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Well-posedness for the viscous rotating shallow water equations with friction terms

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We consider the Cauchy problem for viscous rotating shallow water equations with friction terms. The global existence of the solution in some hybrid spaces is shown for the initial data close to a constant equilibrium state away from the vacuum. © *2011 American Institute of Physics*. [doi:10.1063/1.3553187]

I. INTRODUCTION

The nonlinear shallow water equation is used to model the motion of a shallow layer of homogeneous incompressible fluid in a three-dimensional rotating subdomain and, in particular, to simulate the vertical average dynamics of the fluid in terms of the horizontal velocity and depth variation. The shallow water system is extensively used in environmental studies to model hydrodynamics in lakes, estuaries, coastal regions, and other applications. In general, it is modeled by the three-dimensional incompressible Navier–Stokes–Coriolis system in a rotating subdomain of $\mathbb{R}^3$ together with a (nonlinear) free moving surface boundary condition for which the stress tension is evolved at the air–fluid interface from above and the Navier boundary condition of wall-law type holds at the bottom. Under a large-scale assumption and hydrostatic approximation, the nonlinear shallow water equation has been derived recently.[11,15,17] When the viscosity is of order of the aspect ratio, we cannot take an approximation of the Coriolis force at the first order, while the second order can produce new terms depending on the cosine of the latitude in the two-dimensional Shallow Water model. Thus it is necessary to go to the second order of the system, and then we must take into account the cosine effect of the Coriolis terms. In this case, the viscous shallow water system[15] reads

$$\begin{cases} \partial_t H + \operatorname{div}(H\mathbf{u}) = 0, \\ \partial_t(H\mathbf{u}) + \operatorname{div}(H\mathbf{u}\otimes\mathbf{u}) + \dfrac{g}{2}\nabla H^2 = -\alpha_0(H)\mathbf{u} - \alpha_1(H)H|\mathbf{u}|\mathbf{u} + aH\nabla\Delta H \\ \qquad + 2\mu\nabla(H\operatorname{div}\mathbf{u}) + 2\mu\operatorname{div}(HD\mathbf{u}) + \Omega\cos\theta\nabla(u_1H^2) + \Omega\cos\theta H^2\mathbf{e}_1\operatorname{div}\mathbf{u} \\ \qquad - 2\Omega\sin\theta H\mathbf{u}^\perp - 2\Omega\cos\theta H\mathbf{e}_1\nabla b\cdot\mathbf{u} + 2\Omega\cos\theta u_1H\nabla b + aH\nabla\Delta b - gH\nabla b, \end{cases} \tag{1}$$

where H is the water height, $\mathbf{u} = (u_1, u_2)^\intercal$ denotes the mean velocity, $g > 0$ is the gravitational acceleration, and $\mu > 0$ is the dynamical viscosity. The angular speed of the Earth is $\Omega > 0, \theta \in (0, \pi/2)$ represents the latitude and will be considered as a constant,[15] $\alpha_0(H) = (1 + \kappa_l H/(3\mu))^{-1}\kappa_l$, $\alpha_1(H) = \kappa_t(1 + \kappa_l H/(3\mu))^{-2}$, $\kappa_l > 0$ and $\kappa_t \geqslant 0$ are the laminar and the turbulent friction coefficients which are obtained from the friction condition on the bottom, $D\mathbf{u} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^\intercal)/2$ is the deformation tensor, and the constant $a > 0$ is the capillary coefficient. $b(x)$ is the known bottom topography.

The Coriolis force is due to the Earth's rotation. On the Earth an object that moves along a north–south path, or longitudinal line, will undergo apparent deflection to the right in the Northern

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Hemisphere and to the left in the Southern Hemisphere. Interest has recently grown in the effects of the Coriolis terms that are neglected under a traditional approximation.

In the present paper, we consider the case with a constant latitude and a flat bottom, that is, θ and $b(x)$ are constants in the system (1). More precisely, we investigate the global existence of the Cauchy problem to the following viscous rotating shallow water equations with friction terms

$$\begin{cases} \partial_t H + \operatorname{div}(H\mathbf{u}) = 0, & \text{(2a)}\\ \partial_t(H\mathbf{u}) + \operatorname{div}(H\mathbf{u}\otimes\mathbf{u}) + \dfrac{g}{2}\nabla H^2 = -\alpha_0(H)\mathbf{u} - \alpha_1(H)H|\mathbf{u}|\mathbf{u} & \\ \qquad +2\mu\nabla(H\operatorname{div}\mathbf{u}) + 2\mu\operatorname{div}(HD\mathbf{u}) + \Omega\cos\theta\nabla(u_1H^2) + \Omega\cos\theta H^2\mathbf{e}_1\operatorname{div}\mathbf{u} & \\ \qquad -2\Omega\sin\theta H\mathbf{u}^\perp + aH\nabla\Delta H, & \text{(2b)}\\ H|_{t=0} = H_0(x), \quad \mathbf{u}|_{t=0} = \mathbf{u}_0(x). & \text{(2c)} \end{cases}$$

We note that the system with a smooth bottom topography $b(x)$ can be treated in a similar way without any other essential difficulties.

For the case of a viscosity of order one, the first order of the system is sufficient, and the Coriolis terms reduce to the usual ones. That is, the nonlinear shallow water equations take the following form of compressible Navier–Stokes equations

$$\begin{cases} \partial_t H + \operatorname{div}(H\mathbf{u}) = 0, \\ \partial_t(H\mathbf{u}) + \operatorname{div}(H\mathbf{u}\otimes\mathbf{u}) + gH\nabla H + f(H\mathbf{u})^\perp = \operatorname{div}(2\xi(H)D(\mathbf{u})) + \nabla(\lambda(H)\operatorname{div}\mathbf{u}), \\ H(0) = H_0, \quad \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{3}$$

where $f = 2\Omega\sin\theta > 0$ is the Coriolis frequency, $\xi \geqslant 0$ and λ are the dynamical viscosities satisfying $\lambda + \xi \geqslant 0$.

For the shallow water system with the form in (3), there is a mount of work to deal with the global well-posedness of strong solutions subject to some small initial perturbation of a constant state or the global existence of weak solutions for large initial data. When the viscosities satisfy $\xi(H) = H$ and $\lambda(H) = 0$, and the effect of the Coriolis force is omitted ($f = 0$), the local existence and uniqueness of classical solutions to the Cauchy–Dirichlet problem for the shallow water equations with initial data in $C^{2+\alpha}$ was studied by using Lagrangian coordinates and Hölder space estimates.[4] Kloeden and Sundbye[14,21] proved the global existence and uniqueness of classical solutions to the initial-boundary-value problem with Dirichlet boundary conditions using Sobolev space estimates by following the energy method of Matsumura and Nishida.[18] Sundbye[22] proved also the existence and uniqueness of classical solutions to the Cauchy problem using the similar method.[18]

Recently, Wang and Xu[24] obtained local solutions for any initial data and global solutions for small initial data $H_0 - \bar{H}$, $\mathbf{u}_0 \in H^{2+s}(\mathbb{R}^2)$ with $s > 0$. The result was improved by Haspot[13] to get global existence in time for small initial data $H_0 - \bar{H} \in \dot{B}^0 \cap \dot{B}^1$ and $\mathbf{u}_0 \in \dot{B}^0$ as a special case, and by Chen *et al.*[6] to prove the local existence in time for general initial data and the global existence in time for small initial data where $H_0 - \bar{H} \in \dot{B}^0 \cap \dot{B}^1$ and $\mathbf{u}_0 \in \dot{B}^0$ with additional conditions that $H \geqslant \bar{H}$ and $\bar{H}$ is a strictly positive constant. The related systems with a third-order term stemming from the capillary tensor have also been considered by Danchin and Desjardins[10] for a compressible fluid model of Korteweg type with constant viscosity coefficients, and the global existence of strong solution was shown.

