On the global well-posedness of bipolar Navier-Stokes-Poisson equations with damping

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Abstract

In this paper, we consider the initial value problem for multi-dimensional bipolar compressible Navier-Stokes-Poisson equations with damping terms. We show the global existence and uniqueness of the strong solutions in hybrid Besov spaces with the initial data close to a stable equilibrium.

Keywords: bipolar compressible Navier-Stokes-Poisson system, global existence.

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1 Introduction

The Cauchy problem of the isentropic bipolar compressible Navier-Stokes-Poisson equations with damping terms is described as follow

$$\begin{cases} \partial_t\rho_1 + \mathrm{div}(\rho_1 u_1) = 0, \\ \partial_t(\rho_1 u_1) + \mathrm{div}(\rho_1 u_1 \otimes u_1) + \nabla P_1 = -\rho_1 u_1 \\ \qquad +\rho_1\nabla\Phi + \mu_1\Delta u_1 + (\mu_1+\lambda_1)\nabla\mathrm{div} u_1, \\ \partial_t\rho_2 + \mathrm{div}(\rho_2 u_2) = 0, \\ \partial_t(\rho_2 u_2) + \mathrm{div}(\rho_2 u_2 \otimes u_2) + \nabla P_2 = -\rho_2 u_2 \\ \qquad -\rho_2\nabla\Phi + \mu_2\Delta u_2 + (\mu_2+\lambda_2)\nabla\mathrm{div} u_2, \\ \Delta\Phi = \rho_1 - \rho_2, \\ (\rho_1,\rho_2,u_1,u_2)(0) = (\rho_{10},\rho_{20},u_{10},u_{20}), \end{cases} \tag{1.1}$$

where the unknown functions are the charge densities $\rho_1 > 0$ and $\rho_2 > 0$, the velocities u_1, u_2 and the electrostatic potential Φ. $P_1(\rho_1)$ and $P_2(\rho_2)$ are the pressure functions satisfying $P_1(\rho_1) > 0, P_2(\rho_2) > 0$. And the viscosity coefficients μ_i and λ_i satisfy $2\mu_i + N\lambda_i > 0$ for $i = 1, 2$.

The bipolar Navier-Stokes-Poisson system (BNSP) has been used to simulate the transport of charged particles (ions and electrons for instance) under the influence of the electrostatic force governed by the self-consistent Poisson equation. There are many interesting results devoted to the unipolar Navier-Stokes-Poisson system (NSP). The global existence of weak solutions to NSP with general initial data was proved in [7, 20]. The quasi-neutral and some related asymptotic limits were studied in [6, 8, 13, 19]. In the case when the Poisson equation describes the self-gravitational force for stellar gases, the global existence of weak solutions and asymptotic behavior were also investigated together with the stability analysis in [10, 11, 14] and the references therein. The global existence and the optimal time convergence rates of the classical solution were obtained recently in [15]. In addition, the global well-posedness of NSP was proved in the Besov type space in [12, 16]. As a continuation of the study in this direction, our original aim is to study the well-posedness of BNSP. But in this paper, we can only deal with BNSP with damping terms since we have to use the damping terms to control the electrostatic force appear in momentum equations in the case of low frequencies.

Our main result reads

Theorem 1.1. *Let $N \geq 3$, $P_i \in C^{[\frac{N}{2}]+3}$ with $P_i'(\bar\rho) > 0$, $\mu_i > 0$ and $2\mu_i + N\lambda_i > 0$ for $i = 1, 2$. Assume that $(\rho_{10}-\bar\rho, \rho_{20}-\bar\rho, \rho_{10}-\rho_{20}, u_{10}, u_{20}) \in W_0 := B^{\frac{N}{2}-1,\frac{N}{2}} \times B^{\frac{N}{2}-1,\frac{N}{2}} \times B^{\frac{N}{2}-2,\frac{N}{2}} \times B^{\frac{N}{2}-1} \times B^{\frac{N}{2}-1}$ satisfies for a constant $\delta > 0$ small enough, that*

$$\|(\rho_{10}-\bar\rho, \rho_{20}-\bar\rho, \rho_{10}-\rho_{20}, u_{10}, u_{20})\|_{W_0} \leq \delta, \tag{1.2}$$

then (1.1) has a unique global solution $(\rho_1, \rho_2, u_1, u_2, \Phi)$ such that

$$\|\rho_1(t)-\bar\rho\|_{B^{\frac{N}{2}-1,\frac{N}{2}}} + \|\rho_2(t)-\bar\rho\|_{B^{\frac{N}{2}-1,\frac{N}{2}}}$$

$$
\begin{aligned}
&+\|u_1(t)\|_{B^{\frac{N}{2}-1}}+\|u_2(t)\|_{B^{\frac{N}{2}-1}}+\|\Phi(t)\|_{B^{\frac{N}{2},\frac{N}{2}+2}}\\
&+\int_0^t\big(\|\rho_1(\tau)-\bar\rho\|_{B^{\frac{N}{2}+1,\frac{N}{2}}}+\|\rho_2(\tau)-\bar\rho\|_{B^{\frac{N}{2}+1,\frac{N}{2}}}+\|u_1(\tau)\|_{B^{\frac{N}{2}+1}}\\
&+\|u_2(\tau)\|_{B^{\frac{N}{2}+1}}+\|\Phi(\tau)\|_{B^{\frac{N}{2}+1,\frac{N}{2}+2}}+\|(u_2-u_1)(\tau)\|_{B^{\frac{N}{2},\frac{N}{2}+1}}\big)d\tau\\
&\le M\|(\rho_{10}-\bar\rho,\rho_{20}-\bar\rho,\rho_{10}-\rho_{20},u_{10},u_{20})\|_{W_0} \qquad (1.3)
\end{aligned}
$$

where $M>0$ is independent of the initial data, and B^s and B^{s_1,s_2} are the homogeneous Besov space and homogeneous hybrid Besov space (see section 2), respectively.

The paper is arranged as follows. In section 2, we recall some basic theories for some homogeneous Besov spaces and given the definitions and some properties of hybrid Besov spaces. In Section 3, we reformulate the original problem and establish the a-priori estimates for the reformulated system. In the section 4, we prove the global existence and uniqueness of the solutions of the original Cauchy problem.

2 Littlewood-Paley decomposition and Besov spaces

We introduce the definition of the homogeneous Besov space B^s and the hybrid homogeneous Besov space B^{s_1,s_2} and list the related embedding inequalities.

Let $\psi:\mathbb{R}^N\to[0,1]$ be a radial smooth cut-off function valued in $[0,1]$ such that

$$
\psi(\xi)=\begin{cases}1, & |\xi|\le\frac34,\\ \textit{smooth}, & \frac34<|\xi|<\frac43,\\ 0, & |\xi|\ge\frac43.\end{cases} \qquad (2.1)
$$

Let φ be the function

$$
\varphi(\xi):=\psi(\xi/2)-\psi(\xi). \qquad (2.2)
$$

Thus, ψ is supported in the ball $\{\xi\in\mathbb{R}^N:|\xi|\le4/3\}$, and φ is also a smooth cut-off function valued in $[0,1]$ and supported in the annulus $\{\xi\in\mathbb{R}^N:3/4\le|\xi|\le8/3\}$. By the construction, we have

$$
\sum_{k\in\mathbb{Z}}\varphi(2^{-k}\xi)=1,\qquad\forall\xi\ne0.
$$

Thus, one can define the dyadic blocks as follows. For $k\in\mathbb{Z}$ and $f\in\mathscr{S}'(\mathbb{R}^N)$, let

$$
\Delta_k f:=\mathscr{F}^{-1}\varphi(2^{-k}\xi)\mathscr{F}f.
$$

Definition 2.1. *For $s\in\mathbb{R}$ and $u\in\mathscr{S}'(\mathbb{R}^N)$, we set*

$$
\|u\|_{B^s}:=\sum_{k\in\mathbb{Z}}2^{kq}\|\Delta_q u\|_{L^2}.
$$

A difficulty stems from the choice of homogeneous spaces arises at this point $\|\cdot\|_{B^s}$ can not be a norm on $\{u \in \mathscr{S}'(\mathbb{R}^N) : \|u\|_{B^s} < +\infty\}$ because $\|u\|_{B^s}$ vanishes if and only if u is a polynomial. This leads us to adopt the following definition for homogeneous Besov spaces [4].

Definition 2.2 (Homogeneous Besov space)**.** *Let $s \in \mathbb{R}$ and $m = -[\frac{N}{2} + 1 - s]$, then we define the space $B^s(\mathbb{R}^N)$ as*

$$B^s(\mathbb{R}^N) = \Big\{ u \in \mathscr{S}'(\mathbb{R}^N) : \|u\|_{B^s} < +\infty \Big\}, \qquad m < 0,$$

$$B^s(\mathbb{R}^N) = \Big\{ u \in \mathscr{S}'(\mathbb{R}^N)/\mathcal{P}_m : \|u\|_{B^s} < +\infty \Big\}, \qquad m \geq 0,$$

where $\mathcal{P}_m$ denotes the set of N variables polynomials of degree less than or equal to m.

We have the following estimates.

Lemma 2.1 ([3, Lemma 2.7])**.** *Let $s > 0$ and $u \in B^s \cap L^\infty$.*

(i) *Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ with $F(0) = 0$. Then $F(u) \in B^s$. Moreover, there exists a function of one variable C_0 depending only on s and F, and such that*

$$\|F(u)\|_{B^s} \leq C_0(\|u\|_{L^\infty})\|u\|_{B^s}.$$

(ii) *Let $G \in W^{[\frac{N}{2}]+3,\infty}_{loc}(\mathbb{R}^N)$ satisfy $G'(0) = 0$. Suppose that $u, v \in B^{\frac{N}{2}}$ and $(u - v) \in B^s$ for $s \in (-\frac{N}{2}, \frac{N}{2}]$. Then $G(u) - G(v) \in B^s$ and there exits a function of two variables C depending only on s, N and G, and such that*

$$\|G(u) - G(v)\|_{B^s} \leq C\Big(\|u\|_{L^\infty}, \|v\|_{L^\infty}\Big)\Big(\|u\|_{B^{\frac{N}{2}}} + \|v\|_{B^{\frac{N}{2}}}\Big)\|u - v\|_{B^s}.$$

Next, we define hybrid Besov spaces which have different regularities in low frequencies and high frequencies (see [4]).

