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Wellposedness for the fourth order nonlinear Schrödinger equations

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Abstract

We study the local smoothing effects and wellposedness of Cauchy problem for the fourth order nonlinear Schrödinger equations in 1D

$$i\partial_t u = \partial_x^4 u + P\big((\partial_x^\alpha u)_{|\alpha|\leqslant 2}, (\partial_x^\alpha \bar{u})_{|\alpha|\leqslant 2}\big), \quad t, x \in \mathbb{R},$$

where $P(\cdot)$ is a polynomial excluding constant or linear terms.

Keywords: Fourth order nonlinear Schrödinger equations; Cauchy problem; Wellposedness; Smoothing effects

1. Introduction

We consider the Cauchy problem for the fourth order nonlinear Schrödinger equations

$$i\partial_t u = \partial_x^4 u + P\big((\partial_x^\alpha u)_{|\alpha|\leqslant 2}, (\partial_x^\alpha \bar{u})_{|\alpha|\leqslant 2}\big), \quad t, x \in \mathbb{R}, \tag{1.1}$$

$$u(0, x) = u_0(x), \tag{1.2}$$

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where $u = u(t, x)$ is a complex valued wave function, and $P(\cdot)$ is a complex valued polynomial defined in $\mathbb{C}^6$ such that

$$P(\vec{z}) = P(z_1, z_2, \dots, z_6) = \sum_{\substack{\beta \leqslant |\alpha| \leqslant \gamma \\ \alpha \in \mathbb{Z}^6}} a_\alpha z^\alpha, \tag{1.3}$$

for $\beta \geqslant 2$ and there exists $a_{\alpha_0} \neq 0$ for some $\alpha_0 \in \mathbb{Z}^6$ with $|\alpha_0| = \beta$.

This class of nonlinear Schrödinger equations comes from the infinite hierarchy of commuting flows arising from the 1D cubic nonlinear Schrödinger equation. A large amount of interesting work has been devoted to the study of the Cauchy problem for dispersive equations. One can see [1,3–17] and references cited therein.

In order to study the influence of higher order dispersion on solitary waves, instability and the collapse phenomena, Karpman [9] introduced a class of nonlinear Schrödinger equations

$$i\Psi_t + \frac{1}{2}\Delta\Psi + \frac{\gamma}{2}\Delta^2\Psi + f\big(|\Psi|^2\big)\Psi = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

This system with different nonlinearities was discussed by several authors. In [16], by using the method of Fourier restriction norm, Segata studied a special fourth order nonlinear Schrödinger equation in one-dimensional space and considered the three-dimensional motion of an isolated vortex filament, which was introduced by Da Rios, embedded in inviscid incompressible fluid fulfilled in an infinite region. And the results have been improved in [8,17].

In [1], Ben-Artzi et al. discussed the sharp space–time decay properties of fundamental solutions to the linear equation

$$i\Psi_t - \varepsilon\Delta\Psi + \Delta^2\Psi = 0, \quad \varepsilon \in \{-1, 0, 1\}.$$

In [6], Guo and Wang considered the existence and scattering theory for the Cauchy problem of nonlinear Schrödinger equations with the form

$$iu_t + (-\Delta)^m u + f(u) = 0, \tag{1.4}$$

$$u(0, x) = \varphi(x), \tag{1.5}$$

where $m \geqslant 1$ is an integer. Pecher and von Wahl in [15] proved the existence of classical global solutions of (1.4)–(1.5) for the space dimensions $n \leqslant 7m$ for the case $m \geqslant 1$.

In the present paper we deal with Eq. (1.1) in which the difficulty arises from the fact that the nonlinearity of P involves the first and second derivatives $\partial_x u$, $\partial_x \bar{u}$, $\partial_x^2 u$ and $\partial_x^2 \bar{u}$. This could cause the so-called loss of derivatives so long as we make direct use of the standard methods, such as the energy estimates, the space–time estimates, etc.

In [14], Kenig et al. made a great progress on the nonlinear Schrödinger equation of the form

$$\partial_t u = i\Delta u + P(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

and proved that the Cauchy problem is locally well-posed for small data in the Sobolev space $H^s(\mathbb{R}^n)$ and in its weighted version by pushing forward the linear estimates associated with the Schrödinger group $\{e^{it\Delta}\}_{-\infty}^{\infty}$ and by introducing suitable function spaces where these estimates act naturally. In the one-dimensional case, $n = 1$, the smallness assumption on the size of the data was removed by Hayashi and Ozawa [7] by using a change of variable to obtain an equivalent system with a nonlinear term independent of $\partial_x u$, where the new system can be treated by the standard energy method. But this method might not be able to be applied to the fourth order nonlinear Schrödinger equation including terms $\partial_x^2 u$ and $\partial_x^2 \bar{u}$.

Kato studied in [10] the Cauchy problem for the (generalized) Korteweg–de Vries equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad t, x \in \mathbb{R},$$

by using the smoothing effect method.

In the present paper, we will discuss the local smoothing effects of the unitary group $\{S(t)\}_{-\infty}^{\infty}$ in order to overcome the loss of derivatives. More precisely, we will prove, in Section 2, that

$$\sup_x \left\| \partial_x^j \int_0^t S(t-\tau)F(\tau, x)\,\mathrm{d}\tau \right\|_{L_t^2} \leqslant CT^{\frac{3-j}{4}} \|F\|_{L_x^1 L_t^2} + \left\| \partial_x^{j-\frac{3}{2}} u(0, x) \right\|_{L_x^2}.$$

To construct the work space, we have to study, in Section 3, the boundedness properties of the maximal function $\sup_{[0,T]} |S(t) \cdot|$. This idea is implicit in the splitting argument introduced by Ginibre and Tsutsumi [5] to deal with uniqueness for the generalized KdV equation. Finally, we will consider some special cases in Section 4 to apply the estimates we have obtained. More precisely, we study the Cauchy problem (1.1)–(1.2) without the restriction on the size of the initial data and get the local wellposedness by the fixed point argument.

For convenience, we first introduce some notations. $S(t) := e^{-it\partial_x^4}$ denotes the unitary group generated by $-i\partial_x^4$ in $L^2(\mathbb{R})$. $\bar{z}$ denotes the conjugate of the complex number z. $\mathscr{F}u$ or $\hat{u}$ ($\mathscr{F}^{-1}u$, respectively) denotes the Fourier (inverse, respectively) transform of u with respect to all variables. $\mathscr{S}$ denotes the space of Schwartz' functions. And we denote $\|\cdot\|_{s,2} = \|\cdot\|_{H^s}$, $\|\cdot\|_p = \|\cdot\|_{L^p}$ for $1 \leqslant p < \infty$ and

$$\|f\|_{l,2,j} = \|f\|_{H^l(\mathbb{R};|x|^j\,\mathrm{d}x)} = \sum_{|\gamma|\leqslant l} \left(\int_{-\infty}^{\infty} \left|\partial_x^\gamma f(x)\right|^2 |x|^j\,\mathrm{d}x \right)^{1/2}. \tag{1.6}$$

Throughout the paper, the constant C might be different from each other.

Now we state the main results of this paper.