For the shallow water model (3) including a rotational term, an existence theorem is developed in a bounded domain.[19] Cheng and Tadmor[7] discussed the long time existence of approximate periodic solutions for the rapidly rotating shallow water for initial data $(H_0, \mathbf{u}_0) \in H^m(\mathbb{T}^2)$ with $m > 5$ where the viscous terms are absent (where $f \neq 0$ and $\xi = \lambda = 0$). For the case when $f \neq 0$, $\lambda(H) = 2\xi(H) = 2H$ and an additional third-order surface tension term involved, Hao *et al.*[12] proved the global well-posedness of the Cauchy problem near a constant equilibrium with small data where $H_0 - \bar{H} \in \dot{B}^\varepsilon \cap \dot{B}^{1+\varepsilon}$ and $\mathbf{u}_0 \in \dot{B}^\varepsilon$ for any $\varepsilon \in (0, 1)$ where we did not deal with the friction terms and the cosine effects of the Coriolis force. In this paper, we will use Banach's fixed point theorem to prove the existence and uniqueness instead of the Friedrich's regularization and compactness arguments used by Hao *et al.*[12]

The existence of global weak solutions of a viscous shallow water model with the presence of a friction term is demonstrated in a bounded two-dimensional domain with periodic boundary conditions.[2] It should also be mentioned that the global existence of weak solutions does not apply here since the Bresch–Desjardins entropy[3] is not satisfied for Eq. (2). In addition, the classical theory does not cover the case with coriolis force, friction terms, and capillarity term involved.

We consider the global existence of strong solution in Besov spaces. We focus on the effects of the Coriolis force and the friction. The Coriolis force terms cannot contribute to the energy, but the friction terms will make the solution lose some regularities, especially for low frequencies. In order to get better estimates, we consider the problem in the frame of hybrid Besov spaces. And the Besov–Chemin–Lerner spaces are necessary for using the interpolation theory of time-spatial spaces involving hybrid Besov spaces.

The main results of this paper reads as follows:

Theorem 1: *Let $p \in [1, +\infty]$ and $r \in [2, +\infty]$. Let $g, \mu, a, \Omega, \kappa_l$ be positive constants, $\theta \in (0, \pi/2)$ and $\kappa_t \geqslant 0$ be constants. Assume that $H_0 - 1 \in \tilde{B}^{0,1}$ and $H_0\mathbf{u}_0 \in \dot{B}^0$. Then, there exists a suitable small constant $\varepsilon > 0$ such that the system (2) yields a unique global solution $(H, \mathbf{u})$ satisfying*

$$\|H - 1\|_{\tilde{L}^p(0,\infty;\tilde{B}^{2/p,1+2/p})} + \|H\mathbf{u}\|_{\tilde{L}^p(0,\infty;\dot{B}^{2/p})\cap\tilde{L}^r(0,\infty;\tilde{B}^{0,1+2/r})} \leqslant CE,$$

if $E := \|H_0 - 1\|_{\tilde{B}^{0,1}} + \|H_0\mathbf{u}_0\|_{\dot{B}^0} \leqslant \varepsilon$, where C is independent of the initial data.

Remark 1: For the variable latitude θ, it can also be treated in a similar way due to the boundedness of sine and cosine functions.

The paper is organized as follows. We recall some Littlewood–Paley theories for homogeneous Besov spaces and give the definitions and some properties of hybrid Besov spaces and Besov–Chemin–Lerner spaces in the second section. In Sect. III, we are dedicated into deriving an *a priori* estimate by investigating the linearized system. Finally, we construct a contraction map and use the Banach fixed point theorem to obtain the existence and uniqueness of the solution.

II. LITTLEWOOD–PALEY THEORY AND BESOV SPACES

Let $\psi : \mathbb{R}^2 \to [0, 1]$ be a radial smooth cutoff function valued in [0, 1] such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leqslant 3/4, \\ \text{smooth}, & 3/4 < |\xi| < 4/3, \\ 0, & |\xi| \geqslant 4/3. \end{cases}$$

Let $\varphi(\xi)$ be the function

$$\varphi(\xi) := \psi(\xi/2) - \psi(\xi).$$

Thus, ψ is supported in the ball $\{\xi \in \mathbb{R}^2 : |\xi| \leqslant 4/3\}$, and φ is also a smooth cutoff function valued in [0, 1] and supported in the annulus $\{\xi : 3/4 \leqslant |\xi| \leqslant 8/3\}$. By construction, we have

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1, \quad \forall \xi \neq 0.$$

One can define the dyadic blocks as follows. For $k \in \mathbb{Z}$, let

$$\triangle_k f := \mathscr{F}^{-1}\varphi(2^{-k}\xi)\mathscr{F} f,$$

where $\mathscr{F}$ ($\mathscr{F}^{-1}$) stands for the Fourier (inverse) transform.

The formal decomposition

$$f = \sum_{k\in\mathbb{Z}} \triangle_k f \tag{4}$$

is called homogeneous Littlewood–Paley decomposition. Actually, this decomposition works for just about any locally integrable function which yields some decay at infinity, and one usually has all the convergence properties of the summation that one needs. Thus, the rhs of (4) does not necessarily converge in $\mathscr{S}'(\mathbb{R}^2)$. Even if it does, the equality is not always true in $\mathscr{S}'(\mathbb{R}^2)$. For instance, if $f \equiv 1$, then all the projections $\triangle_k f$ vanish. Nevertheless, (4) is true modulo polynomials, in other words,[9,20] if $f \in \mathscr{S}'(\mathbb{R}^2)$, then $\sum_{k\in\mathbb{Z}} \triangle_k f$ converges modulo $\mathscr{P}[\mathbb{R}^2]$ and (4) holds in $\mathscr{S}'(\mathbb{R}^2)/\mathscr{P}[\mathbb{R}^2]$.

Definition 1: Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. For $f \in \mathscr{S}'(\mathbb{R}^2)$, we write

$$\|f\|_{\dot{B}^s} = \sum_{k\in\mathbb{Z}} 2^{ks} \|\triangle_k f\|_{L^2}.$$

A difficulty comes from the choice of homogeneous spaces at this point. Indeed, $\|\cdot\|_{\dot{B}^s}$ cannot be a norm on $\{f \in \mathscr{S}'(\mathbb{R}^2) : \|f\|_{\dot{B}^s} < \infty\}$ because $\|f\|_{\dot{B}^s} = 0$ means that f is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces.[8]

Definition 2: Let $s \in \mathbb{R}$ and $m = -[2 - s]$. If $m < 0$, then we define $\dot{B}^s(\mathbb{R}^2)$ as

$$\dot{B}^s = \Big\{ f \in \mathscr{S}'(\mathbb{R}^2) : \|f\|_{\dot{B}^s} < \infty \text{ and } f = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^2) \Big\}.$$

If $m \geqslant 0$, we denote by $\mathscr{P}_m$ the set of two variables polynomials of degree less than or equal to m and define

$$\dot{B}^s = \Big\{ f \in \mathscr{S}'(\mathbb{R}^2)/\mathscr{P}_m : \|f\|_{\dot{B}^s} < \infty \text{ and } f = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^2)/\mathscr{P}_m \Big\}.$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies.[8] We are going to recall the definition of these new spaces and some of their main properties.