Definition 2.3 (Hybrid homogeneous Besov space)**.** *For $s, t \in \mathbb{R}$, denote*

$$\|u\|_{B^{s,t}} = \sum_{k \leq 0} 2^{ks}\|\Delta_k u\|_{L^2} + \sum_{k > 0} 2^{kt}\|\Delta_k u\|_{L^2}.$$

Let $m = -[\frac{N}{2} + 1 - s]$, then we define the space $B^{s,t}(\mathbb{R}^N)$ as

$$B^{s,t}(\mathbb{R}^N) = \Big\{ u \in \mathscr{S}'(\mathbb{R}^N) : \|u\|_{B^{s,t}} < +\infty \Big\}, \qquad m < 0,$$

$$B^{s,t}(\mathbb{R}^N) = \Big\{ u \in \mathscr{S}'(\mathbb{R}^N)/\mathcal{P}_m : \|u\|_{B^{s,t}} < +\infty \Big\}, \qquad m \geq 0.$$

Remark 2.2. *The following conclusions to be used in this paper hold obviously.*

(i) *We have* $B^{s,s} = B^s$.

(ii) *If* $s \leq t$ *then* $B^{s,t} = B^s \cap B^t$. *Otherwise,* $B^{s,t} = B^s + B^t$.

(iii) *The space* $B^{0,s}$ *coincides with the usual inhomogeneous Besov space* $B^s_{2,1}$.

(iv) *If* $s_1 \leq s_2$ *and* $t_1 \geq t_2$, *then* $B^{s_1,t_1} \hookrightarrow B^{s_2,t_2}$.

(v) $B^{\frac{N}{2}} \hookrightarrow L^\infty$.

Let us state some estimates for the product in hybrid Besov spaces.

Lemma 2.3 ([3, Lemma 2.7]). *For all* $s, t > 0$,

$$\|uv\|_{B^{s,t}} \lesssim \|u\|_{L^\infty}\|v\|_{B^{s,t}} + \|v\|_{L^\infty}\|u\|_{B^{s,t}}.$$

For all $s_1, s_2, t_1, t_2 \leq \frac{N}{2}$ *such that* $\min(s_1+t_1, s_2+t_2) > 0$,

$$\|uv\|_{B^{s_1+t_1-\frac{N}{2}, s_2+t_2-\frac{N}{2}}} \lesssim \|u\|_{B^{s_1,s_2}}\|v\|_{B^{t_1,t_2}}.$$

3 Reformulation and the a-priori estimates

Let $c_1 = \rho_1 - \bar{\rho}$ and $c_2 = \rho_2 - \bar{\rho}$, then (1.1) can be rewritten as

$$\begin{cases} \partial_t c_1 + u_1 \cdot \nabla c_1 + \bar{\rho}\mathrm{div} u_1 = -c_1 \mathrm{div} u_1 =: F_1, \\ \partial_t u_1 + u_1 \cdot \nabla u_1 - \frac{\mu_1}{\bar{\rho}}\Delta u_1 - \frac{\mu_1+\lambda_1}{\bar{\rho}}\nabla \mathrm{div} u_1 + \frac{1}{\bar{\rho}}P_1'(\bar{\rho})\nabla c_1 \\ \qquad +\nabla\Lambda^{-2}(c_1 - c_2) + u_1 = -\frac{c_1}{c_1+\bar{\rho}}A_1 u_1 + K_1(c_1)\nabla c_1, \\ \partial_t c_2 + u_2 \cdot \nabla c_2 + \bar{\rho}\mathrm{div} u_2 = -c_2 \mathrm{div} u_2 =: F_2, \\ \partial_t u_2 + u_2 \cdot \nabla u_2 - \frac{\mu_2}{\bar{\rho}}\Delta u_2 - \frac{\mu_2+\lambda_2}{\bar{\rho}}\nabla \mathrm{div} u_2 + \frac{1}{\bar{\rho}}P_2'(\bar{\rho})\nabla c_2 \\ \qquad -\nabla\Lambda^{-2}(c_1 - c_2) + u_2 = -\frac{c_2}{c_2+\bar{\rho}}A_2 u_2 + K_2(c_2)\nabla c_2, \\ \Delta\Phi = c_1 - c_2, \\ (c_1, c_2, u_1, u_2)(0) = (\rho_{10} - \bar{\rho}, \rho_{20} - \bar{\rho}, u_{10}, u_{20}), \end{cases} \tag{3.1}$$

where

$$K_i(c) = \frac{P_i'(c+\bar{\rho})}{c+\bar{\rho}} - \frac{P_i'(\bar{\rho})}{\bar{\rho}}, \quad i = 1, 2. \tag{3.2}$$

Denote

$$\bar{\mu}_i = \frac{\mu_i}{\bar{\rho}}, \quad \bar{\lambda}_i = \frac{\lambda_i}{\bar{\rho}}, \quad \kappa_i = P_i'(\bar{\rho}), \tag{3.3}$$

$$A_i := \mu_i\Delta + (\lambda_i + \mu_i)\nabla\mathrm{div}, \quad i = 1, 2, \tag{3.4}$$

$$\Lambda^s f := \mathscr{F}^{-1}|\xi|^s\mathscr{F}f, \quad E := \Lambda\Phi, \tag{3.5}$$

Let $d_i = \Lambda^{-1}\mathrm{div}u_i$ and $\Omega_i = \Lambda^{-1}\mathrm{curl}u_i$ be the "compressible part" and "incompressible part" of u_i, respectively. Then,

$$u_i = -\Lambda^{-1}\nabla d_i - \Lambda^{-1}\mathrm{div}\Omega_i, \tag{3.6}$$

Then $(3.1)_1$ and $(3.1)_3$ turn into

$$\partial_t c_1 + u_1 \cdot \nabla c_1 + \bar{\rho}\Lambda d_1 = F_1, \quad \partial_t c_2 + u_2 \cdot \nabla c_2 + \bar{\rho}\Lambda d_2 = F_2. \tag{3.7}$$

Applying $\Lambda^{-1}\mathrm{div}$ and $\Lambda^{-1}\mathrm{curl}$ to both sides of $(3.1)_2$ and $(3.1)_4$, respectively, we get

$$\begin{cases} \partial_t d_1 + u_1 \cdot \nabla d_1 - (2\bar{\mu}_1 + \bar{\lambda}_1)\Delta d_1 \\ \quad -\kappa_1 \Lambda c_1 + \bar{\rho}E + d_1 = G_1, \\ \partial_t \Omega_1 - \bar{\mu}_1 \Delta\Omega_1 + \Omega_1 = \bar{G}_1, \\ \partial_t d_2 + u_2 \cdot \nabla d_2 - (2\bar{\mu}_2 + \bar{\lambda}_2)\Delta d_2 \\ \quad -\kappa_2 \Lambda c_2 - \bar{\rho}E + d_2 = G_2, \\ \partial_t \Omega_2 - \bar{\mu}_2 \Delta\Omega_2 + \Omega_2 = \bar{G}_2, \end{cases} \tag{3.8}$$

where

$$G_i = u_i \cdot \nabla d_i - \Lambda^{-1}\mathrm{div}\Big(u_i \cdot \nabla u_i - \frac{c_i}{c_i + \bar{\rho}} A_i u_i - K_i(c_i)\nabla c_i\Big), \tag{3.9}$$

$$\bar{G}_i = -\Lambda^{-1}\mathrm{curl}\Big(u_i \cdot \nabla u_i - \frac{c_i}{c_i + \bar{\rho}} A_i u_i\Big). \tag{3.10}$$

By $(3.1)_5$ and (3.5), we obtain the equation

$$E = -\Lambda^{-1}(c_1 - c_2). \tag{3.11}$$

Thus, the system (3.1) can be written as

$$\begin{cases} \partial_t c_1 + u_1 \cdot \nabla c_1 + \bar{\rho}\Lambda d_1 = F_1, \\ \partial_t d_1 + u_1 \cdot \nabla d_1 - (2\bar{\mu}_1 + \bar{\lambda}_1)\Delta d_1 - \kappa_1 \Lambda c_1 + \bar{\rho}E + d_1 = G_1, \\ \partial_t \Omega_1 - \bar{\mu}_1 \Delta\Omega_1 + \Omega_1 = \bar{G}_1, \\ \partial_t c_2 + u_2 \cdot \nabla c_2 + \bar{\rho}\Lambda d_2 = F_2, \\ \partial_t d_2 + u_2 \cdot \nabla d_2 - (2\bar{\mu}_2 + \bar{\lambda}_2)\Delta d_2 - \kappa_2 \Lambda c_2 - \bar{\rho}E + d_2 = G_2, \\ \partial_t \Omega_2 - \bar{\mu}_2 \Delta\Omega_2 + \Omega_2 = \bar{G}_2, \\ E = -\Lambda^{-1}(c_1 - c_2), \\ u_i = -\Lambda^{-1}\nabla d_i - \Lambda^{-1}\mathrm{div}\Omega_i, \\ (c_1, c_2, d_1, d_2, \Omega_1, \Omega_2)(0) = (\rho_{10} - \bar{\rho}, \rho_{20} - \bar{\rho}, \Lambda^{-1}\mathrm{div}u_{10}, \\ \qquad\qquad \Lambda^{-1}\mathrm{div}u_{20}, \Lambda^{-1}\mathrm{curl}u_{10}, \Lambda^{-1}\mathrm{curl}u_{20}). \end{cases} \tag{3.12}$$

We will prove the following proposition.

Proposition 3.1. *Let (c_1, c_2, d_1, d_2, E) be a solution of*

$$\begin{cases} \partial_t c_1 + u_1 \cdot \nabla c_1 + \bar{\rho}\Lambda d_1 = F_1, \\ \partial_t d_1 + u_1 \cdot \nabla d_1 - (2\bar{\mu}_1 + \bar{\lambda}_1)\Delta d_1 - \kappa_1 \Lambda c_1 + \bar{\rho} E + d_1 = G_1, \\ \partial_t c_2 + u_2 \cdot \nabla c_2 + \bar{\rho}\Lambda d_2 = F_2, \\ \partial_t d_2 + u_2 \cdot \nabla d_2 - (2\bar{\mu}_2 + \bar{\lambda}_2)\Delta d_2 - \kappa_2 \Lambda c_2 - \bar{\rho} E + d_2 = G_2, \\ E = -\Lambda^{-1}(c_1 - c_2), \\ (c_1, c_2, d_1, d_2)(0) = (c_{10}, c_{20}, d_{10}, d_{20}), \end{cases} \tag{3.13}$$

on $[0, T)$, $N \geq 3$, $2 - \frac{N}{2} < s \leq \frac{N}{2} + 1$ *and*

$$V(t) = \int_0^t \big(\|u_1(\tau)\|_{B^{\frac{N}{2}+1}} + \|u_2(\tau)\|_{B^{\frac{N}{2}+1}} + \|(u_1 - u_2)(\tau)\|_{B^{\frac{N}{2}, \frac{N}{2}+1}} \big) d\tau.$$

Then the following estimate holds on $[0, T)$*:*

$$\begin{aligned} &\|c_1(t)\|_{B^{s-1,s}} + \|c_2(t)\|_{B^{s-1,s}} + \|d_1(t)\|_{B^{s-1}} + \|d_2(t)\|_{B^{s-1}} \\ &\quad + \|E(t)\|_{B^{s-1,s+1}} + \int_0^t \big(\|c_1(t)\|_{B^{s+1,s}} + \|c_2(t)\|_{B^{s+1,s}} + \|d_1(\tau)\|_{B^{s+1}} \\ &\quad + \|d_2(\tau)\|_{B^{s+1}} + \|E(\tau)\|_{B^{s,s+1}} + \|(d_1 - d_2)(\tau)\|_{B^{s,s+1}} \big) d\tau \\ &\leq C e^{CV(t)} \bigg(\|c_{10}\|_{B^{s-1,s}} + \|c_{20}\|_{B^{s-1,s}} + \|d_{10}\|_{B^{s-1}} + \|d_{20}\|_{B^{s-1}} \\ &\quad + \|c_{20} - c_{10}\|_{B^{s-2,s}} + \int_0^t e^{-CV(\tau)} \big(\|F_1(\tau)\|_{B^{s-1,s}} + \|F_2(\tau)\|_{B^{s-1,s}} \\ &\quad \|G_1(\tau)\|_{B^{s-1}} + \|G_2(\tau)\|_{B^{s-1}} + \|F_2(\tau) - F_1(\tau)\|_{B^{s-2,s}} \big) d\tau \bigg), \end{aligned} \tag{3.14}$$

where C depends only on N and s.