Theorem 1.1 *(Case $\beta \geqslant 3$). Given any polynomial P as in* (1.3) *with $\beta \geqslant 3$, then, for any $u_0 \in H^{s+1}(\mathbb{R})$ with $s \geqslant 7/2$, there exists a $T = T(\|u_0\|_{s+1,2}) > 0$ such that the Cauchy problem* (1.1)–(1.2) *has a unique solution $u(t)$ defined in the time interval* $[0, T]$ *and satisfying*

$$u \in C\big([0, T]; H^s(\mathbb{R})\big) \tag{1.7}$$

and

$$\partial_x^{s+1/2} u \in L^\infty\big(\mathbb{R}; L^2\big([0, T]\big)\big). \tag{1.8}$$

Theorem 1.2 *(Case $\beta = 2$). Given any polynomial P as in* (1.3) *with $\beta = 2$, then, for any $u_0 \in H^{s+1}(\mathbb{R}) \cap H^6(\mathbb{R}; x^2\,\mathrm{d}x)$ with $s \geqslant 11 + 1/2$, there exists a $T = T(\|u_0\|_{s+1,2}, \|u_0\|_{6,2,2}) > 0$ such that the Cauchy problem* (1.1)–(1.2) *has a unique solution $u(t)$ defined in the time interval $[0, T]$ and satisfying*

$$u \in C\big([0, T]; H^s(\mathbb{R}) \cap H^6\big(\mathbb{R}; x^2\,\mathrm{d}x\big)\big) \tag{1.9}$$

and

$$\partial_x^{s+1/2} u \in L^\infty\big(\mathbb{R}; L^2\big([0, T]\big)\big). \tag{1.10}$$

2. Local smoothing effects

We will prove the local smoothing effects of Kato type exhibited by the group $\{S(t)\}_{-\infty}^{\infty}$ in this section.

Lemma 2.1 *(Local smoothing effect: homogeneous case). We have the following estimate*

$$\big\|\partial_x^{3/2} S(t)u_0(x)\big\|_{L_x^\infty(\mathbb{R}; L_t^2([0,T]))} \leqslant C\|u_0\|_2, \tag{2.1}$$

and the corresponding dual version

$$\left\|\partial_x^{3/2} \int_0^T S(-\tau) f(\tau, \cdot)\,\mathrm{d}\tau\right\|_2 \leqslant C\|f\|_{L_x^1(\mathbb{R}; L_t^2([0,T]))}. \tag{2.2}$$

Proof. (2.1) and (2.2) can be derived from [12, Theorem 4.1] for which we omit the details. □

More precisely, (2.2) yields, for $t \in [0, T]$, that

$$\left\|\partial_x^{3/2} \int_0^t S(t-\tau) f(\tau, \cdot)\,\mathrm{d}\tau\right\|_2 \leqslant C\|f\|_{L_x^1(\mathbb{R}; L_t^2([0,T]))}. \tag{2.3}$$

Now we turn to consider the inhomogeneous Cauchy problem:

$$i\partial_t u = \partial_x^4 u + F(t, x), \quad t, x \in \mathbb{R}, \tag{2.4}$$

$$u(0, x) = 0, \tag{2.5}$$

with $F \in \mathscr{S}(\mathbb{R} \times \mathbb{R})$. We have the following estimate on the local smoothing effect in this inhomogeneous case.

Lemma 2.2 *(Local smoothing effect: inhomogeneous case). The solution $u(t, x)$ of the Cauchy problem* (2.4)–(2.5) *satisfies*

$$\big\|\partial_x^j u(t, x)\big\|_{L_x^\infty(\mathbb{R}; L_t^2([0,T]))} \leqslant CT^{\frac{3-j}{4}}\big\|F(t, x)\big\|_{L_x^1(\mathbb{R}; L_t^2([0,T]))}, \quad j = 1, 2, 3. \tag{2.6}$$

Proof. We formally take Fourier transform in both variables t and x in Eq. (2.4) and obtain

$$\hat{u} = \frac{\hat{F}(\tau, \xi)}{\tau - \xi^4}.$$

Consequently, we have

$$\partial_x^j u(t,x) = C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it\tau} e^{ix\xi} \frac{\xi^j}{\tau - \xi^4} \hat{F}(\tau, \xi)\, d\xi\, d\tau \quad \text{for } j = 1, 2, 3.$$

By the Plancherel theorem in the time variable, we can get

$$\begin{aligned}
\left\| \partial_x^j u(t,x) \right\|_{L_t^2} &= C \left\| \mathscr{F}_\tau^{-1} \int_{-\infty}^{\infty} e^{ix\xi} \frac{\xi^j}{\tau - \xi^4} \hat{F}(\tau, \xi)\, d\xi \right\|_{L_t^2} \\
&= C \left\| \int_{-\infty}^{\infty} e^{ix\xi} \frac{\xi^j}{\tau - \xi^4} \hat{F}(\tau, \xi)\, d\xi \right\|_{L_\tau^2} \\
&= C \left\| \int_{-\infty}^{\infty} e^{ix\xi} \frac{\xi^j}{\tau - \xi^4} \int_{-\infty}^{\infty} e^{-iy\xi} \hat{F}^{(t)}(\tau, y)\, dy\, d\xi \right\|_{L_\tau^2} \\
&= C \left\| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-y)\xi} \frac{\xi^j}{\tau - \xi^4}\, d\xi\, \hat{F}^{(t)}(\tau, y)\, dy \right\|_{L_\tau^2} \\
&= C \left\| \int_{-\infty}^{\infty} K_j(\tau, x - y) \hat{F}^{(t)}(\tau, y)\, dy \right\|_{L_\tau^2},
\end{aligned} \tag{2.7}$$

where $\hat{F}^{(t)}$ denotes the Fourier transform of F in the time variable,

$$K_j(\tau, l) = \int_{-\infty}^{\infty} e^{il\xi} \frac{\xi^j}{\tau - \xi^4}\, d\xi,$$

and the integral is understood in the principal value sense.

In order to continue the above estimate, we have to estimate $K_j(\tau, l)$ next. Since the proof for the case $\tau < 0$ is similar to the case $\tau > 0$, we only give the proof for the case $\tau > 0$. In fact, by changing the variables, we have, for $\tau > 0$, that

$$\begin{aligned}
K_j(\tau, l) &= \int_{-\infty}^{\infty} e^{il\xi} \frac{\xi^j}{\tau - \xi^4}\, d\xi \\
&= \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{(\tau^{1/4}\eta)^j}{\tau - (\tau^{1/4}\eta)^4} \tau^{1/4}\, d\eta
\end{aligned}$$

$$= \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{\tau^{\frac{1}{4}(j+1)}\eta^j}{\tau(1-\eta^4)}\,\mathrm{d}\eta$$

$$= \tau^{\frac{1}{4}j-\frac{3}{4}} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{\eta^j}{1-\eta^4}\,\mathrm{d}\eta.$$

We distinguish the discussion into three cases according to the value of j.