Definition 3: Let $s, t \in \mathbb{R}$. We define

$$\|f\|_{\tilde{B}^{s,t}} = \sum_{k\leqslant 0} 2^{ks} \|\triangle_k f\|_{L^2} + \sum_{k>0} 2^{kt} \|\triangle_k f\|_{L^2}.$$

Let $m = -[2 - s]$, we then define

$$\tilde{B}^{s,t}(\mathbb{R}^2) = \left\{ f \in \mathscr{S}'(\mathbb{R}^2) : \|f\|_{\tilde{B}^{s,t}} < \infty \right\}, \quad if\ m < 0,$$
$$\tilde{B}^{s,t}(\mathbb{R}^2) = \left\{ f \in \mathscr{S}'(\mathbb{R}^2)/\mathscr{P}_m : \|f\|_{\tilde{B}^{s,t}} < \infty \right\}, \quad if\ m \geqslant 0.$$

Lemma 1: We have the following inclusions.

(i) *We have $\tilde{B}^{s,s} = \dot{B}^s$.*
(ii) *If $s \leqslant t$ then $\tilde{B}^{s,t} = \dot{B}^s \cap \dot{B}^t$. Otherwise, $\tilde{B}^{s,t} = \dot{B}^s + \dot{B}^t$.*
(iii) *The space $\tilde{B}^{0,s}$ coincides with the usual inhomogeneous Besov space $B^s_{2,1}$.*
(iv) *If $s_1 \leqslant s_2$ and $t_1 \geqslant t_2$, then $\tilde{B}^{s_1,t_1} \hookrightarrow \tilde{B}^{s_2,t_2}$.*

Let us now recall some useful estimates for the product in hybrid Besov spaces.

Lemma 2 (Proposition 2.10; Ref. 8): *Let $s_1, s_2 > 0$ and $f, g \in L^\infty \cap \tilde{B}^{s_1,s_2}$. Then $fg \in \tilde{B}^{s_1,s_2}$ and*

$$\|fg\|_{\tilde{B}^{s_1,s_2}} \lesssim \|f\|_{L^\infty} \|g\|_{\tilde{B}^{s_1,s_2}} + \|f\|_{\tilde{B}^{s_1,s_2}} \|g\|_{L^\infty}.$$

Let $s_1, s_2, t_1, t_2 \leqslant 1$ *such that* $\min(s_1+s_2, t_1+t_2) > 0$, $f \in \tilde{B}^{s_1,t_1}$ *and* $g \in \tilde{B}^{s_2,t_2}$. *Then* $fg \in \tilde{B}^{s_1+s_2-1,t_1+t_2-1}$ *and*

$$\|fg\|_{\tilde{B}^{s_1+s_2-1,t_1+t_2-1}} \lesssim \|f\|_{\tilde{B}^{s_1,t_1}}\|g\|_{\tilde{B}^{s_2,t_2}}.$$

In the context of this paper, we also need to use the interpolation spaces of hybrid Besov spaces together with a time space such as $L^p(0,T;\tilde{B}^{s,t})$. Thus, we have to introduce the Besov–Chemin–Lerner space[5] which is a refinement of the space $L^p(0,T;\tilde{B}^{s,t})$.

Definition 4: Let $p \in [1,\infty]$, $T \in (0,\infty]$, *and* s_1, $s_2 \in \mathbb{R}$. *Then we define*

$$\|f\|_{\tilde{L}^p_T(\tilde{B}^{s,t})} = \sum_{k\leqslant 0} 2^{ks}\|\Delta_k f\|_{L^p(0,T;L^2)} + \sum_{k>0} 2^{kt}\|\Delta_k f\|_{L^p(0,T;L^2)}.$$

Noting that Minkowski's inequality yields $\|f\|_{L^p_T(\tilde{B}^{s,t})} \leqslant \|f\|_{\tilde{L}^p_T(\tilde{B}^{s,t})}$, we define spaces $\tilde{L}^p_T(\tilde{B}^{s,t})$ as follows:

$$\tilde{L}^p_T(\tilde{B}^{s,t}) = \left\{ f \in L^p_T(\tilde{B}^{s,t}) : \|f\|_{\tilde{L}^p_T(\tilde{B}^{s,t})} < \infty \right\}.$$

If $T=\infty$, then we omit the subscript T from the notation $\tilde{L}^p_T(\tilde{B}^{s,t})$, that is, $\tilde{L}^p(\tilde{B}^{s,t})$ for simplicity. We will denote by $\tilde{\mathcal{C}}([0,T];\tilde{B}^{s,t})$ the subset of functions of $\tilde{L}^\infty_T(\tilde{B}^{s,t})$ which are continuous on $[0,T]$ with values in $\tilde{B}^{s,t}$. Let us observe that $L^1_T(\tilde{B}^{s,t}) = \tilde{L}^1_T(\tilde{B}^{s,t})$, but the embedding $\tilde{L}^p_T(\tilde{B}^{s,t}) \subset L^p_T(\tilde{B}^{s,t})$ is strict if $p>1$.

For the composition of functions, we have the following estimates:

Lemma 3 (Lemma 1; Ref. 10): *Let* $s>0$, $p\in[1,+\infty]$, *and* $u \in \tilde{L}^p_T(\dot{B}^s) \cap L^\infty_T(L^\infty)$.

(i) *Let* $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^2)$ *such that* $F(0)=0$. *Then* $F(u) \in \tilde{L}^p_T(\dot{B}^s)$. *More precisely, there exists a function* C *depending only on* s *and* F *such that*

$$\|F(u)\|_{\tilde{L}^p_T(\dot{B}^s)} \leqslant C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\tilde{L}^p_T(\dot{B}^s)}.$$

(ii) *If* v *also belongs to* $\tilde{L}^p_T(\dot{B}^s) \cap L^\infty_T(L^\infty)$ *and* $G \in W^{[s]+3,\infty}_{loc}(\mathbb{R}^2)$, *then* $G(u)-G(v)$ *belongs to* $\tilde{L}^p_T(\dot{B}^s)$ *and there exists a function* C *depending only on* s *and* G *such that*

$$\begin{aligned}
&\|G(u)-G(v)\|_{\tilde{L}^p_T(\dot{B}^s)}\\
&\leqslant C(\|u\|_{L^\infty_T(L^\infty)}, \|v\|_{L^\infty_T(L^\infty)})\big((1+\|u\|_{L^\infty_T(L^\infty)}+\|v\|_{L^\infty_T(L^\infty)})\|u-v\|_{\tilde{L}^p_T(\dot{B}^s)}\\
&\qquad + (\|u\|_{\tilde{L}^p_T(\dot{B}^s)}+\|v\|_{\tilde{L}^p_T(\dot{B}^s)})\|u-v\|_{L^\infty_T(L^\infty)}\big).
\end{aligned}$$