Proof. Let (c_1, c_2, d_1, d_2, E) be a solution of (3.13). Denote $f_k := \Delta_k f$ and $\tilde{f} := e^{-KV(t)} f$, where K is a large constant.

Applying the operator Δ_k to (3.1), we infer that $(\tilde{c}_{1k}, \tilde{c}_{2k}, \tilde{d}_{1k}, \tilde{d}_{2k}, \tilde{E}_k)$ satisfies

$$\begin{cases} \partial_t \tilde{c}_{1k} + \Delta_k(u_1 \cdot \nabla \tilde{c}_1) + \bar{\rho}\Lambda \tilde{d}_{1k} = \tilde{F}_{1k} - KV'(t)\tilde{c}_{1k}, \\ \partial_t \tilde{d}_{1k} + \Delta_k(u_1 \cdot \nabla \tilde{d}_1) - (2\bar{\mu}_1 + \bar{\lambda}_1)\Delta \tilde{d}_{1k} - \kappa_1 \Lambda \tilde{c}_{1k} \\ \quad + \bar{\rho}\tilde{E}_k + \tilde{d}_{1k} = \tilde{G}_{1k} - KV'(t)\tilde{d}_{1k}, \\ \partial_t \tilde{c}_{2k} + \Delta_k(u_2 \cdot \nabla \tilde{c}_2) + \bar{\rho}\Lambda \tilde{d}_{2k} = \tilde{F}_{2k} - KV'(t)\tilde{c}_{2k}, \\ \partial_t \tilde{d}_{2k} + \Delta_k(u_2 \cdot \nabla \tilde{d}_2) - (2\bar{\mu}_2 + \bar{\lambda}_2)\Delta \tilde{d}_{2k} - \kappa_2 \Lambda \tilde{c}_{2k} \\ \quad - \bar{\rho}\tilde{E}_k + \tilde{d}_{2k} = \tilde{G}_{2k} - KV'(t)\tilde{d}_{2k}, \\ \tilde{E}_k = -\Lambda^{-1}(\tilde{c}_{1k} - \tilde{c}_{2k}). \end{cases} \tag{3.15}$$

$\underline{\textit{Step 1. Low frequencies } (k \le k_0(\textit{will be determined}))}$. Taking the L^2 scalar product of $(3.15)_1$ with $\tilde{c}_{1k}$, $(3.15)_2$ with $\tilde{d}_{1k}$, $(3.15)_3$ with $\tilde{c}_{2k}$, and $(3.15)_4$ with $\tilde{d}_{2k}$, we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\tilde{c}_{1k}\|_2^2 + (\Delta_k(u_1\cdot\nabla\tilde{c}_1),\tilde{c}_{1k}) + \bar{\rho}(\Lambda\tilde{d}_{1k},\tilde{c}_{1k}) \\ \quad = (\tilde{F}_{1k},\tilde{c}_{1k}) - KV'(t)\|\tilde{c}_{1k}\|_2^2, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{d}_{1k}\|_2^2 + (\Delta_k(u_1\cdot\nabla\tilde{d}_1),\tilde{d}_{1k}) + (2\bar{\mu}_1+\bar{\lambda}_1)\|\tilde{\Lambda}d_{1k}\|_2^2 - \kappa_1(\Lambda\tilde{c}_{1k},\tilde{d}_{1k}) \\ \quad +\bar{\rho}(\tilde{E}_k,\tilde{d}_{1k}) + \|\tilde{d}_{1k}\|_2^2 = (\tilde{G}_{1k},\tilde{d}_{1k}) - KV'(t)\|\tilde{d}_{1k}\|_2^2, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{c}_{2k}\|_2^2 + (\Delta_k(u_2\cdot\nabla\tilde{c}_2),\tilde{c}_{2k}) + \bar{\rho}(\Lambda\tilde{d}_{2k},\tilde{c}_{2k}) \\ \quad = (\tilde{F}_{2k},\tilde{c}_{2k}) - KV'(t)\|\tilde{c}_{2k}\|_2^2, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{d}_{2k}\|_2^2 + (\Delta_k(u_2\cdot\nabla\tilde{d}_2),\tilde{d}_{2k}) + (2\bar{\mu}_2+\bar{\lambda}_2)\|\Lambda\tilde{d}_{2k}\|_2^2 - \kappa_2(\Lambda\tilde{c}_{2k},\tilde{d}_{2k}) \\ \quad -\bar{\rho}(\tilde{E}_k,\tilde{d}_{2k}) + \|\tilde{d}_{2k}\|_2^2 = (\tilde{G}_{2k},\tilde{d}_{2k}) - KV'(t)\|\tilde{d}_{2k}\|_2^2. \end{cases} \tag{3.16}$$

By $(3.15)_1$, $(3.15)_3$ and $(3.15)_5$, we can easily get

$$\begin{aligned} &\partial_t E_k + \Lambda^{-1}\Delta_k(u_2\cdot\nabla\Lambda\tilde{E}) + \Lambda^{-1}\Delta_k((u_2-u_1)\cdot\nabla\tilde{c}_1) \\ &\quad + \bar{\rho}(\tilde{d}_{2k}-\tilde{d}_{1k}) = \Lambda^{-1}(\tilde{F}_{2k}-\tilde{F}_{1k}) - KV'(t)\tilde{E}_k, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\tilde{E}_k\|_2^2 + (\Lambda^{-1}\Delta_k(u_2\cdot\nabla\Lambda\tilde{E}),\tilde{E}_k) + (\Lambda^{-1}\Delta_k((u_2-u_1)\cdot\nabla\tilde{c}_1),\tilde{E}_k) \\ &\quad + \bar{\rho}(\tilde{d}_{2k}-\tilde{d}_{1k},\tilde{E}_k) = (\Lambda^{-1}(\tilde{F}_{2k}-\tilde{F}_{1k}),\tilde{E}_k) - KV'(t)\|\tilde{E}_k\|_2^2. \end{aligned} \tag{3.18}$$

In order to get the smooth proposition of $\tilde{c}_1$ and $\tilde{c}_2$, we need an equality involving $\|\Lambda\tilde{c}_{1k}\|_2^2$ and $\|\Lambda\tilde{c}_{2k}\|_2^2$. To achieve it, we take L^2 scalar product of $(3.15)_1$ with $\Lambda\tilde{d}_k$ and $(3.15)_2$ with $\Lambda\tilde{c}_k$ then sum up both equalities to get

$$\begin{aligned} &\frac{d}{dt}(\tilde{c}_{1k},\Lambda\tilde{d}_{1k}) + (\Delta_k(u_1\cdot\nabla\tilde{c}_1),\Lambda\tilde{d}_{1k}) + \bar{\rho}\|\Lambda\tilde{d}_{1k}\|_2^2 + (2\bar{\mu}_1+\bar{\lambda}_1)(\Lambda^2\tilde{d}_{1k},\Lambda\tilde{c}_{1k}) \\ &\quad + (\Delta_k(u_1\cdot\nabla\tilde{d}_1),\Lambda\tilde{c}_{1k}) - \kappa_1\|\Lambda\tilde{c}_{1k}\|_2^2 + \bar{\rho}(\tilde{E}_k,\Lambda\tilde{c}_{1k}) + (\tilde{d}_{1k},\Lambda\tilde{c}_{1k}) \\ &\quad = (\tilde{F}_{1k},\Lambda\tilde{d}_{1k}) + (\tilde{G}_{1k},\Lambda\tilde{c}_{1k}) - KV'(t)(\tilde{c}_{1k},\Lambda\tilde{d}_{1k}). \end{aligned} \tag{3.19}$$

At the same way, we have

$$\begin{aligned} &\frac{d}{dt}(\tilde{c}_{2k},\Lambda\tilde{d}_{2k}) + (\Delta_k(u_2\cdot\nabla\tilde{c}_2),\Lambda\tilde{d}_{2k}) + \bar{\rho}\|\Lambda\tilde{d}_{2k}\|_2^2 + (2\bar{\mu}_2+\bar{\lambda}_2)(\Lambda^2\tilde{d}_{2k},\Lambda\tilde{c}_{2k}) \\ &\quad + (\Delta_k(u_2\cdot\nabla\tilde{d}_2),\Lambda\tilde{c}_{2k}) - \kappa_2\|\Lambda\tilde{c}_{2k}\|_2^2 - \bar{\rho}(\tilde{E}_k,\Lambda\tilde{c}_{2k}) + (\tilde{d}_{2k},\Lambda\tilde{c}_{2k}) \\ &\quad = (\tilde{F}_{2k},\Lambda\tilde{d}_{2k}) + (\tilde{G}_{2k},\Lambda\tilde{c}_{2k}) - KV'(t)(\tilde{c}_{2k},\Lambda\tilde{d}_{2k}). \end{aligned} \tag{3.20}$$

Using $(3.15)_5$, we obtain

$$\bar{\rho}(\tilde{E}_k,\Lambda\tilde{c}_{1k}-\Lambda\tilde{c}_{2k}) = -\bar{\rho}(\tilde{E}_k,\Lambda^2\tilde{E}_k) = -\bar{\rho}\|\Lambda\tilde{E}_k\|_2^2. \tag{3.21}$$

Define

$$
\begin{aligned}
f_k^2 = {} & \frac{(\kappa_1 + A)}{\bar{\rho}} \|\tilde{c}_{1k}\|_2^2 + \frac{(\kappa_2 + A)}{\bar{\rho}} \|\tilde{c}_{2k}\|_2^2 + \|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2 \\
& + \|\tilde{E}_k\|_2^2 - 2A\big((\tilde{c}_{1k}, \Lambda \tilde{d}_{1k}) + (\tilde{c}_{2k}, \Lambda \tilde{d}_{2k})\big).
\end{aligned}
\tag{3.22}
$$