In the case $j = 1$, we have

$$\begin{aligned}
K_1(\tau,l) &= \tau^{-1/2} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{\eta}{1-\eta^4}\,\mathrm{d}\eta \\
&= \tau^{-1/2} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \left(-\frac{1}{4}\cdot\frac{1}{1+\eta} + \frac{1}{4}\cdot\frac{1}{1-\eta} + \frac{1}{2}\cdot\frac{1}{\eta} - \frac{1}{2}\cdot\frac{1}{(1+\eta^2)\eta} \right) \mathrm{d}\eta \\
&= \tau^{-1/2} \Bigg[-\frac{1}{4} \int_{-\infty}^{\infty} e^{il\tau^{1/4}(\xi-1)} \frac{1}{\xi}\,\mathrm{d}\xi + \frac{1}{4} \int_{-\infty}^{\infty} e^{il\tau^{1/4}(1-\xi)} \frac{1}{\xi}\,\mathrm{d}\xi \\
&\quad + \frac{1}{2} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{1}{\eta}\,\mathrm{d}\eta - \frac{1}{2} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{1}{(1+\eta^2)\eta}\,\mathrm{d}\eta \Bigg] \\
&= \tau^{-1/2} \Bigg[-\frac{1}{4} i \left(e^{-il\tau^{1/4}} + e^{il\tau^{1/4}} - 2 \right) \operatorname{sgn}\left(l\tau^{1/4}\right) \\
&\quad - \frac{1}{2} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{1}{(1+\eta^2)\eta}\,\mathrm{d}\eta \Bigg],
\end{aligned}$$

where we used the fact that the Fourier transform of $1/x$ (i.e., the kernel of the Hilbert transform) is equal to $-i\,\operatorname{sgn}(\xi)$.

For the estimate of the second term in the right-hand side of the above, we introduce an auxiliary function. Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subseteq [-2,2]$, $\varphi \equiv 1$ in $[-1,1]$ and $0 \leqslant \varphi \leqslant 1$, we have

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} e^{ix\eta} \frac{\varphi(\eta)}{(1+\eta^2)\eta}\,\mathrm{d}\eta \right| \\
&\quad = \left| \mathscr{F}_\eta^{-1}\left(\frac{1}{\eta}\cdot\frac{\varphi(\eta)}{1+\eta^2} \right) \right| = \left| \mathscr{F}_\eta^{-1}\left(\frac{1}{\eta}\right) * \mathscr{F}_\eta^{-1}\left(\frac{\varphi(\eta)}{1+\eta^2} \right) \right| \\
&\quad = \left| -i\,\operatorname{sgn}(x) * \mathscr{F}_\eta^{-1}\left(\frac{\varphi(\eta)}{1+\eta^2} \right) \right| = \left| -i \int_{-\infty}^{\infty} \operatorname{sgn}(x-y) \mathscr{F}_\eta^{-1}\left(\frac{\varphi(\eta)}{1+\eta^2} \right)(y)\,\mathrm{d}y \right|
\end{aligned}$$

$$\leqslant \int_{-\infty}^{\infty} \left| \mathscr{F}_\eta^{-1}\left(\frac{\varphi(\eta)}{1+\eta^2}\right)(y)\,\mathrm{d}y \right| \leqslant C$$

and

$$\left| \int_{-\infty}^{\infty} e^{ix\eta} \frac{1-\varphi(\eta)}{(1+\eta^2)\eta}\,\mathrm{d}\eta \right| \leqslant \int_{|\eta|\geqslant 1} \frac{1}{1+\eta^2}\,\mathrm{d}\eta \leqslant C.$$

Thus, we can get

$$\sup_{\tau,l} \left| \tau^{1/2} K_1(\tau,l) \right| \leqslant C. \tag{2.8}$$

Next, we consider the case $j=2$. We have as above

$$\begin{aligned}
K_2(\tau,l) &= \tau^{-1/4} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{\eta^2}{1-\eta^4}\,\mathrm{d}\eta \\
&= \tau^{-1/4} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \left(\frac{1}{4}\cdot\frac{1}{1+\eta} + \frac{1}{4}\cdot\frac{1}{1-\eta} - \frac{1}{2}\cdot\frac{1}{1+\eta^2}\right)\mathrm{d}\eta \\
&= \tau^{-1/4} \left[\frac{1}{4} \int_{-\infty}^{\infty} e^{il\tau^{1/4}(\xi-1)} \frac{1}{\xi}\,\mathrm{d}\xi + \frac{1}{4} \int_{-\infty}^{\infty} e^{il\tau^{1/4}(1-\xi)} \frac{1}{\xi}\,\mathrm{d}\xi \right. \\
&\qquad \left. - \frac{1}{2} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{1}{1+\eta^2}\,\mathrm{d}\eta \right] \\
&= \tau^{-1/4} \left[\frac{1}{4} i \left(e^{-il\tau^{1/4}} - e^{il\tau^{1/4}} \right) \operatorname{sgn}\left(l\tau^{1/4}\right) - \frac{1}{2} \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{1}{1+\eta^2}\,\mathrm{d}\eta \right]
\end{aligned}$$

and consequently

$$\left| K_2(\tau,l) \right| \leqslant \tau^{-1/4} \left[\frac{1}{2} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+\eta^2}\,\mathrm{d}\eta \right] = \frac{1+\pi}{2} \tau^{-1/4}.$$

Thus, we obtain

$$\sup_{\tau,l} \left| \tau^{1/4} K_2(\tau,l) \right| \leqslant C. \tag{2.9}$$

For the case $j=3$, we have

$$K_3(\tau,l) = \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta} \frac{\eta^3}{1-\eta^4}\,\mathrm{d}\eta$$

$$= \int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta}\left(-\frac{1}{4}\cdot\frac{1}{1+\eta}+\frac{1}{4}\cdot\frac{1}{1-\eta}-\frac{1}{2}\cdot\frac{1}{\eta}+\frac{1}{2}\cdot\frac{1}{(1+\eta^2)\eta}\right)\mathrm{d}\eta$$

$$= \left[-\frac{1}{4}i\left(e^{-il\tau^{1/4}}+e^{il\tau^{1/4}}+2\right)\operatorname{sgn}\left(l\tau^{1/4}\right)+\frac{1}{2}\int_{-\infty}^{\infty} e^{il\tau^{1/4}\eta}\frac{1}{(1+\eta^2)\eta}\,\mathrm{d}\eta\right].$$

Similar to the case $j=1$, we can get

$$\sup_{\tau,l}\left|K_3(\tau,l)\right| \leqslant C. \tag{2.10}$$

Combining (2.8), (2.9), and (2.10), it yields

$$\sup_{\tau,l}|\tau|^{\frac{3-j}{4}}\left|K_j(\tau,l)\right| \leqslant C. \tag{2.11}$$

Now we turn to the estimate of $\partial_x^j u(t,x)$. Due to (2.7), the above estimates, the Young inequality, the Sobolev embedding theorem and the Hölder inequality, we obtain, for $t \in [0,T]$, that

$$\begin{aligned}
\sup_x\left\|\partial_x^j u(t,x)\right\|_{L_t^2} &\leqslant C\int_{-\infty}^{\infty}\sup_x\left\|K_j(\tau,x-y)\hat{F}^{(t)}(\tau,y)\right\|_{L_\tau^2}\mathrm{d}y \\
&\leqslant C\int_{-\infty}^{\infty}\sup_{\tau,x}\left(|\tau|^{\frac{3-j}{4}}\left|K_j(\tau,x-y)\right|\right)\left\||\tau|^{-\frac{3-j}{4}}\hat{F}^{(t)}(\tau,y)\right\|_{L_\tau^2}\mathrm{d}y \\
&\leqslant C\sup_{\tau,x}|\tau|^{\frac{3-j}{4}}\left|K_j(\tau,x)\right|\|F\|_{L_x^1\left(\dot{H}_t^{-\frac{3-j}{4}}\right)} \leqslant C\|F\|_{L_x^1\left(L_t^{\frac{4}{5-j}}\right)} \\
&\leqslant CT^{\frac{3-j}{4}}\|F\|_{L_x^1L_t^2}.
\end{aligned}$$