III. *A PRIORI* ESTIMATES

Let $H\mathbf{u}=\mathbf{m}$ and $H=h+1$. Noticing that $2\mathrm{div}D(\mathbf{m}) = \nabla\mathrm{div}\mathbf{m} + \Delta\mathbf{m}$, we can rewrite the system as follows:

$$\begin{cases}
\partial_t h + \mathrm{div}\mathbf{m} = 0, & \text{(5a)}\\
\partial_t \mathbf{m} - \mu\Delta\mathbf{m} - 3\mu\nabla\mathrm{div}\mathbf{m} - \Omega\cos\theta\nabla m_1 - \Omega\cos\theta\mathbf{e}_1\mathrm{div}\mathbf{m} + \alpha_0(1)\mathbf{m} & \\
\quad +2\Omega\sin\theta\mathbf{m}^\perp + g\nabla h - a\nabla\Delta h = -\mathrm{div}\left(\dfrac{\mathbf{m}\otimes\mathbf{m}}{h+1}\right) - gh\nabla h + ah\nabla\Delta h & \\
\quad -\left(\dfrac{\alpha_0(h+1)}{h+1} - \alpha_0(1)\right)\mathbf{m} - \dfrac{\alpha_1(h+1)}{h+1}|\mathbf{m}|\mathbf{m} - 2\mu\nabla\left(\dfrac{\mathbf{m}\cdot\nabla h}{h+1}\right) & \\
\quad -\mu\mathrm{div}\left(\dfrac{\mathbf{m}\otimes\nabla h + (\mathbf{m}\otimes\nabla h)^\intercal}{h+1}\right) & \\
\quad +\Omega\cos\theta\nabla(hm_1) + \Omega\cos\theta h\mathbf{e}_1\mathrm{div}\mathbf{m} - \Omega\cos\theta\mathbf{e}_1\mathbf{m}\cdot\nabla h, & \text{(5b)}\\
h|_{t=0} = h_0(x) := H_0(x) - 1, \quad \mathbf{m}|_{t=0} = \mathbf{m}_0(x) = H_0(x)\mathbf{u}_0(x). & \text{(5c)}
\end{cases}$$

In order to derive the *a priori* estimates, we consider the following linear system:

$$\begin{cases} \partial_t h + \operatorname{div}\mathbf{m} = 0, & \text{(6a)} \\ \partial_t \mathbf{m} - \mu\Delta\mathbf{m} - 3\mu\nabla\operatorname{div}\mathbf{m} - \Omega\cos\theta\nabla m_1 - \Omega\cos\theta\mathbf{e}_1\operatorname{div}\mathbf{m} + \alpha_0(1)\mathbf{m} & \\ \qquad +2\Omega\sin\theta\mathbf{m}^\perp + g\nabla h - a\nabla\Delta h = \mathbf{F}, & \text{(6b)} \\ h|_{t=0} = h_0(x), \quad \mathbf{m}|_{t=0} = \mathbf{m}_0(x). & \text{(6c)} \end{cases}$$

We have the following proposition:

Proposition 1: Let $s \in \mathbb{R}$, $1 \leqslant q \leqslant p \leqslant +\infty$, $\max(q,2) \leqslant r \leqslant +\infty$, and $T \in (0,+\infty]$. If $(h_0, \mathbf{m}_0) \in \tilde{B}^{s-1,s} \times (\dot{B}^{s-1})^2$ and $\mathbf{F} \in L^q_T((\dot{B}^{s-3+2/q})^2)$, then the linear system (6) has a unique solution $(h, \mathbf{m}) \in \tilde{C}_T(\tilde{B}^{s-1,s} \times (\dot{B}^{s-1})^2) \cap \tilde{L}_T(\tilde{B}^{s-1+2/p,s+2/p} \times (\dot{B}^{s-1+2/p})^2)$. Moreover, there exists a constant C depending only on p, q, μ, Ω, θ, κ_l, g, and a such that the following inequality holds:

$$\begin{aligned} &\|h\|_{\tilde{L}^p_T(\tilde{B}^{s-1+2/p,s+2/p})} + \|\mathbf{m}\|_{\tilde{L}^p_T(\dot{B}^{s-1+2/p})} + \|\mathbf{m}\|_{\tilde{L}^r_T(\tilde{B}^{s-1,s-1+2/r})} \\ \leqslant\ & C\left(\|h_0\|_{\tilde{B}^{s-1,s}} + \|\mathbf{m}_0\|_{\dot{B}^{s-1}} + \|\mathbf{F}\|_{\tilde{L}^q_T(\dot{B}^{s-3+2/q})}\right). \end{aligned} \tag{7}$$

Proof: Denote $f_k = \triangle_k f$. Applying the operator $\triangle_k$ to (6), we have

$$\begin{cases} \partial_t h_k + \operatorname{div}\mathbf{m}_k = 0, & \text{(8a)} \\ \partial_t \mathbf{m}_k - \mu\Delta\mathbf{m}_k - 3\mu\nabla\operatorname{div}\mathbf{m}_k - \Omega\cos\theta\nabla m_{1k} - \Omega\cos\theta\mathbf{e}_1\operatorname{div}\mathbf{m}_k + \alpha_0(1)\mathbf{m}_k & \\ \qquad +2\Omega\sin\theta\mathbf{m}_k^\perp + g\nabla h_k - a\nabla\Delta h_k = \mathbf{F}_k, & \text{(8b)} \\ h_k|_{t=0} = h_0(x), \quad \mathbf{m}_k|_{t=0} = \mathbf{m}_0(x). & \text{(8c)} \end{cases}$$

The L^2 scalar product of (8a) with h_k yields

$$\frac{1}{2}\frac{d}{dt}\|h_k\|_2^2 = -(\operatorname{div}\mathbf{m}_k, h_k) = (\nabla h_k, \mathbf{m}_k). \tag{9}$$

From (8a) and integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla h_k\|_2^2 = (\nabla h_k, \nabla\partial_t h_k) = -(\nabla h_k, \nabla\operatorname{div}\mathbf{m}_k) = -(\nabla\Delta h_k, \mathbf{m}_k). \tag{10}$$

Take the L^2 scalar product of (8b) with $\mathbf{m}_k$, it implies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\mathbf{m}_k\|_2^2 + g\|h_k\|_2^2 + a\|\nabla h_k\|_2^2) \\ &\qquad + \mu\|\nabla\mathbf{m}_k\|_2^2 + 3\mu\|\operatorname{div}\mathbf{m}_k\|_2^2 + \alpha_0(1)\|\mathbf{m}_k\|_2^2 = (\mathbf{F}_k, \mathbf{m}_k). \end{aligned} \tag{11}$$

Taking the L^2 scalar product of (8b) with ∇h_k, we can get

$$\begin{aligned} &(\partial_t\mathbf{m}_k, \nabla h_k) + 2\mu\frac{d}{dt}\|\nabla h_k\|_2^2 + \frac{1}{2}\alpha_0(1)\frac{d}{dt}\|h_k\|_2^2 + g\|\nabla h_k\|_2^2 + a\|\Delta h_k\|_2^2 \\ =&(\mathbf{F}_k, \nabla h_k) + \Omega\cos\theta(\nabla m_{1k}, \nabla h_k) + \Omega\cos\theta(\mathbf{e}_1\operatorname{div}\mathbf{m}_k, \nabla h_k) - 2\Omega\sin\theta(\mathbf{m}_k^\perp, \nabla h_k). \end{aligned}$$

Taking the L^2 scalar product of (8a) with $-\operatorname{div}\mathbf{m}_k$, we obtain

$$(\partial_t(\nabla h_k), \mathbf{m}_k) = \|\operatorname{div}\mathbf{m}_k\|_2^2.$$