By the linear combination of $(3.16)_1 - (3.16)_4$ and $(3.18) - (3.20)$, we have

$$
\begin{aligned}
& \frac{d}{dt} f_k^2 + A\kappa_1 \|\Lambda \tilde{c}_{1k}\|_2^2 + A\kappa_2 \|\Lambda \tilde{c}_{2k}\|_2^2 + (2\bar{\mu}_1 + \bar{\lambda}_1 - A\bar{\rho}) \|\Lambda \tilde{d}_{1k}\|_2^2 \\
& \quad + (2\bar{\mu}_2 + \bar{\lambda}_2 - A\bar{\rho}) \|\Lambda \tilde{d}_{2k}\|_2^2 + A\bar{\rho} \|\Lambda \tilde{E}_k\|_2^2 + \|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2 \\
& \quad - A(2\bar{\mu}_1 + \bar{\lambda}_1)(\Lambda^2 \tilde{d}_{1k}, \Lambda \tilde{c}_{1k}) - A(2\bar{\mu}_2 + \bar{\lambda}_2)(\Lambda^2 \tilde{d}_{2k}, \Lambda \tilde{c}_{2k}) \\
& \quad = \frac{\kappa_1 + A}{\bar{\rho}} (\tilde{F}_{1k}, \tilde{c}_{1k}) + \frac{\kappa_2 + A}{\bar{\rho}} (\tilde{F}_{2k}, \tilde{c}_{2k}) + (\tilde{G}_{1k}, \tilde{d}_{1k}) + (\tilde{G}_{2k}, \tilde{d}_{2k}) \\
& \quad - \frac{\kappa_1 + A}{\bar{\rho}} (\Delta_k(u_1 \cdot \nabla \tilde{c}_1), \tilde{c}_{1k}) - \frac{\kappa_2 + A}{\bar{\rho}} (\Delta_k(u_2 \cdot \nabla \tilde{c}_2), \tilde{c}_{2k}) \\
& \quad - (\Delta_k(u_1 \cdot \nabla \tilde{d}_1), \tilde{d}_{1k}) - (\Delta_k(u_2 \cdot \nabla \tilde{d}_2), \tilde{d}_{2k}) + (\Lambda^{-1}(\tilde{F}_{2k} - \tilde{F}_{1k}), \tilde{E}_k) \\
& \quad + A\big[(\Delta_k(u_1 \cdot \nabla \tilde{c}_1), \Lambda \tilde{d}_{1k}) + (\Delta_k(u_1 \cdot \nabla \tilde{d}_1), \Lambda \tilde{c}_{1k})\big] - KV'(t) f_k^2 \\
& \quad + A\big[(\Delta_k(u_2 \cdot \nabla \tilde{c}_2), \Lambda \tilde{d}_{2k}) + (\Delta_k(u_2 \cdot \nabla \tilde{d}_2), \Lambda \tilde{c}_{2k})\big] \\
& \quad - (\Lambda^{-1} \Delta_k(u_2 \cdot \nabla \Lambda \tilde{E}), \tilde{E}_k) - (\Lambda^{-1} \Delta_k((u_2 - u_1) \cdot \nabla \tilde{c}_1), \tilde{E}_k).
\end{aligned}
\tag{3.23}
$$

By Cauchy's inequality and Bernstein's inequality, we get

$$
|2A(\tilde{c}_{1k}, \Lambda \tilde{d}_{1k})| \le A\|\tilde{c}_{1k}\|_2^2 + \frac{2^{2k_0+6} A}{9} \|\tilde{d}_{1k}\|_2^2, \tag{3.24}
$$

$$
|2A(\tilde{c}_{2k}, \Lambda \tilde{d}_{2k})| \le A\|\tilde{c}_{2k}\|_2^2 + \frac{2^{2k_0+6} A}{9} \|\tilde{d}_{2k}\|_2^2, \tag{3.25}
$$

$$
|A(2\bar{\mu}_1 + \bar{\lambda}_1)(\Lambda^2 \tilde{d}_{1k}, \Lambda \tilde{c}_{1k})| \le \frac{A\kappa_1}{2} \|\Lambda \tilde{c}_{1k}\|_2^2 + \frac{2^{2k_0+6} A (2\bar{\mu}_1 + \bar{\lambda}_1)^2}{18\kappa_1} \|\Lambda \tilde{d}_{1k}\|_2^2, \tag{3.26}
$$

$$
|A(2\bar{\mu}_2 + \bar{\lambda}_2)(\Lambda^2 \tilde{d}_{2k}, \Lambda \tilde{c}_{2k})| \le \frac{A\kappa_2}{2} \|\Lambda \tilde{c}_{2k}\|_2^2 + \frac{2^{2k_0+6} A (2\bar{\mu}_2 + \bar{\lambda}_2)^2}{18\kappa_2} \|\Lambda \tilde{d}_{2k}\|_2^2. \tag{3.27}
$$

Taking

$$
\begin{aligned}
A = \min\Big\{ & \frac{2\bar{\mu}_1 + \bar{\lambda}_1}{4\bar{\rho}},\ \frac{2\bar{\mu}_2 + \bar{\lambda}_2}{4\bar{\rho}},\ \frac{9\kappa_1}{2^{2k_0+7}(2\bar{\mu}_1 + \bar{\lambda}_1)}, \\
& \frac{9\kappa_2}{2^{2k_0+7}(2\bar{\mu}_2 + \bar{\lambda}_2)},\ \frac{\kappa_1}{4\bar{\rho}}, \frac{\kappa_2}{4\bar{\rho}},\ \frac{9}{2^{2k_0+8}} \Big\},
\end{aligned}
\tag{3.28}
$$

we have

$$
f_k^2 \sim \|\tilde{c}_{1k}\|_2^2 + \|\tilde{c}_{2k}\|_2^2 + \|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2 + \|\tilde{E}_k\|_2^2, \tag{3.29}
$$

and there exists a constant a such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}f_k^2 + a2^{2k}f_k^2 \lesssim \frac{d}{dt}f_k^2 + A\kappa_1\|\Lambda\tilde{c}_{1k}\|_2^2 + A\kappa_2\|\Lambda\tilde{c}_{2k}\|_2^2 \\
&\quad + (2\bar{\mu}_1 + \bar{\lambda}_1 - A\bar{\rho})\|\Lambda\tilde{d}_{1k}\|_2^2 + (2\bar{\mu}_2 + \bar{\lambda}_2 - A\bar{\rho})\|\Lambda\tilde{d}_{2k}\|_2^2 \\
&\quad + A\bar{\rho}\|\Lambda\tilde{E}_k\|_2^2 + \|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2 - A(2\bar{\mu}_1 + \bar{\lambda}_1)(\Lambda^2\tilde{d}_{1k}, \Lambda\tilde{c}_{1k}) \\
&\quad - A(2\bar{\mu}_2 + \bar{\lambda}_2)(\Lambda^2\tilde{d}_{2k}, \Lambda\tilde{c}_{2k}). \qquad (3.30)
\end{aligned}$$

Applying Lemma 6.2. of [4] to the convection terms of (3.23), we get the following estimates

$$|(\Delta_k(u_i\cdot\nabla\tilde{c}_i), \tilde{c}_{ik})| \lesssim J_k 2^{-k(s-1)}\|u_i\|_{B^{\frac{N}{2}+1}}\|\tilde{c}_i\|_{B^{s-1,s}}\|\tilde{c}_{ik}\|_2, \qquad (3.31)$$

$$|(\Delta_k(u_i\cdot\nabla\tilde{d}_i), \tilde{d}_{ik})| \lesssim J_k 2^{-k(s-1)}\|u_i\|_{B^{\frac{N}{2}+1}}\|\tilde{d}_i\|_{B^{s-1}}\|\tilde{d}_{ik}\|_2, \qquad (3.32)$$

$$\begin{aligned}
&|(\Delta_k(u_i\cdot\nabla\tilde{c}_i), \Lambda\tilde{d}_{ik}) + (\Delta_k(u_i\cdot\nabla\tilde{d}_i), \Lambda\tilde{c}_{ik})| \\
&\quad \lesssim J_k 2^{-k(s-1)}\|u_i\|_{B^{\frac{N}{2}+1}}(\|\tilde{c}_i\|_{B^{s-1,s}}\|\tilde{d}_{ik}\|_2 + \|\tilde{d}_i\|_{B^{s-1}}\|\tilde{c}_{ik}\|_2), \qquad (3.33)
\end{aligned}$$

$$|(\Lambda^{-1}\Delta_k(u_2\cdot\nabla\Lambda\tilde{E}), \tilde{E}_k)| \lesssim J_k 2^{-k(s-1)}\|u_2\|_{B^{\frac{N}{2}+1}}\|\tilde{E}\|_{B^{s-1,s+1}}\|\tilde{E}_k\|_2, \qquad (3.34)$$

where $i = 1, 2$, $\sum_{k\in\mathbb{Z}} J_k \leq 1$. And by the paradifferential calculus and Cauchy's inequality, we get

$$\begin{aligned}
&|(\Lambda^{-1}\Delta_k((u_2 - u_1)\cdot\nabla\tilde{c}_1), \tilde{E}_k)| \\
&\quad \lesssim J_k 2^{-k(s-1)}\|u_2 - u_1\|_{B^{\frac{N}{2},\frac{N}{2}+1}}\|c_1\|_{B^{s-1,s}}\|\tilde{E}_k\|_2. \qquad (3.35)
\end{aligned}$$

Combining (3.23) and (3.30)-(3.35), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}f_k^2 + a2^{2k}f_k^2 \lesssim \big(\|\tilde{F}_{1k}\|_2 + \|\tilde{F}_{2k}\|_2 + \|\tilde{G}_{1k}\|_2 + \|\tilde{G}_{2k}\|_2 \\
&\quad + \|\Lambda^{-1}(\tilde{F}_{1k} - \tilde{F}_{2k})\|_2\big)f_k + J_k 2^{-k(s-1)}\big(\|u_1(\tau)\|_{B^{\frac{N}{2}+1}} \\
&\quad + \|u_2(\tau)\|_{B^{\frac{N}{2}+1}} + \|(u_1 - u_2)(\tau)\|_{B^{\frac{N}{2},\frac{N}{2}+1}}\big)\big(\|\tilde{c}_1(t)\|_{B^{s-1,s}} \\
&\quad + \|\tilde{c}_2(t)\|_{B^{s-1,s}} + \|\tilde{d}_1(t)\|_{B^{s-1}} + \|\tilde{d}_2(t)\|_{B^{s-1}} + \|\tilde{E}(t)\|_{B^{s-1,s+1}}\big) \\
&\quad - KV'(t)f_k^2, \qquad (3.36)
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}f_k + a2^{2k}f_k \lesssim \big(\|\tilde{F}_{1k}\|_2 + \|\tilde{F}_{2k}\|_2 + \|\tilde{G}_{1k}\|_2 + \|\tilde{G}_{2k}\|_2 \\
&\quad + \|\Lambda^{-1}(\tilde{F}_{1k} - \tilde{F}_{2k})\|_2\big) + J_k 2^{-k(s-1)}\big(\|u_1(t)\|_{B^{\frac{N}{2}+1}} \\
&\quad + \|u_2(t)\|_{B^{\frac{N}{2}+1}} + \|(u_1 - u_2)(t)\|_{B^{\frac{N}{2},\frac{N}{2}+1}}\big)\big(\|\tilde{c}_1(t)\|_{B^{s-1,s}} \\
&\quad + \|\tilde{c}_2(t)\|_{B^{s-1,s}} + \|\tilde{d}_1(t)\|_{B^{s-1}} + \|\tilde{d}_2(t)\|_{B^{s-1}} + \|\tilde{E}(t)\|_{B^{s-1,s+1}}\big) \\
&\quad - KV'(t)f_k. \qquad (3.37)
\end{aligned}$$