The desired result follows then. □

In general, the solution $u(t,x)$ of (2.4) may not vanish at $t=0$. However, by using the Parseval identity, we are able to show

$$\begin{aligned}
u(0,x) &= u(t,x)|_{t=0} \\
&= C\int_{-\infty}^{\infty} e^{ix\xi}\int_{-\infty}^{\infty}\frac{1}{\tau-\xi^4}\hat{F}(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&= C\int_{-\infty}^{\infty} e^{ix\xi}\int_{-\infty}^{\infty}\frac{1}{\tau-\xi^4}\int_{-\infty}^{\infty} e^{-i\tau s}\hat{F}^{(x)}(s,\xi)\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&= C\int_{-\infty}^{\infty} e^{ix\xi}\int_{-\infty}^{\infty}\hat{F}^{(x)}(s,\xi)\int_{-\infty}^{\infty}\frac{e^{-i\tau s}}{\tau-\xi^4}\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}\xi
\end{aligned}$$

$$= C \int_{-\infty}^{\infty} e^{ix\xi} \int_{-\infty}^{\infty} \hat{F}^{(x)}(s,\xi) \left(\int_{-\infty}^{\infty} \frac{e^{-iys}}{y} \,\mathrm{d}y \right) e^{-is\xi^4} \,\mathrm{d}s \,\mathrm{d}\xi$$

$$= C \int_{-\infty}^{\infty} e^{ix\xi} \int_{-\infty}^{\infty} \hat{F}^{(x)}(s,\xi) \operatorname{sgn}(s) e^{-is\xi^4} \,\mathrm{d}s \,\mathrm{d}\xi$$

$$= C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix\xi} e^{-is\xi^4} \operatorname{sgn}(s) \hat{F}^{(x)}(s,\xi) \,\mathrm{d}\xi \,\mathrm{d}s$$

$$= C \int_{-\infty}^{\infty} S(s) \operatorname{sgn}(s) \hat{F}^{(x)}(s,x) \,\mathrm{d}s,$$

which, combined with (2.2), implies $\partial_x^{3/2} u(0,x) \in L^2(\mathbb{R})$.

Thus, by (2.1), the function $w(t)$,

$$w(t) := u(t) - S(t)u_0 = -i \int_0^t S(t-\tau) F(\tau) \,\mathrm{d}\tau,$$

is the solution of (2.4)–(2.5) and satisfies the estimate (2.6). Hence, for $j = 1, 2, 3$,

$$\sup_x \left\| \partial_x^j \int_0^t S(t-\tau) F(\tau, x) \,\mathrm{d}\tau \right\|_{L_t^2} = \sup_x \left\| \partial_x^j w(t,x) \right\|_{L_t^2} \leqslant C T^{\frac{3-j}{4}} \|F\|_{L_x^1 L_t^2}. \tag{2.12}$$

3. Estimates for the maximal function

We start by stating an L^2-continuity result for the maximal function $\sup_{[0,T]} |S(t) \cdot |$.

Lemma 3.1. *For any $s > 1$ and any $\rho > 1/4$, it holds*

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t| \leqslant T} \sup_{j \leqslant x < j+1} \left| S(t) u_0(x) \right|^2 \right)^{1/2} \leqslant C(1+T)^{\rho} \|u_0\|_{s,2}. \tag{3.1}$$

Proof. This is a special case of [13, Corollary 2.8]. For the convenience of readers, we give a direct and simple proof here. Denote $t = Tt'$ with $|t'| \leqslant 1$. We have

$$S(t)u_0(x) = S(Tt')u_0(x) = C \int_{-\infty}^{\infty} e^{i(x\xi - \xi^4 T t')} \,\mathrm{d}\xi * u_0(x)$$

$$= C T^{-1/4} \int_{-\infty}^{\infty} e^{i(\frac{x}{T^{1/4}}\eta - \eta^4 t')} \,\mathrm{d}\eta * u_0(x)$$

$$= CT^{-1/4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\frac{x-y}{T^{1/4}}\eta - \eta^4 t')} \, \mathrm{d}\eta \, u_0(y) \, \mathrm{d}y$$

$$= C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i((\frac{x}{T^{1/4}} - z)\eta - \eta^4 t')} \, \mathrm{d}\eta u_0(T^{1/4} z) \, \mathrm{d}z$$

$$= \big(S(t')v\big)\big(T^{-1/4}x\big),$$

where $v(x) = u_0(T^{1/4}x)$.

As the same way as the proofs in [13, Theorem 2.7], we can check that

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t| \leqslant 1} \sup_{j \leqslant x < j+1} \left| S(t) u_0(x) \right|^2 \right)^{1/2} \leqslant C \|u_0\|_{s,2}$$

holds for $s > 1$.

Thus, we obtain

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t| \leqslant T} \sup_{j \leqslant x < j+1} \left| S(t) u_0(x) \right|^2 \right)^{1/2}$$

$$= \left(\sum_{j=-\infty}^{\infty} \sup_{|t'| \leqslant 1} \sup_{j \leqslant T^{1/4} y < j+1} \left| S(t') v(y) \right|^2 \right)^{1/2}$$

$$\leqslant C \left(T^{1/4} \sum_{k=-\infty}^{\infty} \sup_{|t'| \leqslant 1} \sup_{k \leqslant x < k+1} \left| S(t') v(x) \right|^2 \right)^{1/2}$$

$$\leqslant CT^{1/8} \|v\|_{s,2} = CT^{1/8} \left\| u_0(T^{1/4} \cdot) \right\|_{s,2}$$

$$\leqslant C(1+T)^{s/4} \|u_0\|_{s,2}.$$

For the case $\rho \geqslant s/4$, we have the desired result since

$$(1+T)^{s/4} \leqslant (1+T)^{\rho}.$$

For the case $1/4 < \rho < s/4$, let $s_0 = 4\rho$. Then, following the above process and using the Sobolev embedding theorem, we obtain

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t| \leqslant T} \sup_{j \leqslant x < j+1} \left| S(t) u_0(x) \right|^2 \right)^{1/2} \leqslant C(1+T)^{\rho} \|u_0\|_{s_0,2} \leqslant C(1+T)^{\rho} \|u_0\|_{s,2}.$$

Thus, the proof of Lemma 3.1 is completed. □

It is clear that (3.1) yields

$$\left(\int_{-\infty}^{\infty} \sup_{[0,T]} \left| S(t) u_0(x) \right|^2 \mathrm{d}x \right)^{1/2} \leqslant C(1+T)^{\rho} \|u_0\|_{s,2}. \tag{3.2}$$

To estimate the maximal function $\sup_{[0,T]}|S(t)u_0(x)|$ in the L^1- and l^1-norms, we shall use the following weighted inequality.