Summing up the above two equalities, we reach

$$\begin{aligned}
&\frac{d}{dt}(\mathbf{m}_k, \nabla h_k) + 2\mu\frac{d}{dt}\|\nabla h_k\|_2^2 + \frac{1}{2}\alpha_0(1)\frac{d}{dt}\|h_k\|_2^2 \\
&\quad + g\|\nabla h_k\|_2^2 + a\|\Delta h_k\|_2^2 - \|\mathrm{div}\mathbf{m}_k\|_2^2 \\
&= (\mathbf{F}_k, \nabla h_k) + \Omega\cos\theta(\nabla m_{1k}, \nabla h_k) + \Omega\cos\theta(\mathbf{e}_1\mathrm{div}\mathbf{m}_k, \nabla h_k) \\
&\quad -2\Omega\sin\theta(\mathbf{m}_k^\perp, \nabla h_k).
\end{aligned} \tag{12}$$

Let $\beta > 0$ be a constant to be chosen later and denote

$$f_k^2 = \|\mathbf{m}_k\|_2^2 + (g + \beta\alpha_0(1))\|h_k\|_2^2 + (a + 4\beta\mu)\|\nabla h_k\|_2^2 + 2\beta(\mathbf{m}_k, \nabla h_k).$$

From (11) and (12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}f_k^2 + \mu\|\nabla\mathbf{m}_k\|_2^2 + (3\mu - \beta)\|\mathrm{div}\mathbf{m}_k\|_2^2 + \alpha_0(1)\|\mathbf{m}_k\|_2^2 + \beta g\|\nabla h_k\|_2^2 + \beta a\|\Delta h_k\|_2^2 \\
&= (\mathbf{F}_k, \mathbf{m}_k + \beta\nabla h_k) + \beta\Omega\cos\theta(\nabla m_{1k}, \nabla h_k) \\
&\quad +\beta\Omega\cos\theta(\mathbf{e}_1\mathrm{div}\mathbf{m}_k, \nabla h_k) - 2\beta\Omega\sin\theta(\mathbf{m}_k^\perp, \nabla h_k).
\end{aligned} \tag{13}$$

Since we have for some $M_i > 0$

$$\begin{aligned}
2|(\mathbf{m}_k, \nabla h_k)| &\leqslant \frac{1}{M_1}\|\mathbf{m}_k\|_2^2 + M_1\|\nabla h_k\|_2^2, \\
|(\nabla m_{1k}, \nabla h_k)| &\leqslant \frac{1}{2M_2}\|\nabla\mathbf{m}_k\|_2^2 + \frac{M_2}{2}\|\nabla h_k\|_2^2, \\
|(\mathbf{e}_1\mathrm{div}\mathbf{m}_k, \nabla h_k)| &\leqslant \frac{1}{2M_3}\|\mathrm{div}\mathbf{m}_k\|_2^2 + \frac{M_3}{2}\|\nabla h_k\|_2^2, \\
2|(\mathbf{m}_k^\perp, \nabla h_k)| &\leqslant \frac{1}{M_4}\|\mathbf{m}_k\|_2^2 + M_4\|\nabla h_k\|_2^2,
\end{aligned}$$

we have to choose β and M_i such that

$$\beta < M_1 < \frac{a}{\beta} + 4\mu, \quad \mu - \frac{\beta\Omega\cos\theta}{2M_2} > 0, \quad 3\mu - \beta - \frac{\beta\Omega\cos\theta}{2M_3} > 0,$$
$$\alpha_0(1) - \frac{\beta\Omega\sin\theta}{M_4} > 0, \qquad g - \Omega\cos\theta\frac{M_2 + M_3}{2} - \Omega\sin\theta M_4 > 0.$$

We can verify that the above inequalities hold if we choose

$$\begin{aligned}
\beta &= \frac{\mu g}{2\Omega^2}\min\left(\frac{\kappa_l}{3\mu + \kappa_l}, \frac{4\Omega^2}{g + \Omega^2}\right), \\
M_1 &= 4\mu + \frac{a}{8\mu}, \quad M_2 = M_3 = \frac{3}{\Omega\cos\theta}, \quad M_4 = \frac{3}{\Omega\sin\theta}.
\end{aligned}$$

Hence, we deduce that

$$\frac{1}{\delta}f_k^2 \leqslant \|\mathbf{m}_k\|_2^2 + g\|h_k\|_2^2 + a\|\nabla h_k\|_2^2 \leqslant \delta f_k^2,$$

for some positive constant δ. Thus, there exists a constant $\gamma > 0$ such that

$$\frac{1}{2}\frac{d}{dt}f_k^2 + \gamma 2^{2k}f_k^2 \leqslant \|\mathbf{F}_k\|_2(\|\mathbf{m}_k\|_2 + \|\nabla h_k\|_2).$$

Dividing the above inequality by f_k, we get

$$\frac{d}{dt}f_k + \gamma 2^{2k}f_k \leqslant C\|\mathbf{F}_k\|_2.$$

By Gronwall inequality, we obtain

$$f_k(t) \leqslant e^{-\gamma 2^{2k} t} f_k(0) + C \int_0^t e^{-\gamma 2^{2k}(t-\tau)} \|\mathbf{F}_k(\tau)\|_2 d\tau.$$

By the definition of f_k, it yields

$$\begin{aligned}
&\|\mathbf{m}_k(t)\|_2 + \|h_k(t)\|_2 + \|\nabla h_k(t)\|_2 \\
\leqslant& C e^{-\gamma 2^{2k} t}(\|\mathbf{m}_{0k}\|_2 + \|h_{0k}\|_2 + \|\nabla h_{0k}\|_2) + C \int_0^t e^{-\gamma 2^{2k}(t-\tau)} \|\mathbf{F}_k(\tau)\|_2 d\tau.
\end{aligned}$$

Taking the L^p-norm on $[0, T]$ with respect to the time, we have for $1 \leqslant q \leqslant p$, from Young's inequality, that

$$\begin{aligned}
&\|\mathbf{m}_k(t)\|_{L^p(0,T;L^2)} + \|h_k(t)\|_{L^p(0,T;L^2)} + \|\nabla h_k(t)\|_{L^p(0,T;L^2)} \\
\leqslant& C\|e^{-\gamma 2^{2k} t}\|_p(\|\mathbf{m}_{0k}\|_2 + \|h_{0k}\|_2 + \|\nabla h_{0k}\|_2) + C\|e^{-\gamma 2^{2k} t} * \|\mathbf{F}_k(t)\|_2\|_{L^p(0,T)} \\
\leqslant& C(\gamma p)^{-1/p} 2^{-2k/p}(\|\mathbf{m}_{0k}\|_2 + \|h_{0k}\|_2 + \|\nabla h_{0k}\|_2) \\
&+ C\left(\frac{1}{\gamma}(\frac{1}{p} - \frac{1}{q} + 1)\right)^{\frac{1}{p}-\frac{1}{q}+1} 2^{-2k(\frac{1}{p}-\frac{1}{q}+1)} \|F_k(t)\|_{L^q(0,T;L^2)}.
\end{aligned}$$

Multiplying the above with $2^{(s-1+2/p)k}$ and summing with respect to $k \in \mathbb{Z}$, we obtain for any $1 \leqslant q \leqslant p \leqslant +\infty$

$$\begin{aligned}
&\sum_{k\leqslant 0} 2^{k(s-1+2/p)} \|h_k\|_{L^p_T L^2_x} + \sum_{k>0} 2^{k(s+2/p)} \|h_k\|_{L^p_T L^2_x} + \sum_{k\in\mathbb{Z}} 2^{k(s-1+2/p)} \|\mathbf{m}_k\|_{L^p_T L^2_x} \\
\lesssim& \|h_0\|_{\tilde{B}^{s-1,s}} + \sum_{k\in\mathbb{Z}} 2^{k(s-1)} \|\mathbf{m}_k(0)\|_2 + \sum_{k\in\mathbb{Z}} 2^{k(s-3+2/q)} \|\mathbf{F}\|_{L^q_T L^2_x}. \qquad (14)
\end{aligned}$$