Thus $\int_0^t \sum_{k\le k_0} 2^{(s-1)k} \times$ (3.37)$d\tau$ and (3.29) imply

$$
\begin{aligned}
&\sum_{k\le k_0} 2^{(s-1)k}\big(\|\tilde c_{1k}(t)\|_2 + \|\tilde c_{2k}(t)\|_2 + \|\tilde d_{1k}(t)\|_2 + \|\tilde d_{2k}(t)\|_2 + \|\tilde E_k(t)\|_2\big)\\
&\int_0^t \sum_{k\le k_0} 2^{(s+1)k}\big(\|\tilde c_{1k}(\tau)\|_2 + \|\tilde c_{2k}(\tau)\|_2 + \|\tilde d_{1k}(\tau)\|_2 + \|\tilde d_{2k}(\tau)\|_2 + \|\tilde E_k(\tau)\|_2\big)d\tau\\
&\lesssim \|\tilde c_{10}\|_{B^{s-1,s}} + \|\tilde c_{20}\|_{B^{s-1,s}} + \|\tilde d_{10}\|_{B^{s-1}} + \|\tilde d_{20}\|_{B^{s-1}} + \|\tilde E_0\|_{B^{s-1,s+1}}\\
&\quad + \int_0^t \big(\|\tilde F_1(\tau)\|_{B^{s-1,s}} + \|\tilde F_2(\tau)\|_{B^{s-1,s}} + \|\tilde G_1(\tau)\|_{B^{s-1}} + \|\tilde G_2(\tau)\|_{B^{s-1}}\\
&\quad + \|(\tilde F_2 - \tilde F_1)(\tau)\|_{B^{s-2,s}}\big)d\tau + \int_0^t V'(\tau)\big(\|\tilde c_1(\tau)\|_{B^{s-1,s}} + \|\tilde c_2(\tau)\|_{B^{s-1,s}}\\
&\quad + \|\tilde d_1(\tau)\|_{B^{s-1}} + \|\tilde d_2(\tau)\|_{B^{s-1}} + \|\tilde E(\tau)\|_{B^{s-1,s+1}} - K\sum_{k\le k_0} 2^{k(s-1)} f_k(\tau)\big)d\tau. \qquad (3.38)
\end{aligned}
$$

Now, we want to get the estimate of $(\tilde d_2 - \tilde d_1)$. By $(3.15)_2$ and $(3.15)_4$, it holds that

$$
\begin{aligned}
&\partial_t(\tilde d_{2k} - \tilde d_{1k}) + \Delta_k(u_2\cdot\nabla(\tilde d_2 - \tilde d_1)) + \Delta_k((u_2 - u_1)\cdot\nabla\tilde d_1)\\
&\quad - 2\bar\rho\tilde E_k + (\tilde d_{2k} - \tilde d_{1k}) = (2\bar\mu_2 + \bar\lambda_2)\Delta\tilde d_{2k} - (2\bar\mu_1 + \bar\lambda_1)\Delta\tilde d_{1k}\\
&\quad + \kappa_2\Lambda\tilde c_{2k} - \kappa_1\Lambda\tilde c_{1k} + \tilde G_{2k} - \tilde G_{1k} - KV'(t)(\tilde d_{2k} - \tilde d_{1k}), \qquad (3.39)
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\tilde d_{2k} - \tilde d_{1k}\|_2^2 - 2\bar\rho(\tilde E_k, \tilde d_{2k} - \tilde d_{1k}) + \|\tilde d_{2k} - \tilde d_{1k}\|_2^2\\
&\quad = (\tilde G_{2k} - \tilde G_{1k}, \tilde d_{2k} - \tilde d_{1k}) - KV'(t)\|\tilde d_{2k} - \tilde d_{1k}\|_2^2\\
&\quad - \big(\Delta_k(u_2\cdot\nabla(\tilde d_2 - \tilde d_1)), \tilde d_{2k} - \tilde d_{1k}\big) + (\kappa_2\Lambda\tilde c_{2k}, \tilde d_{2k} - \tilde d_{1k})\\
&\quad - \big(\Delta_k((u_2 - u_1)\cdot\nabla\tilde d_1), \tilde d_{2k} - \tilde d_{1k}\big) - (\kappa_1\Lambda\tilde c_{1k}, \tilde d_{2k} - \tilde d_{1k})\\
&\quad + \big((2\bar\mu_2 + \bar\lambda_2)\Delta\tilde d_{2k} - (2\bar\mu_1 + \bar\lambda_1)\Delta\tilde d_{1k}, \tilde d_{2k} - \tilde d_{1k}\big). \qquad (3.40)
\end{aligned}
$$

By (3.17) and (3.39), we have

$$
\begin{aligned}
&\frac{d}{dt}(\tilde d_{2k} - \tilde d_{1k}, \tilde E_k) + \bar\rho\|\tilde d_{2k} - \tilde d_{1k}\|_2^2 - 2\bar\rho\|\tilde E_k\|_2^2 + (\tilde E_k, \tilde d_{2k} - \tilde d_{1k})\\
&\quad = (\tilde G_{2k} - \tilde G_{1k}, \tilde E_k) - 2KV'(t)(\tilde d_{2k} - \tilde d_{1k}, \tilde E_k) + \kappa_2(\Lambda c_{2k}, \tilde E_k)\\
&\quad + (\Lambda^{-1}(\tilde F_{2k} - \tilde F_{1k}), \tilde d_{2k} - \tilde d_{1k}) - \kappa_1(\Lambda c_{1k}, \tilde E_k) + (2\mu_1 + \lambda_1)(\Lambda^2\tilde d_{1k}, \tilde E_k)\\
&\quad - (2\mu_2 + \lambda_2)(\Lambda^2\tilde d_{2k}, \tilde E_k) - \big(\Delta_k(u_2\cdot\nabla(\tilde d_2 - \tilde d_1)), \tilde E_k\big)\\
&\quad - \big(\Lambda^{-1}\Delta_k(u_2\cdot\nabla\Lambda E), \tilde d_{2k} - \tilde d_{1k}\big) - \big(\Delta_k((u_2 - u_1)\cdot\nabla\tilde d_1), \tilde E_k\big)\\
&\quad - \big(\Delta_k((u_2 - u_1)\cdot\nabla\tilde c_1), \tilde d_{2k} - \tilde d_{1k}\big). \qquad (3.41)
\end{aligned}
$$

Applying Lemma 6.2. of [4] to the convection terms of (3.40) and (3.41), we get the following estimates

$$
\begin{aligned}
&|\big(\Delta_k(u_2\cdot\nabla(\tilde d_2-\tilde d_1)),\tilde d_{2k}-\tilde d_{1k}\big)| \\
&\quad\lesssim J_k 2^{-sk}\|u_2\|_{B^{\frac N2+1}}\|\tilde d_2-d_1\|_{B^{s,s-1}}\|\tilde d_{2k}-\tilde d_{1k}\|_2, &(3.42)\\
&|(\Lambda^{-1}\Delta_k(u_2\cdot\nabla\Lambda\tilde E),\tilde E_k)|\lesssim J_k 2^{-ks}\|u_2\|_{B^{\frac N2+1}}\|\tilde E\|_{B^{s,s+1}}\|\tilde E_k\|_2, &(3.43)\\
&|\big(\Delta_k(u_2\cdot\nabla(\tilde d_2-\tilde d_1)),\tilde E_k\big)+\big(\Lambda^{-1}\Delta_k(u_2\cdot\nabla\Lambda E),\tilde d_{2k}-\tilde d_{1k}\big)| \\
&\quad\lesssim J_k 2^{-sk}\|u_2\|_{B^{\frac N2+1}}\big(\|\tilde d_2-d_1\|_{B^{s,s-1}}\|\tilde E_k+\|\tilde E\|_{B^{s,s+1}}\|\tilde d_{2k}-\tilde d_{1k}\|_2\big). &(3.44)
\end{aligned}
$$

And using the paradifferential calculus and Cauchy's inequality, we get

$$
\begin{aligned}
&|\big(\Delta_k((u_2-u_1)\cdot\nabla\tilde d_1),\tilde d_{2k}-\tilde d_{1k}\big)| \\
&\quad\lesssim J_k 2^{-ks}\|u_2-u_1\|_{B^{\frac N2,\frac N2+1}}\|d_1\|_{B^{s-1}}\|\tilde d_{2k}-\tilde d_{1k}\|_2, &(3.45)\\
&|\big(\Delta_k((u_2-u_1)\cdot\nabla\tilde c_1),\tilde d_{2k}-\tilde d_{1k}\big)| \\
&\quad\lesssim J_k 2^{-ks}\|u_2-u_1\|_{B^{\frac N2,\frac N2+1}}\|c_1\|_{B^{s-1,s}}\|\tilde d_{2k}-\tilde d_{1k}\|_2, &(3.46)\\
&|\big(\Delta_k((u_2-u_1)\cdot\nabla\tilde d_1),\tilde E_k\big)| \\
&\quad\lesssim J_k 2^{-ks}\|u_2-u_1\|_{B^{\frac N2,\frac N2+1}}\|d_1\|_{B^{s-1}}\|\tilde E_k\|_2. &(3.47)
\end{aligned}
$$