Lemma 3.2. *We have the estimate*

$$\int_{-\infty}^{\infty}\sup_{[0,T]}\big|S(t)u_0(x)\big|\,dx \leqslant C(1+T)^2\big(\|u_0\|_{7,2}+\|u_0\|_{4,2,2}\big), \tag{3.3}$$

where $\|\cdot\|_{l,2,j}$ *is defined as in* (1.6)*.*

Proof. We first derive an estimate which will be used in the next derivation. Taking $t_0 \in [0,T]$ such that

$$\big|f(t_0)\big| \leqslant \frac{\int_0^T |f(t)|\,dt}{T},$$

we have for any t

$$f(t) = f(t_0) + f(t) - f(t_0) = f(t_0) + \int_{t_0}^{t} f'(s)\,ds.$$

Thus, we can get

$$\big|f(t)\big| \leqslant \frac{1}{T}\int_0^T\big|f(t)\big|\,dt + \int_{t_0}^{t}\big|f'(s)\big|\,ds \leqslant \frac{1}{T}\int_0^T\big|f(t)\big|\,dt + \int_0^T\big|\partial_t f(t)\big|\,dt,$$

namely,

$$\sup_{t\in[0,T]}\big|f(t)\big| \leqslant \frac{1}{T}\int_0^T\big|f(t)\big|\,dt + \int_0^T\big|\partial_t f(t)\big|\,dt. \tag{3.4}$$

We turn to prove (3.3). Noticing that

$$\int_{-\infty}^{\infty}\big|f(x)\big|\,dx \leqslant C\|f\|_2 + C\|xf\|_2$$

and

$$xS(t)u_0 = S(t)(xu_0) + 4itS(t)\big(\partial_x^3 u_0\big), \tag{3.5}$$

we have, from (3.4) and the Fubini theorem, that

$$\int_{-\infty}^{\infty}\sup_{[0,T]}\big|S(t)u_0(x)\big|\,dx$$

$$\leqslant \frac{1}{T}\int_{-\infty}^{\infty}\int_0^T \left|S(t)u_0(x)\right| \mathrm{d}t\,\mathrm{d}x + \int_{-\infty}^{\infty}\int_0^T \left|\partial_t S(t)u_0(x)\right| \mathrm{d}t\,\mathrm{d}x$$
$$= \frac{1}{T}\int_0^T\int_{-\infty}^{\infty} \left|S(t)u_0(x)\right| \mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{-\infty}^{\infty} \left|S(t)\partial_x^4 u_0(x)\right| \mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{C}{T}\int_0^T \left(\left\|S(t)u_0\right\|_2 + \left\|xS(t)u_0\right\|_2\right) \mathrm{d}t + C\int_0^T \left(\left\|S(t)\partial_x^4 u_0\right\|_2 + \left\|xS(t)\partial_x^4 u_0\right\|_2\right) \mathrm{d}t$$
$$\leqslant \frac{C}{T}\int_0^T \left(\|u_0\|_2 + \left\|S(t)(xu_0)\right\|_2 + 4t\left\|S(t)\left(\partial_x^3 u_0\right)\right\|_2\right) \mathrm{d}t$$
$$+ C\int_0^T \left(\left\|\partial_x^4 u_0\right\|_2 + \left\|S(t)\left(x\partial_x^4 u_0\right)\right\|_2 + 4t\left\|S(t)\left(\partial_x^7 u_0\right)\right\|_2\right) \mathrm{d}t$$
$$\leqslant C\left(\|u_0\|_2 + \|xu_0\|_2 + T\left\|\partial_x^3 u_0\right\|_2 + T\left\|\partial_x^4 u_0\right\|_2 + T\left\|x\partial_x^4 u_0\right\|_2 + T^2\left\|\partial_x^7 u_0\right\|_2\right)$$
$$\leqslant C(1+T)^2\left(\|u_0\|_{7,2} + \|u_0\|_{4,2,2}\right).$$

Thus, the desired result is obtained. □

4. Wellposedness

We will give the proofs of the main theorems in this section.

Proof of Theorem 1.1. Similar to the proof in [14, Theorem 4.1], we shall only consider the case $s = 7/2$. The general case follows by combining this result with the fact that the highest derivatives involved in that proof always appear linearly and with some commutator estimates (see [11]) for the cases where $s \neq k + 1/2$, $k \in \mathbb{Z}^+$.

For $u_0 \in H^{9/2}(\mathbb{R})$, we denote by $u = \mathscr{T}(v) = \mathscr{T}_{u_0}(v)$ the solution of the linear inhomogeneous Cauchy problem

$$i\partial_t u = \partial_x^4 u + P\left(\left(\partial_x^\alpha v\right)_{|\alpha|\leqslant 2}, \left(\partial_x^\alpha \bar{v}\right)_{|\alpha|\leqslant 2}\right), \quad t, x \in \mathbb{R}, \tag{4.1}$$
$$u(0,x) = u_0(x). \tag{4.2}$$

In order to construct $\mathscr{T}$ as a contraction mapping in some space, we use the integral equation

$$u(t) = \mathscr{T}(v)(t) = S(t)u_0 - i\int_0^t S(t-\tau)P\left(\left(\partial_x^\alpha v(\tau)\right)_{|\alpha|\leqslant 2}, \left(\partial_x^\alpha \bar{v}(\tau)\right)_{|\alpha|\leqslant 2}\right) \mathrm{d}\tau. \tag{4.3}$$

We introduce the following work space:

$$Z_T^D = \left\{ w:[0,T]\times\mathbb{R}\to\mathbb{C}\colon\ \left\|\partial_x^4 w\right\|_{L_x^\infty L_t^2} \leqslant T^\delta;\ \sup_{t\in[0,T]}\left\|w(t)\right\|_{\frac{7}{2},2}\leqslant D; \right.$$
$$\left. (1+T)^{-\rho}\left(\int_{-\infty}^{\infty}\sup_t\left(|w|^2+|w_x|^2+|w_{xx}|^2\right)\mathrm{d}x\right)^{1/2}\leqslant D\right\},$$

where $\delta < 1/4$ is a constant.

We notice that

$$\partial_x^2 P\left(\left(\partial_x^\alpha v\right)_{|\alpha|\leqslant 2}, \left(\partial_x^\alpha \bar v\right)_{|\alpha|\leqslant 2}\right) = \partial_x^4 v R_1(\cdot) + \partial_x^4 \bar v R_2(\cdot) + R_0(\cdot), \tag{4.4}$$

where

$$R_j(\cdot) = R_j\left(\left(\partial_x^\alpha v\right)_{|\alpha|\leqslant 2}, \left(\partial_x^\alpha \bar v\right)_{|\alpha|\leqslant 2}\right) \quad \text{for } j=1,2,$$

and

$$R_0(\cdot) = R_0\left(\left(\partial_x^\alpha v\right)_{|\alpha|\leqslant 3}, \left(\partial_x^\alpha \bar v\right)_{|\alpha|\leqslant 3}\right).$$

