}}\Big\{\Big(1+T^{\frac{1}{2}}+4\mu\bar{h}_0^{-1}\big(1+\|h_2\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}\big)\Big)\\ &\cdot\|h_1\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}+Z(T)\Big\}\big\|(\delta h,\delta\mathbf{u})\big\|_{E^1_T},\end{aligned}$$

where $\limsup_{T\to0^+} Z(T)=0$.

Supposing that $\frac{A^2+A+2}{A(A+2)}(A+1)\tilde{C}E(0) < \frac{1}{4}$ besides (37) for $E(0)$ and taking $0<T\leqslant 1$ small enough such that $C\|\mathbf{u}_2\|_{L^1_T(\dot{B}^2_{2,1})} \leqslant \ln 2$ and $Z(T) < \frac{1}{2}$, we obtain $\|(\delta h,\delta\mathbf{u})\|_{E^1_T} \equiv 0$. Hence, $(h_1,\mathbf{u}_1)\equiv(h_2,\mathbf{u}_2)$ on $[0,T]$.

Let T_m (supposedly finite) be the largest time such that the two solutions coincide on $[0,T_m]$. If we denote

$$\big(\tilde{h}_i(t),\tilde{\mathbf{u}}_i(t)\big) := \big(h_i(t+T_m),\mathbf{u}_i(t+T_m)\big), \quad i=1,2,$$

we can use the above arguments and the fact that

$$\|\tilde{h}_i\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2)} \leqslant \frac{1}{2}\bar{h}_0 \quad\text{and}\quad \|\tilde{h}_i\|_{L^\infty(\mathbb{R}^+;\dot{B}^0_{2,1}\cap\dot{B}^1_{2,1})} \leqslant A\tilde{C}E(0)$$

to prove that $(\tilde{h}_1,\tilde{\mathbf{u}}_1)=(\tilde{h}_2,\tilde{\mathbf{u}}_2)$ on the interval $[0,T_m]$ with the same T_m as in the above. Therefore, we complete the proofs.

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