Define $\bar f_k^2=\|\tilde d_{2k}-\tilde d_{1k}\|_2^2+2\|\tilde E_k\|_2^2-2B(\tilde d_{2k}-\tilde d_{1k},\tilde E_k)$. Combining (3.18), (3.35) and (3.40) – (3.47), we get

$$
\begin{aligned}
&\frac12\frac{d}{dt}\bar f_k^2+(1-B\bar\rho)\|\tilde d_{2k}-\tilde d_{1k}\|_2^2+2B\bar\rho\|\tilde E_k\|_2^2-B(\tilde E_k,\tilde d_{2k}-\tilde d_{1k})\\
&\quad\lesssim \big(\|\Lambda^2\tilde d_{1k}\|_2+\|\Lambda^2\tilde d_{2k}\|_2+\|\Lambda\tilde c_{1k}\|_2+\|\Lambda\tilde c_{2k}\|_2+\|\tilde G_{1k}\|_2\\
&\quad+\|\tilde G_{2k}\|_2\big)\|\tilde E_k\|_2+\|\Lambda^{-1}(\tilde F_{2k}-\tilde F_{1k})\|_2\|\tilde d_{2k}-\tilde d_{1k}\|_2\\
&\quad+J_k2^{-ks}V'(t)\big(\|\tilde d_2-\tilde d_1\|_{B^{s,s-1}}+\|\tilde E\|_{B^{s,s+1}}\big)\bar f_k-KV'(t)f_k^2\\
&\quad+J_k2^{-ks}V'(t)\big(\|\tilde d_1\|_{B^{s-1}}+\|\tilde c_1\|_{B^{s-1,s}}\big). &(3.48)
\end{aligned}
$$

By Cauchy's inequality, we get

$$
|2B(\tilde d_{2k}-\tilde d_{1k},\tilde E_k)|\le\frac{B}{\bar\rho}\|\tilde d_{2k}-\tilde d_{1k}\|_2^2+B\bar\rho\|\tilde E_k\|_2^2. \tag{3.49}
$$

Taken $B=\min\{\frac{1}{2\bar\rho},\ \frac{\bar\rho}{4}\}$, then there exists a constant a such that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\bar f_k^2+a\bar f_k^2\lesssim\big(\|\Lambda^2\tilde d_{1k}\|_2+\|\Lambda^2\tilde d_{2k}\|_2+\|\Lambda\tilde c_{1k}\|_2+\|\Lambda\tilde c_{2k}\|_2\\
&\quad+\|\tilde G_{1k}\|_2+\|\tilde G_{2k}\|_2+\|\Lambda^{-1}(\tilde F_{2k}-\tilde F_{1k})\|_2\big)\big(\|\tilde d_{2k}-\tilde d_{1k}\|_2+\|\tilde E_k\|_2\big)\\
&\quad+J_k2^{-ks}V'(t)\big(\|\tilde d_2-\tilde d_1\|_{B^{s,s-1}}+\|\tilde E\|_{B^{s,s+1}}\big)\bar f_k-KV'(t)\bar f_k^2\\
&\quad+J_k2^{-ks}V'(t)\big(\|\tilde d_1\|_{B^{s-1}}+\|\tilde c_1\|_{B^{s-1,s}}\big)\bar f_k, &(3.50)
\end{aligned}
$$

thus it holds

$$
\begin{aligned}
\frac{d}{dt}\bar{f}_k + a\bar{f}_k \lesssim\ & \|\Lambda^2\tilde{d}_{1k}\|_2 + \|\Lambda^2\tilde{d}_{2k}\|_2 + \|\Lambda\tilde{c}_{1k}\|_2 + \|\Lambda\tilde{c}_{2k}\|_2 \\
& + \|\tilde{G}_{1k}\|_2 + \|\tilde{G}_{2k}\|_2 + \|\Lambda^{-1}(\tilde{F}_{2k} - \tilde{F}_{1k})\|_2 \\
& + J_k 2^{-ks} V'(t)\big(\|\tilde{d}_2 - \tilde{d}_1\|_{B^{s,s-1}} + \|\tilde{E}\|_{B^{s,s+1}}\big) - KV'(t)\bar{f}_k \\
& + J_k 2^{-ks} V'(t)\big(\|\tilde{d}_1\|_{B^{s-1}} + \|\tilde{c}_1\|_{B^{s-1,s}}\big).
\end{aligned} \tag{3.51}
$$

Then $\int_0^t \sum_{k\le k_0} 2^{sk} \times$ (3.51)$d\tau$ implies

$$
\begin{aligned}
&\int_0^t \sum_{k\le k_0} 2^{sk}\big(\|(\tilde{d}_{2k} - \tilde{d}_{1k})(\tau)\|_2 + \|\tilde{E}_k(\tau)\|_2\big)d\tau \\
&\lesssim \|\tilde{d}_{10}\|_{B^{s-1}} + \|\tilde{d}_{20}\|_{B^{s-1}} + \|\tilde{E}_0\|_{B^{s-1,s+1}} \\
&+ \int_0^t \sum_{k\le k_0} 2^{sk}\big(\|\Lambda^2\tilde{d}_{1k}\|_2 + \|\Lambda^2\tilde{d}_{2k}\|_2 + \|\Lambda\tilde{c}_{1k}\|_2 + \|\Lambda\tilde{c}_{2k}\|_2\big)d\tau \\
&+ \int_0^t \big(\|\tilde{G}_1(\tau)\|_{B^{s-1}} + \|\tilde{G}_2(\tau)\|_{B^{s-1}} + \|(\tilde{F}_2 - \tilde{F}_1)(\tau)\|_{B^{s-2,s}}\big)d\tau \\
&+ \int_0^t \sum_{k\le k_0}\big(V'(\tau)\big(\|(\tilde{d}_2 - \tilde{d}_1)(\tau)\|_{B^{s,s-1}} + \|\tilde{E}(\tau)\|_{B^{s,s+1}}\big) - K2^{sk}\bar{f}_k(\tau)\big) \\
&+ V(t)\sup_{\tau\in[0,t]}\big(\|\tilde{d}_1(\tau)\|_{B^{s-1}} + \|\tilde{c}_1(\tau)\|_{B^{s-1,s}}\big).
\end{aligned} \tag{3.52}
$$

Step 2. High frequencies ($k \ge k_0 + 1$). By (3.15), we can get the following estimates

$$
\begin{cases}
\frac{1}{2}\frac{d}{dt}\|\Lambda\tilde{c}_{1k}\|_2^2 + (\Lambda\Delta_k(u_1\cdot\nabla\tilde{c}_1), \Lambda\tilde{c}_{1k}) + \bar{\rho}(\Lambda^2\tilde{d}_{1k}, \Lambda\tilde{c}_{1k}) \\
\quad = (\Lambda\tilde{F}_{1k}, \Lambda\tilde{c}_{1k}) - KV'(t)\|\Lambda\tilde{c}_{1k}\|_2^2, \\
\frac{1}{2}\frac{d}{dt}\|\tilde{d}_{1k}\|_2^2 + (\Delta_k(u_1\cdot\nabla\tilde{d}_1), \tilde{d}_{1k}) + (2\bar{\mu}_1 + \bar{\lambda}_1)\|\tilde{\Lambda}d_{1k}\|_2^2 - \kappa_1(\Lambda\tilde{c}_{1k}, \tilde{d}_{1k}) \\
\quad -\bar{\rho}(\Lambda^{-1}(\tilde{c}_{1k} - \tilde{c}_{2k}), \tilde{d}_{1k}) + \|\tilde{d}_{1k}\|_2^2 = (\tilde{G}_{1k}, \tilde{d}_{1k}) - KV'(t)\|\tilde{d}_{1k}\|_2^2, \\
\frac{1}{2}\frac{d}{dt}\|\Lambda\tilde{c}_{2k}\|_2^2 + (\Lambda\Delta_k(u_2\cdot\nabla\tilde{c}_2), \Lambda\tilde{c}_{2k}) + \bar{\rho}(\Lambda^2\tilde{d}_{2k}, \Lambda\tilde{c}_{2k}) \\
\quad = (\Lambda\tilde{F}_{2k}, \Lambda\tilde{c}_{2k}) - KV'(t)\|\Lambda\tilde{c}_{2k}\|_2^2, \\
\frac{1}{2}\frac{d}{dt}\|\tilde{d}_{2k}\|_2^2 + (\Delta_k(u_2\cdot\nabla\tilde{d}_2), \tilde{d}_{2k}) + (2\bar{\mu}_2 + \bar{\lambda}_2)\|\Lambda\tilde{d}_{2k}\|_2^2 - \kappa_2(\Lambda\tilde{c}_{2k}, \tilde{d}_{2k}) \\
\quad +\bar{\rho}(\Lambda^{-1}(\tilde{c}_{1k} - \tilde{c}_{2k}), \tilde{d}_{2k}) + \|\tilde{d}_{2k}\|_2^2 = (\tilde{G}_{2k}, \tilde{d}_{2k}) - KV'(t)\|\tilde{d}_{2k}\|_2^2.
\end{cases} \tag{3.53}
$$

Define $f_k^2 = \frac{2\bar{\mu}_1+\bar{\lambda}_1}{\bar{\rho}}\|\Lambda\tilde{c}_{1k}\|_2^2 + \frac{2\bar{\mu}_2+\bar{\lambda}_2}{\bar{\rho}}\|\Lambda\tilde{c}_{2k}\|_2^2 + D\|\tilde{d}_{1k}\|_2^2 + D\|\tilde{d}_{2k}\|_2^2 - 2(\Lambda\tilde{c}_{1k}, \tilde{d}_{1k}) - 2(\Lambda\tilde{c}_{2k}, \tilde{d}_{2k})$. By $(3.15)_5$, (3.19), (3.20) and (3.53), we get

$$
\begin{aligned}
\frac{d}{dt}f_k^2 &+ \kappa_1\|\Lambda\tilde{c}_{1k}\|_2^2 + \kappa_2\|\Lambda\tilde{c}_{2k}\|_2^2 + (D(2\bar{\mu}_1 + \bar{\lambda}_1) - \bar{\rho})\|\Lambda\tilde{d}_{1k}\|_2^2 \\
&+ (D(2\bar{\mu}_2 + \bar{\lambda}_2) - \bar{\rho})\|\Lambda\tilde{d}_{2k}\|_2^2 + D\|\tilde{d}_{1k}\|_2^2 + D\|\tilde{d}_{2k}\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&-(D\kappa_1+1)(\tilde{d}_{1k},\Lambda\tilde{c}_{1k})-(D\kappa_2+1)(\tilde{d}_{2k},\Lambda\tilde{c}_{2k})\\
&-D\bar{\rho}(\Lambda^{-1}(\tilde{c}_{1k}-\tilde{c}_{2k}),\tilde{d}_{1k})+D\bar{\rho}(\Lambda^{-1}(\tilde{c}_{1k}-\tilde{c}_{2k}),\tilde{d}_{2k})+\bar{\rho}\|\Lambda\tilde{E}_k\|_2^2\\
&=\frac{2\bar{\mu}_1+\bar{\lambda}_1}{\bar{\rho}}(\Lambda\tilde{F}_{1k},\Lambda\tilde{c}_{1k})+\frac{2\bar{\mu}_2+\bar{\lambda}_2}{\bar{\rho}}(\Lambda\tilde{F}_{2k},\Lambda\tilde{c}_{2k})+D(\tilde{G}_{1k},\tilde{d}_{1k})\\
&+D(\tilde{G}_{2k},\tilde{d}_{2k})-(\tilde{F}_{1k},\Lambda\tilde{d}_{1k})-(\tilde{G}_{1k},\Lambda\tilde{c}_{1k})-(\tilde{F}_{2k},\Lambda\tilde{d}_{2k})\\
&-(\tilde{G}_{2k},\Lambda\tilde{c}_{2k})-KV'(t)f_k^2-\frac{2\bar{\mu}_1+\bar{\lambda}_1}{\bar{\rho}}(\Lambda\Delta_k(u_1\cdot\nabla\tilde{c}_1),\Lambda\tilde{c}_{1k})\\
&-\frac{2\bar{\mu}_2+\bar{\lambda}_2}{\bar{\rho}}(\Lambda\Delta_k(u_2\cdot\nabla\tilde{c}_2),\Lambda\tilde{c}_{2k})-D(\Delta_k(u_1\cdot\nabla\tilde{d}_1),\tilde{d}_{1k})\\
&-D(\Delta_k(u_2\cdot\nabla\tilde{d}_2),\tilde{d}_{2k})-(\Delta_k(u_1\cdot\nabla\tilde{c}_1),\Lambda\tilde{d}_{1k})\\
&-(\Delta_k(u_1\cdot\nabla\tilde{d}_1),\Lambda\tilde{c}_{1k})-(\Delta_k(u_2\cdot\nabla\tilde{c}_2),\Lambda\tilde{d}_{2k})\\
&-(\Delta_k(u_2\cdot\nabla\tilde{d}_2),\Lambda\tilde{c}_{2k}). \qquad (3.54)
\end{aligned}
$$