Thus, from the integral equation (4.3), the Hölder inequality, the Moser inequality, (4.4) and (2.12), we can get, for $v\in Z_T^D$, that

$$\begin{aligned}
&\left\|\partial_x^4 u\right\|_{L_x^\infty L_t^2} \\
&\quad = \sup_x\left(\int_0^T \left|\partial_x^4 u(t,x)\right|^2 \mathrm{d}t\right)^{1/2} \\
&\quad \leqslant \sup_x\left(\int_0^T \left|\partial_x^4 S(t)u_0\right|^2 \mathrm{d}t\right)^{1/2} + \sup_x\left(\int_0^T\left|\partial_x^4\int_0^t S(t-\tau)P(\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}t\right)^{1/2} \\
&\quad \leqslant T^{1/2}\sup_{x,t}\left|\partial_x^4 S(t)u_0\right| + \sup_x\left(\int_0^T\left|\partial_x^2\int_0^t S(t-\tau)\partial_x^2 P(\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}t\right)^{1/2} \\
&\quad \leqslant T^{1/2}\|u_0\|_{\frac{9}{2},2} + C\sup_x\left(\int_0^T\left|\partial_x^2\int_0^t S(t-\tau)\left(\partial_x^4 v R_1 + \partial_x^4\bar v R_2 + R_0\right)(\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}t\right)^{1/2} \\
&\quad \leqslant T^{1/2}\|u_0\|_{\frac{9}{2},2} + CT^{1/4}\left\|\partial_x^4 v R_1\right\|_{L_x^1 L_t^2} + CT^{1/4}\left\|\partial_x^4 \bar v R_2\right\|_{L_x^1 L_t^2} \\
&\qquad + C\int_0^T\left\|\partial_x^{1/2} R_0(t)\right\|_2\mathrm{d}t \\
&\quad \leqslant T^{1/2}\|u_0\|_{\frac{9}{2},2} + CT^{1/4}\left\|\partial_x^4 v\right\|_{L_x^\infty L_t^2}\sum_{j=1,2}\|R_j\|_{L_x^1 L_t^\infty} + CT\sup_t\left\|\partial_x^{1/2}R_0(t)\right\|_2.
\end{aligned}$$

By using the commutator estimates [11] about the last term and the Sobolev embedding theorem, we may bound this by

$$
\begin{aligned}
&T^{1/2}\|u_0\|_{\frac{9}{2},2} + CT^{1/4}\left\|\partial_x^4 v\right\|_{L_x^\infty L_t^2}\left\||v|^2+|v_x|^2+|v_{xx}|^2\right\|_{L_x^1 L_t^\infty}\\
&\quad\times\left(1+\left\||v|+|v_x|+|v_{xx}|\right\|_{L_x^\infty L_t^\infty}^{\gamma-3}\right) + CT\sup_t\left(\|v\|_{\frac{7}{2},2}^3\left(1+\|v\|_{\frac{7}{2},2}^{\gamma-3}\right)\right)\\
&\leqslant T^{1/2}\|u_0\|_{\frac{9}{2},2} + CT^{1/4}\left\|\partial_x^4 v\right\|_{L_x^\infty L_t^2}\left\||v|^2+|v_x|^2+|v_{xx}|^2\right\|_{L_x^1 L_t^\infty}\\
&\quad\times\left(1+\sup_t\|v\|_{\frac{7}{2},2}^{\gamma-3}\right) + CT\sup_t\left(\|v\|_{\frac{7}{2},2}^3\left(1+\|v\|_{\frac{7}{2},2}^{\gamma-3}\right)\right)\\
&\leqslant T^{1/2}\|u_0\|_{\frac{9}{2},2} + CT^{1/4}T^\delta D^2\left(1+D^{\gamma-3}\right) + CTD^3\left(1+D^{\gamma-3}\right)\\
&\leqslant T^\delta,
\end{aligned}
\tag{4.5}
$$

where we take T so small that

$$
T^{1/2-\delta}\|u_0\|_{\frac{9}{2},2} + CT^{1/4}D^2\left(1+D^{\gamma-3}\right) + CT^{1-\delta}D^3\left(1+D^{\gamma-3}\right) \leqslant 1 \tag{4.6}
$$

in the last step.

By the properties of the Sobolev spaces (cf. [2]), the integral equation (4.3), the group properties (2.3) and (4.5), we have

$$
\begin{aligned}
&\sup_t\left\|u(t)\right\|_{\frac{7}{2},2}\\
&\quad\leqslant \sup_t\left\|u(t)\right\|_2 + \sup_t\left\|\partial_x^{7/2}u(t)\right\|_2\\
&\quad\leqslant \|u_0\|_2 + \sup_t\left\|\int_0^t S(t-\tau)P(\tau)\,\mathrm{d}\tau\right\|_2 + \left\|\partial_x^{7/2}u_0\right\|_2\\
&\qquad + \sup_t\left\|\partial_x^{7/2}\int_0^t S(t-\tau)P(\tau)\,\mathrm{d}\tau\right\|_2\\
&\quad\leqslant \|u_0\|_{\frac{7}{2},2} + \sup_t\int_0^t\left\|S(t-\tau)P(\tau)\right\|_2\mathrm{d}\tau + \sup_t\left\|\partial_x^{3/2}\int_0^t S(t-\tau)\partial_x^2 P(\tau)\,\mathrm{d}\tau\right\|_2\\
&\quad\leqslant \|u_0\|_{\frac{7}{2},2} + T\sup_t\left\|P(t)\right\|_2 + \left\|\partial_x^2 P\right\|_{L_x^1 L_t^2}\\
&\quad\leqslant C\|u_0\|_{\frac{7}{2},2} + CT\sup_t\left(\|v\|_{\frac{7}{2},2}^3\left(1+\|v\|_{\frac{7}{2},2}^{\gamma-3}\right)\right)\\
&\qquad + C\left\|\partial_x^4 v\right\|_{L_x^\infty L_t^2}\sum_{j=1,2}\|R_j\|_{L_x^1 L_t^\infty} + CT^{1/2}\sup_t\left\|R_0(t)\right\|_2\\
&\quad\leqslant C\|u_0\|_{\frac{7}{2},2} + CT\sup_t\left(\|v\|_{\frac{7}{2},2}^3\left(1+\|v\|_{\frac{7}{2},2}^{\gamma-3}\right)\right)\\
&\qquad + C\left\|\partial_x^4 v\right\|_{L_x^\infty L_t^2}\left\||v|^2+|v_x|^2+|v_{xx}|^2\right\|_{L_x^1 L_t^\infty}\cdot\left(1+\sup_t\|v\|_{\frac{7}{2},2}^{\gamma-3}\right)\\
&\qquad + CT^{1/2}\sup_t\left(\|v\|_{\frac{7}{2},2}^3\left(1+\|v\|_{\frac{7}{2},2}^{\gamma-3}\right)\right)\\
&\quad\leqslant C\|u_0\|_{\frac{7}{2},2} + CTD^3\left(1+D^{\gamma-3}\right) + CT^\delta D^2\left(1+D^{\gamma-3}\right) + CT^{1/2}D^3\left(1+D^{\gamma-3}\right)
\end{aligned}
$$

$$\leqslant D, \tag{4.7}$$

where in the last step, we have chosen

$$D = 2C\|u_0\|_{\frac{9}{2},2}, \tag{4.8}$$

and T small enough such that

$$CTD^2\big(1+D^{\gamma-3}\big) + CT^{\delta}D\big(1+D^{\gamma-3}\big) + CT^{1/2}D^2\big(1+D^{\gamma-3}\big) \leqslant \frac{1}{2}. \tag{4.9}$$