Therefore, we have for any $1 \leqslant q \leqslant p \leqslant +\infty$

$$\begin{aligned}
&\|h\|_{\tilde{L}^p_T(\tilde{B}^{s-1+2/p,s+2/p})} + \|\mathbf{m}\|_{\tilde{L}^p_T(\dot{B}^{s-1+2/p})} \\
\lesssim& \|h_0\|_{\tilde{B}^{s-1,s}} + \|\mathbf{m}_0\|_{\dot{B}^{s-1}} + \|\mathbf{F}\|_{\tilde{L}^q_T(\dot{B}^{s-3+2/q})}. \qquad (15)
\end{aligned}$$

But here it is not enough to consider the effect of the friction. In fact, the friction can make regularities of the momentum $\mathbf{m}$ decrease, especially for low frequencies. Now, we suppose that $k \leqslant 0$. From (11), (9) and (10) and by Bernstein's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{m}_k\|_2^2 + \mu\|\nabla \mathbf{m}_k\|_2^2 + 3\mu\|\mathrm{div}\mathbf{m}_k\|_2^2 + \alpha_0(1)\|\mathbf{m}_k\|_2^2 \\
=&(\mathbf{F}_k, \mathbf{m}_k) - g(\nabla h_k, \mathbf{m}_k) + a(\nabla\Delta h_k, \mathbf{m}_k) \\
\leqslant&(\|\mathbf{F}_k\|_2 + g\|\nabla h_k\|_2 + a\|\nabla\Delta h_k\|_2)\|\mathbf{m}_k\|_2 \\
\leqslant&(\|\mathbf{F}_k\|_2 + (\frac{8}{3}2^k g + \frac{8^3}{3^3}2^{3k}a)\|h_k\|_2)\|\mathbf{m}_k\|_2 \\
\leqslant&(\|\mathbf{F}_k\|_2 + (\frac{8}{3}g + \frac{8^3}{3^3}a)2^k\|h_k\|_2)\|\mathbf{m}_k\|_2,
\end{aligned}$$

which yields

$$\frac{d}{dt}\|\mathbf{m}_k\|_2 + \alpha_0(1)\|\mathbf{m}_k\|_2 \lesssim \|\mathbf{F}_k\|_2 + 2^k\|h_k\|_2.$$

From Gronwall's inequality, we get

$$\|\mathbf{m}_k(t)\|_2 \leqslant e^{-\alpha_0(1)t}\|\mathbf{m}_k(0)\|_2 + C\int_0^t e^{-\alpha_0(1)(t-\tau)}(\|\mathbf{F}_k(\tau)\|_2 + 2^k\|h_k(\tau)\|_2)d\tau.$$

Taking the L^r-norm on $[0, T]$ with respect to the time, we have for $1 \leqslant q,\ \tilde{q} \leqslant r$, from Young's inequality, that

$$\begin{aligned}\|\mathbf{m}_k(t)\|_{L^r(0,T;L^2)} \leqslant& \|e^{-\alpha_0(1)t}\|_{L^r(0,T)}\|\mathbf{m}_k(0)\|_2 + C\|e^{-\alpha_0(1)t} * \|\mathbf{F}_k(t)\|_2\|_{L^r(0,T)}\\ &+ C2^k\|e^{-\alpha_0(1)t} * \|h_k(t)\|_2\|_{L^r(0,T)}\\ \lesssim& \|\mathbf{m}_k(0)\|_2 + \|\mathbf{F}_k\|_{L^q(0,T;L^2)} + 2^k\|h_k\|_{L^{\tilde{q}}(0,T;L^2)}.\end{aligned}$$

Multiplying the above with $2^{k(s-1)}$ and summing with respect to $k \leqslant 0$, we obtain, from (14), that for $q \geqslant 1$, $\tilde{q} = 2$, and $r \geqslant 2$

$$\begin{aligned}&\sum_{k\leqslant 0} 2^{k(s-1)}\|\mathbf{m}_k(t)\|_{L^r(0,T;L^2)}\\ &\lesssim \sum_{k\leqslant 0} 2^{k(s-1)}\|\mathbf{m}_k(0)\|_2 + \sum_{k\leqslant 0} 2^{k(s-1)}\|\mathbf{F}_k\|_{L^r(0,T;L^2)} + \sum_{k\leqslant 0} 2^{ks}\|h_k\|_{L^2(0,T;L^2)}\\ &\lesssim \sum_{k\leqslant 0} 2^{k(s-1)}\|\mathbf{m}_k(0)\|_2 + \sum_{k\leqslant 0} 2^{k(s-1)}\|\mathbf{F}_k\|_{L^p(0,T;L^2)}\\ &\quad + \|h_0\|_{\tilde{B}^{s-1,s}} + \sum_{k\in\mathbb{Z}} 2^{k(s-1)}\|\mathbf{m}_k(0)\|_2 + \sum_{k\in\mathbb{Z}} 2^{k(s-3+2/q)}\|\mathbf{F}\|_{L^q_T L^2_x}\\ &\lesssim \|h_0\|_{\tilde{B}^{s-1,s}} + \|\mathbf{m}_0\|_{\dot{B}^{s-1}} + \sum_{k\in\mathbb{Z}} 2^{k(s-3+2/q)}\|\mathbf{F}\|_{L^q_T L^2_x}.\end{aligned} \tag{16}$$

Combining (16) with (14), we get for $1 \leqslant \min(q, 2) \leqslant \max(q, 2) \leqslant r \leqslant +\infty$

$$\begin{aligned}&\|h\|_{\tilde{L}^p_T(\tilde{B}^{s-1+2/p,s+2/p})} + \|\mathbf{m}\|_{\tilde{L}^r_T(\tilde{B}^{s-1,s-1+2/r})}\\ &\lesssim \|h_0\|_{\tilde{B}^{s-1,s}} + \|\mathbf{m}_0\|_{\dot{B}^{s-1}} + \|\mathbf{F}\|_{\tilde{L}^q_T(\dot{B}^{s-3+2/q})}.\end{aligned} \tag{17}$$

From (15) and (17), we have the desired results. □

IV. GLOBAL EXISTENCE AND UNIQUENESS

In this section, we will construct a contraction mapping and use the Banach fixed point theorem to obtain the existence and uniqueness of the solution. We first define the work space.