By Cauchy's inequality, we have

$$
|2(\tilde{c}_{ik},\Lambda\tilde{d}_{ik})|\le\frac{2\bar{\mu}_i+\lambda_i}{4\bar{\rho}}\|\Lambda\tilde{c}_{ik}\|_2^2+\frac{4\bar{\rho}}{2\bar{\mu}_i+\lambda_i}\|\tilde{d}_{ik}\|_2^2, \qquad (3.55)
$$

$$
|(D\kappa_i+1)(\tilde{c}_{ik},\Lambda\tilde{d}_{ik})|\le\frac{\kappa_i}{4}\|\Lambda\tilde{c}_{ik}\|_2^2+\frac{2^{-2k_0+3}(D\kappa_i+1)^2}{\kappa_i}\|\Lambda\tilde{d}_{ik}\|_2^2, \qquad (3.56)
$$

$$
\begin{aligned}
&|D\bar{\rho}(\Lambda^{-1}(\tilde{c}_{1k}-\tilde{c}_{2k}),\tilde{d}_{ik})|\le\frac{2^{-4k_0+6}D\bar{\rho}\kappa_1}{4}\|\Lambda\tilde{c}_{1k}\|_2^2\\
&\quad+\frac{2^{-4k_0+6}D\bar{\rho}\kappa_2}{4}\|\Lambda\tilde{c}_{2k}\|_2^2+\frac{2^{-2k_0+3}D\bar{\rho}(\kappa_1+\kappa_2)}{\kappa_1\kappa_2}\|\Lambda\tilde{d}_{ik}\|_2^2, \qquad (3.57)
\end{aligned}
$$

where $i=1,2$. Taking $D=\frac{8\bar{\rho}}{2\bar{\mu}_1+\bar{\lambda}_1}+\frac{8\bar{\rho}}{2\bar{\mu}_2+\bar{\lambda}_2}$, then

$$
f_k^2\sim\|\Lambda\tilde{c}_{1k}\|_2^2+\|\Lambda\tilde{c}_{2k}\|_2^2+\|\tilde{d}_{1k}\|_2^2+\|\tilde{d}_{2k}\|_2^2, \qquad (3.58)
$$

and it holds

$$
\frac{2^{-2k_0+3}(D\kappa_i+1)^2}{\kappa_i}+\frac{2^{-2k_0+3}D\bar{\rho}(\kappa_1+\kappa_2)}{\kappa_1\kappa_2}<\bar{\rho},\ \ i=1,2, \qquad (3.59)
$$

$$
D\bar{\rho}2^{-4k_0+6}<1, \qquad (3.60)
$$

when k_0 is large enough. Applying Lemma 6.2. of [4] to the convection terms of (3.54), and combining (3.54) − (3.60), we can find a constant a such that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}f_k^2+af_k^2\lesssim\big(\|\Lambda\tilde{F}_{1k}\|_2+\|\Lambda\tilde{F}_{2k}\|_2+\|\tilde{G}_{1k}\|_2+\|\tilde{G}_{2k}\|_2\big)f_k\\
&\quad+J_k2^{-k(s-1)}\big(\|u_1(t)\|_{B^{\frac{N}{2}+1}}+\|u_2(t)\|_{B^{\frac{N}{2}+1}}\big)\big(\|\tilde{c}_1(t)\|_{B^{s-1,s}}\\
&\quad+\|\tilde{c}_2(t)\|_{B^{s-1,s}}+\|\tilde{d}_1(t)\|_{B^{s-1}}+\|\tilde{d}_2(t)\|_{B^{s-1}}\big)f_k-KV'(t)f_k^2. \qquad (3.61)
\end{aligned}
$$

Then

$$\frac{d}{dt}f_k + af_k \lesssim \big(\|\Lambda\tilde{F}_{1k}\|_2 + \|\Lambda\tilde{F}_{2k}\|_2 + \|\tilde{G}_{1k}\|_2 + \|\tilde{G}_{2k}\|_2\big)$$
$$+ J_k 2^{-k(s-1)}\big(\|u_1(t)\|_{B^{\frac{N}{2}+1}} + \|u_2(t)\|_{B^{\frac{N}{2}+1}}\big)\big(\|\tilde{c}_1(t)\|_{B^{s-1,s}}$$
$$+ \|\tilde{c}_2(t)\|_{B^{s-1,s}} + \|\tilde{d}_1(t)\|_{B^{s-1}} + \|\tilde{d}_2(t)\|_{B^{s-1}}\big) - KV'(t)f_k. \tag{3.62}$$

$\int_0^t \sum_{k>k_0} 2^{k(s-1)}(3.62)d\tau$ and (3.58) imply

$$\sum_{k>k_0} 2^{k(s-1)}\big(\|\Lambda\tilde{c}_{1k}(t)\|_2^2 + \|\Lambda\tilde{c}_{2k}(t)\|_2^2 + \|\tilde{d}_{1k}(t)\|_2^2 + \|\tilde{d}_{2k}\|_2^2(t)\big)$$
$$+ \int_0^t \sum_{k>k_0} 2^{k(s-1)}\big(\|\Lambda\tilde{c}_{1k}(\tau)\|_2^2 + \|\Lambda\tilde{c}_{2k}(\tau)\|_2^2 + \|\tilde{d}_{1k}(\tau)\|_2^2$$
$$+ \|\tilde{d}_{2k}\|_2^2(\tau)\big) \lesssim \|\tilde{c}_{10}\|_{B^{s-1,s}} + \|\tilde{c}_{20}\|_{B^{s-1,s}} + \|\tilde{d}_{10}\|_{B^{s-1}}$$
$$+ \|\tilde{d}_{20}\|_{B^{s-1}} + \int_0^t \big(\|\tilde{F}_1(\tau)\|_{B^{s-1,s}} + \|\tilde{F}_2(\tau)\|_{B^{s-1,s}} + \|\tilde{G}_1(\tau)\|_{B^{s-1}}$$
$$+ \|\tilde{G}_2(\tau)\|_{B^{s-1}}\big)d\tau + \int_0^t V'(\tau)\big(\|\tilde{c}_1(\tau)\|_{B^{s-1,s}} + \|\tilde{c}_2(\tau)\|_{B^{s-1,s}}$$
$$+ \|\tilde{d}_1(\tau)\|_{B^{s-1}} + \|\tilde{d}_2(\tau)\|_{B^{s-1}} - K\sum_{k>k_0} 2^{k(s-1)}f_k(\tau)\big)d\tau. \tag{3.63}$$

In fact, we can show the smooth properties on d_1 and d_2 by considering $(3.15)_2$ and $(3.15)_4$. By $(3.53)_2$ and $(3.53)_4$, we get

$$\frac{1}{2}\frac{d}{dt}(\|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2) + (2\bar{\mu}_1 + \bar{\lambda}_1)\|\tilde{\Lambda}d_{1k}\|_2^2 + (2\bar{\mu}_2 + \bar{\lambda}_2)\|\tilde{\Lambda}d_{2k}\|_2^2$$
$$- \kappa_1(\Lambda\tilde{c}_{1k}, \tilde{d}_{1k}) - \kappa_2(\Lambda\tilde{c}_{2k}, \tilde{d}_{2k}) - \bar{\rho}(\Lambda^{-1}(\tilde{c}_{1k} - \tilde{c}_{2k}), \tilde{d}_{1k})$$
$$+ \bar{\rho}(\Lambda^{-1}(\tilde{c}_{1k} - \tilde{c}_{2k}), \tilde{d}_{2k}) + \|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2$$
$$= (\tilde{G}_{1k}, \tilde{d}_{1k}) + (\tilde{G}_{2k}, \tilde{d}_{2k}) - KV'(t)(\|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2)$$
$$- (\Delta_k(u_1\cdot\nabla\tilde{d}_1), \tilde{d}_{1k}) - (\Delta_k(u_2\cdot\nabla\tilde{d}_2), \tilde{d}_{2k}). \tag{3.64}$$

By Cauchy's inequality and Lemma 6.2 of [4], we finally get

$$\frac{1}{2}\frac{d}{dt}(\|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2) + (2\bar{\mu}_1 + \bar{\lambda}_1)\|\tilde{\Lambda}d_{1k}\|_2^2 + (2\bar{\mu}_2 + \bar{\lambda}_2)\|\tilde{\Lambda}d_{2k}\|_2^2$$
$$\lesssim \big(\|\Lambda\tilde{c}_{1k}\|_2 + \|\Lambda\tilde{c}_{2k}\|_2 + \|\tilde{G}_{1k}\|_2 + \|\tilde{G}_{2k}\|_2\big)\big(\|\tilde{d}_{1k}\|_2 + \|\tilde{d}_{2k}\|_2\big)$$
$$+ J_k 2^{-k(s-1)}\big(\|\tilde{d}_1\|_{B^{s-1}} + \|\tilde{d}_2\|_{B^{s-1}}\big)\big(\|\tilde{d}_{1k}\|_2 + \|\tilde{d}_{2k}\|_2\big)$$
$$- KV'(t)(\|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2). \tag{3.65}$$