Similar to the derivation of (4.5) and (4.7), we obtain, from (3.2), that

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} \sup_t\big(|u|^2+|u_x|^2+|u_{xx}|^2\big)\,\mathrm{d}x\right)^{1/2} \\
&\quad\leqslant \left(\int_{-\infty}^{\infty} \sup_t\big(\big|S(t)u_0\big|^2+\big|S(t)\partial_x u_0\big|^2+\big|S(t)\partial_x^2 u_0\big|^2\big)\,\mathrm{d}x\right)^{1/2} \\
&\qquad + \left(\int_{-\infty}^{\infty} \sup_t\left[\left|\int_0^t S(t-\tau)P(\tau)\,\mathrm{d}\tau\right|^2 + \left|\partial_x\int_0^t S(t-\tau)P(\tau)\,\mathrm{d}\tau\right|^2\right.\right. \\
&\qquad \left.\left. + \left|\partial_x^2\int_0^t S(t-\tau)P(\tau)\,\mathrm{d}\tau\right|^2\right]\mathrm{d}x\right)^{1/2} \\
&\quad\leqslant C(1+T)^{\rho}\|u_0\|_{\frac{7}{2},2} + C(1+T)^{\rho}\sum_{j=0}^{2}\sup_t\left\|\int_0^t S(-\tau)\partial_x^j P(\tau)\,\mathrm{d}\tau\right\|_{\frac{3}{2},2} \\
&\quad\leqslant C(1+T)^{\rho}\Big\{\|u_0\|_{\frac{7}{2},2} + C\big\|\partial_x^4 v\big\|_{L_x^\infty L_t^2}\big\||v|^2+|v_x|^2+|v_{xx}|^2\big\|_{L_x^1 L_t^\infty} \\
&\qquad \times\Big(1+\sup_t\|v\|_{\frac{7}{2},2}^{\gamma-3}\Big) + C\big(T^{1/2}+T\big)\sup_t\big(\|v\|_{\frac{7}{2},2}^3\big(1+\|v\|_{\frac{7}{2},2}^{\gamma-3}\big)\big)\Big\} \\
&\quad\leqslant (1+T)^{\rho}D,
\end{aligned} \tag{4.10}$$

where $\rho > 1/4$ and (4.9) has been used.

Therefore, choosing a D as in (4.8) and then taking a T sufficiently small such that both (4.6) and (4.9) hold, we obtain that the mapping

$$\mathscr{T} = \mathscr{T}_{u_0} : Z_T^D \to Z_T^D$$

is well defined.

For convenience, we denote

$$\begin{aligned}
\Lambda_T(w) = \max\bigg\{& DT^{-\delta}\big\|\partial_x^4 w\big\|_{L_x^\infty L_t^2};\ \sup_{t\in[0,T]}\big\|w(t)\big\|_{\frac{7}{2},2}; \\
&(1+T)^{-\rho}\left(\int_{-\infty}^{\infty}\sup_t\big(|w|^2+|w_x|^2+|w_{xx}|^2\big)\,\mathrm{d}x\right)^{1/2}\bigg\}.
\end{aligned} \tag{4.11}$$

To show that $\mathscr{T}$ is a contraction mapping, we apply the estimates obtained in (4.5), (4.7) and (4.10) to the following integral equation:

$$\mathscr{T}v(t) - \mathscr{T}w(t) = \int_0^t S(t-\tau)\big[P\big((\partial_x^\alpha v)_{|\alpha|\leqslant 2}, (\partial_x^\alpha \bar{v})_{|\alpha|\leqslant 2}\big) - P\big((\partial_x^\alpha w)_{|\alpha|\leqslant 2}, (\partial_x^\alpha \bar{w})_{|\alpha|\leqslant 2}\big)\big](\tau)\,\mathrm{d}\tau,$$

and obtain, for $v, w \in Z_T^D$, that

$$\begin{aligned}\Lambda_T(\mathscr{T}v - \mathscr{T}w) &\leqslant CT^\delta \Lambda_T(v-w)\cdot\big[\Lambda_T^2(v) + \Lambda_T^{\gamma-1}(v) + \Lambda_T^2(w) + \Lambda_T^{\gamma-1}(w)\big]\\ &\leqslant 2CT^\delta\big(D^2 + D^{\gamma-1}\big)\Lambda_T(v-w),\end{aligned} \tag{4.12}$$

where the constant C depends only on the form of $P(\cdot)$ and the linear estimates (2.1), (2.3), (2.6) and (3.1).

Thus, we can choose $0 < T \ll 1$ satisfying (4.6), (4.9) and

$$2CT^\delta\big(D^2 + D^{\gamma-1}\big) \leqslant 1/2. \tag{4.13}$$

Therefore, for those T, satisfying (4.6), (4.9) and (4.13), the mapping $\mathscr{T}_{u_0}$ is a contraction mapping in Z_T^D. Consequently, by the Banach contraction mapping principle, there exists a unique function $u \in Z_T^D$ such that $\mathscr{T}_{u_0}u = u$ which solves the Cauchy problem.

By the method given in [14, Theorem 4.1], we can prove the persistence property of $u(t)$ in $H^{7/2}$, i.e.,

$$u(t,x) \in C\big([0,T]; H^{7/2}\big),$$

the uniqueness and the continuous dependence on the initial data of solution. For simplicity, we omit the rest of the proof. □

Proof of Theorem 1.2. For simplicity, we assume

$$P\big((\partial_x^\alpha u)_{|\alpha|\leqslant 2}, (\partial_x^\alpha \bar{u})_{|\alpha|\leqslant 2}\big) = \big(\partial_x^2 u\big)^2.$$

It will be clear, from the argument presented below, that this does not represent any loss of generality. And as in the proof of Theorem 1.1, we consider the case $s = 11 + 1/2$.

For $u_0 \in H^{25/2}(\mathbb{R}) \cap H^3(\mathbb{R}; x^2\,\mathrm{d}x)$, we denote by $u = \mathscr{T}(v) = \mathscr{T}_{u_0}(v)$ the solution of the linear inhomogeneous Cauchy problem

$$i\partial_t u = \partial_x^4 u + \big(\partial_x^2 v\big)^2, \quad t, x \in \mathbb{R}, \tag{4.14}$$

$$u(0,x) = u_0(x). \tag{4.15}$$

We will consider the integral equation

$$u(t) = \mathscr{T}(v)(t) = S(t)u_0 - i\int_0^t S(t-\tau)\big(\partial_x^2 v\big)^2\,\mathrm{d}\tau, \tag{4.16}$$

in the following work space:

$$Z_T^E = \Big\{ v:[0,T]\times\mathbb{R}\to\mathbb{C}\colon\ \sup_t \big\|v(t)\big\|_{\frac{23}{2},2} \leqslant E;\ \sup_t\big\|v(t)\big\|_{6,2,2}\leqslant E;\\ \big\|\partial_x^{12}v\big\|_{L_x^\infty L_t^2}\leqslant T^\delta;\ (1+T)^{-2}\big\|\partial_x^2 v(t,x)\big\|_{L_x^1L_t^\infty}\leqslant E\Big\},$$

where $0<\delta<1/4$ is a constant and the norm is defined as

$$\big\|v(t)\big\|_{Z_T^E} = \max\Big\{ \sup_t\big\|v(t)\big\|_{\frac{23}{2},2};\ \sup_t\big\|v(t)\big\|_{6,2,2};\ ET^{-\delta}\big\|\partial_x^{12}v\big\|_{L_x^\infty L_t^2};\\ (1+T)^{-2}\big\|\partial_x^2 v(t,x)\big\|_{L_x^1L_t^\infty}\Big\}.$$