Denote

$$\begin{aligned}E :=& \|h_0\|_{\tilde{B}^{0,1}} + \|\mathbf{m}_0\|_{\dot{B}^0},\\ \|(h,\mathbf{m})\|_{E_T} :=& \sup_{p\in[1,\infty]}\|h\|_{\tilde{L}^p_T(\tilde{B}^{2/p,1+2/p})} + \sup_{\substack{p\in[1,\infty]\\ r\in[2,\infty]}}\|\mathbf{m}\|_{\tilde{L}^p_T(\dot{B}^{2/p})\cap\tilde{L}^r_T(\tilde{B}^{0,2/r})},\\ \mathscr{D} :=& \left\{(h,\mathbf{m}) \in (\mathscr{S}')^{1+2} : \|(h,\mathbf{m})\|_{E_T} \leqslant AE\right\},\end{aligned}$$

where A is a constant determined later. It is easy to verify that

$$\left\{(h,\mathbf{m}) \in (\mathscr{S}')^{1+2} : \|(h,\mathbf{m})\|_{E_T} < \infty\right\}$$

is a Banach space since $\dot{B}^s$ is a Banach space. Denote by $S(t)$ the semigroup associated with (6). According to Duhamel's formula, (5) can be rewritten as the integral form

$$\begin{pmatrix} h(t)\\ \mathbf{m}(t)\end{pmatrix} = S(t)\begin{pmatrix} h_0\\ \mathbf{m}_0\end{pmatrix} + \int_0^t S(t-\tau)\begin{pmatrix} 0\\ \mathbf{F}(h(\tau),\mathbf{m}(\tau))\end{pmatrix} d\tau, \tag{18}$$

where $\mathbf{F}(\mathbf{h},\mathbf{m})=\sum_{i=1}^{10}\mathbf{F}_i(h,\mathbf{m})$ with

$$\begin{aligned}
\mathbf{F}_1 =& -\operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{h+1}\right), \quad \mathbf{F}_2=-gh\nabla h, \quad \mathbf{F}_3=ah\nabla\Delta h,\\
\mathbf{F}_4 =& -\left(\frac{\alpha_0(h+1)}{h+1}-\alpha_0(1)\right)\mathbf{m}, \quad \mathbf{F}_5=-\frac{\alpha_1(h+1)}{h+1}|\mathbf{m}|\mathbf{m},\\
\mathbf{F}_6 =& -2\mu\nabla\left(\frac{\mathbf{m}\cdot\nabla h}{h+1}\right), \quad \mathbf{F}_7=-\mu\operatorname{div}\left(\frac{\mathbf{m}\otimes\nabla h+(\mathbf{m}\otimes\nabla h)^{\mathsf{T}}}{h+1}\right),\\
\mathbf{F}_8 =&\Omega\cos\theta\nabla(hm_1), \quad \mathbf{F}_9=\Omega\cos\theta h\mathbf{e}_1\operatorname{div}\mathbf{m}, \quad \mathbf{F}_{10}=-\Omega\cos\theta\mathbf{e}_1\mathbf{m}\cdot\nabla h.
\end{aligned}$$

We define the operator $\mathscr{T}$ in $\mathscr{D}$ by

$$\mathscr{T}(h(t),\mathbf{m}(t))=S(t)\begin{pmatrix}h_0\\ \mathbf{m}_0\end{pmatrix}+\int_0^t S(t-\tau)\begin{pmatrix}0\\ \mathbf{F}(h(\tau),\mathbf{m}(\tau))\end{pmatrix}d\tau. \tag{19}$$

To prove the existence part of the theorem, we just have to show that $\mathscr{T}$ has a fixed point in $\mathscr{D}$. First, we need to prove that the space $\mathscr{D}$ is stable under the operator $\mathscr{T}$ for some constant A provided that E is small enough.

Let $(h,\mathbf{m})\in\mathscr{D}$. We prove that the nonlinearities can be controlled by the norms of $(h,\mathbf{m})$. By Lemma 3, we can get, in view of Bernstein's inequalities, that

$$\begin{aligned}
\|\mathbf{F}_1\|_{L^1_T(\dot B^0)} \lesssim&\|\frac{\mathbf{m}\otimes\mathbf{m}}{h+1}\|_{L^1_T(\dot B^1)}\lesssim\|\mathbf{m}\otimes\mathbf{m}\|_{L^1_T(\dot B^1)}+\|\frac{h}{h+1}\mathbf{m}\otimes\mathbf{m}\|_{L^1_T(\dot B^1)}\\
\lesssim&(1+\|h\|_{\tilde L^\infty_T(\dot B^1)})\|\mathbf{m}\|^2_{\tilde L^2_T(\dot B^1)}\lesssim(1+\|(h,\mathbf{m})\|_{E_T})\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_2\|_{L^1_T(\dot B^0)} \lesssim&\|h\|_{\tilde L^\infty_T(\dot B^0)}\|\nabla h\|_{L^1_T(\dot B^1)}\lesssim\|h\|_{\tilde L^\infty_T(\dot B^0)}\|h\|_{L^1_T(\dot B^2)}\lesssim\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_3\|_{L^1_T(\dot B^0)} \lesssim&\|h\|_{\tilde L^\infty_T(\dot B^1)}\|\nabla\Delta h\|_{L^1_T(\dot B^0)}\lesssim\|h\|_{\tilde L^\infty_T(\dot B^1)}\|h\|_{L^1_T(\dot B^3)}\lesssim\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_4\|_{L^1_T(\dot B^0)} =&\frac{3\mu\kappa_l}{3\mu+\kappa_l}\|\frac{(3\mu+2\kappa_l)h+\kappa_l h^2}{3\mu(h+1)+\kappa_l(h+1)^2}\mathbf{m}\|_{L^1_T(\dot B^0)}\lesssim\|h\|_{\tilde L^2_T(\dot B^1)}\|\mathbf{m}\|_{\tilde L^2_T(\dot B^0)}\\
\lesssim&\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_5\|_{L^1_T(\dot B^0)} =&9\mu^2\kappa_t\|\frac{|\mathbf{m}|\mathbf{m}}{(h+1)(3\mu+\kappa_l(h+1))^2}\|_{L^1_T(\dot B^0)}\\
\lesssim&(1+\|h\|_{\tilde L^\infty_T(\dot B^1)})\|\mathbf{m}\|^2_{\tilde L^2_T(\tilde B^{0,1})}\lesssim(1+\|(h,\mathbf{m})\|_{E_T})\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_6\|_{L^1_T(\dot B^0)} \lesssim&\|\frac{\mathbf{m}\cdot\nabla h}{h+1}\|_{L^1_T(\dot B^1)}\lesssim(1+\|h\|_{\tilde L^\infty_T(\dot B^1)})\|h\|_{\tilde L^2_T(\dot B^2)}\|\mathbf{m}\|_{\tilde L^2_T(\dot B^1)}\\
\lesssim&(1+\|(h,\mathbf{m})\|_{E_T})\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_7\|_{L^1_T(\dot B^0)} \lesssim&\|\frac{\mathbf{m}\otimes\nabla h+(\mathbf{m}\otimes\nabla h)^{\mathsf{T}}}{h+1}\|_{L^1_T(\dot B^1)}\\
\lesssim&(1+\|h\|_{\tilde L^\infty_T(\dot B^1)})\|h\|_{\tilde L^2_T(\dot B^2)}\|\mathbf{m}\|_{\tilde L^2_T(\dot B^1)}\lesssim(1+\|(h,\mathbf{m})\|_{E_T})\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_8\|_{L^1_T(\dot B^0)} \lesssim&\|hm_1\|_{L^1_T(\dot B^1)}\lesssim\|h\|_{\tilde L^2_T(\dot B^1)}\|\mathbf{m}\|_{\tilde L^2_T(\dot B^1)}\lesssim\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_9\|_{L^1_T(\dot B^0)} \lesssim&\|h\|_{\tilde L^2_T(\dot B^1)}\|\mathbf{m}\|_{\tilde L^2_T(\dot B^1)}\lesssim\|(h,\mathbf{m})\|^2_{E_T},\\
\|\mathbf{F}_{10}\|_{L^1_T(\dot B^0)} \lesssim&\|h\|_{\tilde L^2_T(\dot B^1)}\|\mathbf{m}\|_{\tilde L^2_T(\dot B^1)}\lesssim\|(h,\mathbf{m})\|^2_{E_T}.
\end{aligned}$$

From these estimates, by choosing $A \geqslant 4C$ and sufficiently small E such that $(1+AE)A^2E \leqslant 3$, we can get

$$\begin{aligned}\|\mathscr{T}(h,\mathbf{m})\|_{E_T} \leqslant& CE + C(1+\|(h,\mathbf{m})\|_{E_T})\|(h,\mathbf{m})\|_{E_T}^2\\ \leqslant& CE + C(1+AE)A^2E^2 \leqslant AE.\end{aligned}$$

Thus, we deduce that $\mathscr{T}(\mathscr{D}) \subset \mathscr{D}$.