Thus it holds

$$\frac{1}{2}\frac{d}{dt}\big(\|\tilde{d}_{1k}\|_2^2 + \|\tilde{d}_{2k}\|_2^2\big)^{1/2} + 2^{2k}\big(\|\tilde{d}_{1k}\|_2 + \|\tilde{d}_{2k}\|_2\big)$$

$$\lesssim \big(\|\Lambda\tilde{c}_{1k}\|_2+\|\Lambda\tilde{c}_{2k}\|_2+\|\tilde{G}_{1k}\|_2+\|\tilde{G}_{2k}\|_2\big)$$
$$+J_k 2^{-k(s-1)}V'(t)\big(\|\tilde{d}_1\|_{B^{s-1}}+\|\tilde{d}_2\|_{B^{s-1}}\big)$$
$$-KV'(t)(\|\tilde{d}_{1k}\|_2+\|\tilde{d}_{2k}\|_2), \tag{3.66}$$

then $\int_0^t \sum_{k>k_0} 2^{k(s-1)}(3.66)d\tau$ implies

$$\begin{aligned}
&\int_0^t \sum_{k>k_0} 2^{k(s+1)}\big(\|\tilde{d}_{1k}(\tau)\|_2+\|\tilde{d}_{2k}(\tau)\|_2\big) \lesssim \|\tilde{d}_{10}\|_{B^{s-1}} \\
&\quad + \|\tilde{d}_{20}\|_{B^{s-1}} + \int_0^t \sum_{k>k_0} 2^{ks}\big(\|\tilde{c}_{1k}(\tau)\|_2+\|\tilde{c}_{2k}(\tau)\|_2\big)d\tau \\
&\quad + \int_0^t \|\tilde{G}_1(\tau)\|_{B^{s-1}} + \|\tilde{G}_2(\tau)\|_{B^{s-1}}d\tau + \int_0^t V'(\tau)\big(\|\tilde{d}_1(\tau)\|_{B^{s-1}} \\
&\quad + \|\tilde{d}_2(\tau)\|_{B^{s-1}} - K\sum_{k>k_0} 2^{k(s-1)}(\|\tilde{d}_{1k}(\tau)\|_2+\|\tilde{d}_{2k}(\tau)\|_2)\big).
\end{aligned} \tag{3.67}$$

Combining (3.38), (3.52), (3.63) and (3.67), and taking K large enough, we complete the proof. □

We can obtain the estimates for the "incompressible part" of u_i $(i=1,2)$ by the following Lemma.

Lemma 3.1. *Let u be the solution of*

$$\begin{cases} \partial_t u - \mu\Delta u + u = f, \\ u(0) = u_0. \end{cases} \tag{3.68}$$

Then for $s\in\mathbb{R}$ and $t>0$, it holds

$$\|u(t)\|_{B^{s-1}} + \int_0^t \|u(\tau)\|_{B^{s-1,s+1}}d\tau \lesssim \|u_0\|_{B^{s-1}} + \int_0^t \|f(\tau)\|_{B^{s-1}}d\tau. \tag{3.69}$$

It is trivial to prove Lemma 3.1, so we omit the details of the proof.

4 Existence and uniqueness

Let us define two norms:

$$\begin{aligned}
&\|(c_1,c_2,u_1,u_2,\Phi)\|_{W_T^s} := \|c_1\|_{L^\infty((0,T);B^{s-1,s})} + \|c_2\|_{L^\infty((0,T);B^{s-1,s})} \\
&\quad + \|u_1\|_{L^\infty((0,T);B^{s-1})} + \|u_2\|_{L^\infty((0,T);B^{s-1})} + \|\Phi\|_{L^\infty((0,T);B^{s,s+2})} \\
&\quad + \|c_1\|_{L^1((0,T);B^{s+1,s})} + \|c_2\|_{L^1((0,T);B^{s+1,s})} + \|u_1\|_{L^1((0,T);B^{s+1})} \\
&\quad + \|u_2\|_{L^1((0,T);B^{s+1})} + \|\Phi\|_{L^1((0,T);B^{s+1,s+2})} + \|u_2-u_1\|_{L^1((0,T);B^{s,s+1})},
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\|(c_1,c_2,u_1,u_2,\Phi)\|_{W^s(0)} &:= \|c_1\|_{B^{s-1,s}}+\|c_2\|_{B^{s-1,s}}\\ &\quad+\|u_1\|_{B^{s-1}}+\|u_2\|_{B^{s-1}}+\|\Phi\|_{B^{s,s+2}}.\end{aligned} \tag{4.2}$$

We have the following estimates.

Theorem 4.1. *Let (c_1,c_2,u_1,u_2,Φ) be a solution of (3.1) on $[0,T]$, $N\geq 3$ and $P_1,\ P_2$ are $C^{[\frac{N}{2}]+3}$ function of $\rho_1,\ \rho_2$ such that $P_1(\rho_1)>0,\ P_2(\rho_2)>0$. Assume that $\|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}})}$ $(i=1,2)$ are small enough. Then (c_1,c_2,u_1,u_2,Φ) satisfies*

$$\begin{aligned}\|(c_1,c_2,u_1,u_2,\Phi)\|_{W_T^{\frac{N}{2}}} &\leq Ce^{CV(T)}\big(\|(c_{10},c_{10},d_{10},d_{20},\Phi_0)\|_{W^{\frac{N}{2}}(0)}\\ &\quad+(1+\|c_1\|_{L^\infty((0,T);B^{\frac{N}{2}})}+\|c_2\|_{L^\infty((0,T);B^{\frac{N}{2}})})^{\frac{N}{2}+3}\|(c_1,c_2,u_1,u_2,\Phi)\|^2_{W_T^{\frac{N}{2}}}\big),\end{aligned} \tag{4.3}$$

where C is a constant depending only on $\bar{\rho},\ \mu_1,\ \mu_2,\ \lambda_1,\ \lambda_2$ and N.

Proof. By (3.12), Proposition 3.1 and Lemma 3.1, we need to show that

$$\begin{aligned}&\|F_1\|_{L^1((0,T);B^{\frac{N}{2}-1,\frac{N}{2}})}+\|F_2\|_{L^1((0,T);B^{\frac{N}{2}-1,\frac{N}{2}})}\|G_1\|_{L^1((0,T);B^{\frac{N}{2}-1})}\\ &\quad+\|G_2\|_{L^1((0,T);B^{\frac{N}{2}-1})}+\|F_1-F_2\|_{L^1((0,T);B^{\frac{N}{2}-2,\frac{N}{2}})}\\ &\quad+\|\bar{G}_1\|_{L^1((0,T);B^{\frac{N}{2}-1})}+\|\bar{G}_2\|_{L^1((0,T);B^{\frac{N}{2}-1})}\\ &\quad\lesssim(1+\|c_1\|_{L^\infty((0,T);B^{\frac{N}{2}})}+\|c_2\|_{L^\infty((0,T);B^{\frac{N}{2}})})^{\frac{N}{2}+3}\|(c_1,c_2,u_1,u_2,\Phi)\|^2_{W_T^{\frac{N}{2}}}.\end{aligned} \tag{4.4}$$

In fact, for $i=1,2$, we have

$$\begin{aligned}\|F_i\|_{L^1((0,T);B^{\frac{N}{2}-1,\frac{N}{2}})} &= \|c_i\mathrm{div}u_i\|_{L^1((0,T);B^{\frac{N}{2}-1,\frac{N}{2}})}\\ &\lesssim\|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}-1,\frac{N}{2}})}\|u_i\|_{L^1((0,T);B^{\frac{N}{2}+1})}\\ &\lesssim\|(c_1,c_2,u_1,u_2,\Phi)\|^2_{W_T^{\frac{N}{2}}},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\|F_1-F_2\|_{L^1((0,T);B^{\frac{N}{2}-2,\frac{N}{2}})} &\lesssim\|(c_1-c_2)\mathrm{div}u_1\|_{L^1((0,T);B^{\frac{N}{2}-2,\frac{N}{2}})}\\ &\quad+\|c_2\mathrm{div}(u_1-u_2)\|_{L^1((0,T);B^{\frac{N}{2}-2,\frac{N}{2}})}\\ &\lesssim\|\Phi\|_{L^\infty((0,T);B^{\frac{N}{2},\frac{N}{2}+2})}\|u_1\|_{L^1((0,T);B^{\frac{N}{2}+1})}\\ &\quad+\|c_2\|_{L^\infty((0,T);B^{\frac{N}{2}-1,\frac{N}{2}})}\|(u_1-u_2)\|_{L^1((0,T);B^{\frac{N}{2},\frac{N}{2}+1})}\\ &\lesssim\|(c_1,c_2,u_1,u_2,\Phi)\|^2_{W_T^{\frac{N}{2}}},\end{aligned} \tag{4.6}$$

and

$$\|u_i\mathrm{div}d_i\|_{L^1((0,T);B^{\frac{N}{2}-1})}+\|u_i\mathrm{div}u_i\|_{L^1((0,T);B^{\frac{N}{2}-1})}$$

$$
\begin{aligned}
&\lesssim \|u_i\|_{L^\infty((0,T);B^{\frac{N}{2}-1})}\|u_i\|_{L^1((0,T);B^{\frac{N}{2}+1})} \\
&\lesssim \|(c_1,c_2,u_1,u_2,\Phi)\|^2_{W_T^{\frac{N}{2}}}, \qquad (4.7) \\
\|\frac{c_i}{c_i+\bar\rho}A_iu_i\|_{L^1((0,T);B^{\frac{N}{2}-1})} &\lesssim \|\frac{c_i}{c_i+\bar\rho}\|_{L^\infty((0,T);B^{\frac{N}{2}})}\|u_i\|_{L^1((0,T);B^{\frac{N}{2}+1})} \\
&\lesssim (1+\|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}})})^{\frac{N}{2}+3}\|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}})}\|u_i\|_{L^1((0,T);B^{\frac{N}{2}+1})} \\
&\lesssim (1+\|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}})})^{\frac{N}{2}+3}\|(c_1,c_2,u_1,u_2,\Phi)\|^2_{W_T^{\frac{N}{2}}}, \qquad (4.8) \\
\|K_i(c_i)\nabla c_i\|_{L^1((0,T);B^{\frac{N}{2}-1})} &\lesssim (1+\|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}})})^{\frac{N}{2}+3} \\
&\times \|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}-1,\frac{N}{2}})}\|c_i\|_{L^1((0,T);B^{\frac{N}{2}+1,\frac{N}{2}})} \\
&\lesssim (1+\|c_i\|_{L^\infty((0,T);B^{\frac{N}{2}})})^{\frac{N}{2}+3}\|(c_1,c_2,u_1,u_2,\Phi)\|^2_{W_T^{\frac{N}{2}}}, \qquad (4.9)
\end{aligned}
$$

At the same way, we can get the desired estimates of $\bar G_1$ and $\bar G_2$. Thus we finish the proof. □

The construction of the approximate solutions to (3.1) is standard (see [12]). Then, Using Theorem 4.1 and the smallness assumption, we can show that the approximate solutions converge to the solution of (3.1) by compactness arguments (see [4, 12]). Using a similar argument as that in [4, 12], we can easily obtain the uniqueness of the solution. Thus, we finish the proof of Theorem 1.1.

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