We notice that

$$\partial_x^{10}\big((\partial_x^2 v)^2\big) = 2\partial_x^2 v\partial_x^{12}v + R_0\big((\partial_x^\alpha v)_{|\alpha|\leqslant 11}\big).$$

From the integral equation (4.3), (2.1), the Sobolev embedding theorem, (4.4), (2.12), the Moser inequality and the commutator estimates[11], we can get, as in (4.5), that

$$\begin{aligned}
&\big\|\partial_x^{12}u\big\|_{L_x^\infty L_t^2}\\
&\quad= \sup_x\left(\int_0^T\big|\partial_x^{12}u(t,x)\big|^2\,\mathrm{d}t\right)^{1/2}\\
&\quad\leqslant \sup_x\left(\int_0^T\big|\partial_x^{12}S(t)u_0\big|^2\,\mathrm{d}t\right)^{1/2} + \sup_x\left(\int_0^T\left|\partial_x^{12}\int_0^t S(t-\tau)\big(\partial_x^2v(\tau)\big)^2\,\mathrm{d}\tau\right|^2\mathrm{d}t\right)^{1/2}\\
&\quad\leqslant T^{1/2}\|u_0\|_{\frac{25}{2},2} + CT^{1/4}\big\|\partial_x^{12}v\partial_x^2v\big\|_{L_x^1L_t^2} + C\int_0^T\big\|\partial_x^{1/2}R_0(t)\big\|_2\,\mathrm{d}t\\
&\quad\leqslant T^{1/2}\|u_0\|_{\frac{25}{2},2} + CT^{1/4}\big\|\partial_x^{12}v\big\|_{L_x^\infty L_t^2}\int_{-\infty}^{\infty}\sup_t\big|\partial_x^2v(x,t)\big|\,\mathrm{d}x + CT\sup_t\big\|v(t)\big\|_{\frac{23}{2},2}\\
&\quad\leqslant T^{1/2}\|u_0\|_{\frac{25}{2},2} + CT^{1/4}T^\delta(1+T)^4E^2 + CTE\\
&\quad\leqslant T^\delta,
\end{aligned}\tag{4.17}$$

if we take T sufficiently small such that

$$T^{1/2-\delta}\|u_0\|_{\frac{25}{2},2} + CT^{1/4}(1+T)^4E^2 + CT^{1-\delta}E\leqslant 1.\tag{4.18}$$

We can rewrite (4.16) as

$$u(t) = S(t)\left(u_0 - i\int_0^t S(-\tau)\big(\partial_x^2 v\big)^2\,\mathrm{d}\tau\right).\tag{4.19}$$

From (3.3) and (3.5), we have

$$
\begin{aligned}
&(1+T)^{-2}\int_{-\infty}^{\infty}\sup_t\left|\partial_x^2 u(x,t)\right|\mathrm{d}x\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+\int_{-\infty}^{\infty}\sup_t\left|\partial_x^2 S(t)\int_0^t S(-\tau)\big(\partial_x^2 v\big)^2\,\mathrm{d}\tau\right|\mathrm{d}x\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+C\sup_t\left\|\int_0^t S(-\tau)\big(\partial_x^2 v\big)^2\,\mathrm{d}\tau\right\|_{9,2}\\
&\qquad+\sup_t\left\|\partial_x^2\int_0^t S(-\tau)\big(\partial_x^2 v\big)^2\,\mathrm{d}\tau\right\|_{4,2,2}\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+CT\|v\|_{11,2}^2+\int_0^T\big\|\partial_x^2 S(-\tau)\big(\partial_x^2 v\big)^2\big\|_{4,2,2}\,\mathrm{d}\tau\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+CT\|v\|_{11,2}^2+\sum_{j=0}^{4}\int_0^T\big\|xS(-\tau)\partial_x^{j+2}\big(\partial_x^2 v\big)^2\big\|_2\,\mathrm{d}\tau\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+CT\|v\|_{11,2}^2\\
&\qquad+\sum_{j=0}^{4}\int_0^T\big\|S(-\tau)\big(x\partial_x^{j+2}\big(\partial_x^2 v\big)^2\big)-4i\tau S(-\tau)\partial_x^{j+5}\big(\partial_x^2 v\big)^2\big\|_2\,\mathrm{d}\tau\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+CT\|v\|_{11,2}^2\\
&\qquad+\sum_{j=0}^{4}\int_0^T\big[\big\|x\partial_x^{j+2}\big(\partial_x^2 v\big)^2\big\|_2+4\tau\big\|\partial_x^{j+5}\big(\partial_x^2 v\big)^2\big\|_2\big]\,\mathrm{d}\tau\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+CT\|v\|_{11,2}^2+T\sup_t\|v\|_{6,2,2}\sup_t\|v\|_{9,2}\\
&\qquad+CT^2\|v\|_{11,2}^2\\
&\quad\leqslant C\big(\|u_0\|_{9,2}+\|u_0\|_{6,2,2}\big)+CT(1+T)\|v\|_{11,2}^2+T\sup_t\|v\|_{6,2,2}\sup_t\|v\|_{9,2}\\
&\quad\leqslant C\big(\|u_0\|_{\frac{25}{2},2}+\|u_0\|_{6,2,2}\big)+CT(1+T)E^2\\
&\quad\leqslant E,
\end{aligned}
\tag{4.20}
$$

where we have chosen $E=2C(\|u_0\|_{\frac{25}{2},2}+\|u_0\|_{6,2,2})$ and T so small that

$$
CT(1+T)E\leqslant\frac{1}{2}. \tag{4.21}
$$

Similar to (4.20), we have

$$\begin{aligned}
\sup_t \big\| u(t) \big\|_{6,2,2} &= \sup_t \sum_{j=0}^{6} \big\| x \partial_x^j u(t) \big\|_2 \\
&\leqslant \sum_{j=0}^{6} \Bigg[\sup_t \big\| S(t) x \partial_x^j u_0 \big\|_2 + 4T \sup_t \big\| S(t) \partial_x^{j+3} u_0 \big\|_2 \\
&\quad + \sup_t \Bigg\| x \int_0^t S(t-\tau) \partial_x^j \big(\partial_x^2 v\big)^2 \, \mathrm{d}\tau \Bigg\|_2 \Bigg] \\
&\leqslant C \|u_0\|_{6,2,2} + T \|u_0\|_{9,2} + T \sup_t \|v\|_{6,2,2} \sup_t \|v\|_{9,2} \\
&\quad + T^2 \sup_t \|v\|_{11,2} \\
&\leqslant C \|u_0\|_{6,2,2} + T \|u_0\|_{9,2} + T E^2 + T^2 E \\
&\leqslant E.
\end{aligned} \tag{4.22}$$

As in the derivation of (4.7), we can get

$$\sup_t \big\| u(t) \big\|_{\frac{23}{2},2} \leqslant E. \tag{4.23}$$

The estimates (4.17), (4.20), (4.22) and (4.23) yield $u \in Z_T^E$. Thus, we can fix E and T as above such that $\mathscr{T}$ is a contraction mapping from Z_T^E to itself. By the standard argument used in the proof of Theorem 1.1, we can complete the proof for which we omit the details. □

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