Next, we consider the contraction properties of the operator $\mathscr{T}$. For two elements $(h_1, \mathbf{m}_1)$ and $(h_2, \mathbf{m}_2)$ in $\mathscr{D}$, according to (19) and to Proposition 1, we get

$$\|\mathscr{T}(h_2,\mathbf{m}_2) - \mathscr{T}(h_1,\mathbf{m}_1)\|_{E_T} \lesssim \|\mathbf{F}(h_2,\mathbf{m}_2) - \mathbf{F}(h_1,\mathbf{m}_1)\|_{L^1_T(\dot{B}^0)}.$$

Applying Proposition 1 and Lemma 3 to

$$\begin{aligned}
&\mathbf{F}_1(h_2,\mathbf{m}_2) - \mathbf{F}_1(h_1,\mathbf{m}_1)\\
&\quad = \operatorname{div}\left(\mathbf{m}_1\otimes\mathbf{m}_1\left(\frac{h_2}{h_2+1} - \frac{h_1}{h_1+1}\right) - \frac{\mathbf{m}_2\otimes(\mathbf{m}_2-\mathbf{m}_1) + (\mathbf{m}_2-\mathbf{m}_1)\otimes\mathbf{m}_1}{h_2+1}\right),\\
&\mathbf{F}_2(h_2,\mathbf{m}_2) - \mathbf{F}_2(h_1,\mathbf{m}_1) = -g(h_2-h_1)\nabla h_2 - gh_1\nabla(h_2-h_1),\\
&\mathbf{F}_3(h_2,\mathbf{m}_2) - \mathbf{F}_3(h_1,\mathbf{m}_1) = a(h_2-h_1)\nabla\Delta h_2 + ah_1\nabla\Delta(h_2-h_1),\\
&\mathbf{F}_4(h_2,\mathbf{m}_2) - \mathbf{F}_4(h_1,\mathbf{m}_1)\\
&\quad = -\left(\frac{\alpha_0(h_2+1)}{h_2+1} - \alpha_0(1)\right)(\mathbf{m}_2-\mathbf{m}_1) - \left(\frac{\alpha_0(h_2+1)}{h_2+1} - \frac{\alpha_0(h_1+1)}{h_1+1}\right)\mathbf{m}_1,\\
&\mathbf{F}_5(h_2,\mathbf{m}_2) - \mathbf{F}_5(h_1,\mathbf{m}_1) = -\frac{\alpha_1(h_2+1)}{h_2+1}|\mathbf{m}_2|(\mathbf{m}_2-\mathbf{m}_1)\\
&\qquad - \left(\frac{\alpha_1(h_2+1)}{h_2+1} - \frac{\alpha_1(h_1+1)}{h_1+1}\right)|\mathbf{m}_2|\mathbf{m}_1 - \frac{\alpha_1(h_1+1)}{h_1+1}(|\mathbf{m}_2| - |\mathbf{m}_1|)\mathbf{m}_1,\\
&\mathbf{F}_6(h_2,\mathbf{m}_2) - \mathbf{F}_6(h_1,\mathbf{m}_1)\\
&\quad = -2\mu\nabla\left(\frac{\mathbf{m}_2\cdot\nabla(h_2-h_1)}{h_2+1} - \left(\frac{h_2}{h_2+1} - \frac{h_1}{h_1+1}\right)\mathbf{m}_2\cdot\nabla h_1 + \frac{(\mathbf{m}_2-\mathbf{m}_1)\cdot\nabla h_1}{h_1+1}\right),\\
&\mathbf{F}_7(h_2,\mathbf{m}_2) - \mathbf{F}_7(h_1,\mathbf{m}_1)\\
&\quad = -\mu\operatorname{div}\Big(\frac{1}{h_2+1}((\mathbf{m}_2-\mathbf{m}_1)\otimes\nabla h_2 + ((\mathbf{m}_2-\mathbf{m}_1)\otimes\nabla h_2)^\top)\\
&\qquad\qquad + \frac{1}{h_2+1}(\mathbf{m}_1\otimes\nabla(h_2-h_1) + (\mathbf{m}_1\otimes\nabla(h_2-h_1))^\top)\\
&\qquad\qquad - \Big(\frac{h_2}{h_2+1} - \frac{h_1}{h_1+1}\Big)(\mathbf{m}_1\otimes\nabla h_1 + (\mathbf{m}_1\otimes\nabla h_1)^\top)\Big),\\
&\mathbf{F}_8(h_2,\mathbf{m}_2) - \mathbf{F}_8(h_1,\mathbf{m}_1) = \Omega\cos\theta\nabla((h_2-h_1)m_{21} + h_1(m_{21}-m_{11})),\\
&\mathbf{F}_9(h_2,\mathbf{m}_2) - \mathbf{F}_9(h_1,\mathbf{m}_1) = \Omega\cos\theta\mathbf{e}_1((h_2-h_1)\operatorname{div}\mathbf{m} + h_1\operatorname{div}(\mathbf{m}_{-}\mathbf{m}_1)),\\
&\mathbf{F}_{10}(h_2,\mathbf{m}_2) - \mathbf{F}_{10}(h_1,\mathbf{m}_1) = \Omega\cos\theta\mathbf{e}_1((\mathbf{m}_2-\mathbf{m}_1)\cdot\nabla h_2 + \mathbf{m}_1\cdot\nabla(h_2-h_1)),
\end{aligned}$$

we can get

$$\begin{aligned}&\|\mathscr{T}(h_2,\mathbf{m}_2) - \mathscr{T}(h_1,\mathbf{m}_1)\|_{E_T}\\ &\leqslant C(\|(h_1,\mathbf{m}_1)\|_{E_T} + \|(h_2,\mathbf{m}_2)\|_{E_T})\|(h_2-h_1,\mathbf{m}_2-\mathbf{m}_1)\|_{E_T}.\end{aligned}$$

Now, we can choose E small enough and A (which may be greater than the previous one) such that

$$\|\mathscr{T}(h_2,\mathbf{m}_2) - \mathscr{T}(h_1,\mathbf{m}_1)\|_{E_T} \leqslant \frac{1}{2}\|(h_2-h_1,\mathbf{m}_2-\mathbf{m}_1)\|_{E_T},$$

which implies the operator $\mathscr{T}$ is a contraction map in $\mathscr{D}$. By the fixed point theorem, there exists a unique fixed point $(h, \mathbf{m}) \in \mathscr{D}$ for (19). Indeed, $(h, \mathbf{m})$ is a global solution of (18) since all constants are independent of $T \in (0, +\infty]$ (we can take $T = +\infty$) in the above derivations.

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