Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
J. Differential Equations 228 (2006) 311–338
Journal of Differential Equations
www.elsevier.com/locate/jde

Energy scattering theory for the nonlinear Schrödinger equations with exponential growth in lower spatial dimensions

Wang Baoxiang [a,*], Hao Chengchun [b], Henryk Hudzik [c]

[a] LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China
[b] Academy of Mathematics and Systems Science, CAS, Beijing 100080, People's Republic of China
[c] Faculty of Mathematics and Computer Sciences, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland

Received 27 July 2005; revised 3 May 2006
Available online 22 June 2006

Abstract

For one and two spatial dimensions, we show the existence of the scattering operators for the nonlinear Schrödinger equation with exponential nonlinearity in the whole energy spaces.

Keywords: Nonlinear Schrödinger equation; Exponential growth; Energy scattering; Separation of localized mass

1. Introduction

In this paper we study the existence of the scattering operators in energy spaces for the nonlinear Schrödinger equation (NLS):

$$\mathrm{i}u_t + \Delta u - f(u) = 0, \tag{1.1}$$

where $u(t,x)$ is a complex valued function of $(t,x) \in \mathbb{R}^{1+n}$, $\mathrm{i} = \sqrt{-1}$, $u_t = \partial u/\partial t$, $\Delta u = \sum_{i=1}^{n} \partial^2 u/\partial x_i^2$, $n = 1, 2$. $f(u)$ is a nonlinear function with exponential growth, say

* Corresponding author.
E-mail addresses: wbx@math.pku.edu.cn (B.X. Wang), hcc@amss.ac.cn (C.C. Hao), hudzik@amu.edu.pl (H. Hudzik).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.05.010

$$f(u)=\begin{cases}\mu\big(e^{\lambda|u|^2}-1-\lambda|u|^2-\frac{\lambda^2}{2}|u|^4\big)u, & n=1,\\ \mu(\cosh\lambda|u|-1-\lambda^2|u|^2)u, & n=2,\end{cases} \tag{1.2}$$

for some $\lambda,\mu>0$, $\cosh v=(e^v+e^{-v})/2$. For convenience, we say that H^1 is the energy space for NLS. We will show that the scattering operators for Eq. (1.1) with the nonlinearity as in (1.2) are well defined and bijective in the energy spaces.

If $f(u)$ is a power function, say $f(u)=|u|^\alpha u$, a large amount of work has been devoted to the study of the scattering theory of the nonlinear Schrödinger equation; cf. [2,5,7,8,14,18,19,23,25, 29,30]. If $n\geqslant 3$ and α is a subcritical power in H^1, i.e., $4/n<\alpha<4/(n-2)$, the energy scattering was obtained by Ginibre and Velo [7,8] and Tsutsumi [25]. Bourgain [2] considered the critical NLS with $f(u)=|u|^{4/(n-2)}u$ in three and four spatial dimensions and obtained the existence of the scattering operators in energy spaces for the radial solutions, where a new method so-called "separation of localized energy" was invented (Grillakis [9] gave a different approach which recovered the global well-posedness for the smooth radial solutions in 3D). Applying this argument and setting up a new Morawetz-type inequality, Nakanishi [18,19] was able to show the energy scattering in one and two spatial dimensions for $4/n<\alpha<\infty$. Recently, Colliander, Keel, Staffilani, Takaoka and Tao [5] developed the localization techniques in both physical and frequency spaces. By establishing a frequency-localized interaction Morawetz-type estimate, they obtained the energy scattering for the critical NLS in three spatial dimensions and removed the radial assumption in [2]; one can consult their paper for details. Recently, Ryckman and Visan have generalized their work to higher spatial dimensions; cf. [21,26]. The regularity of the scattering operator was also shown in [2,5–7,29].

If the nonlinearity has the exponential growth, Nakamura and Ozawa [16] considered the small data scattering for NLS in the critical space $H^{n/2}$. Nakamura and Ozawa [17], Wang [30] showed that the scattering operator carries a band in H^s into H^s for $s\geqslant n/2$. We will use Bourgain's localization arguments (separation of localized mass) to study the energy scattering of Eq. (1.1) with the nonlinearity as in (1.2). In order to state our results more precisely, we will use Taylor's expansion of $f(u)$ and consider a generalized version of $f(u)$:

$$f(u)=\mu\sum_{k\geqslant k(n)}\frac{\lambda_k^k}{k!}|u|^{2k}u,\quad k(n):=1+\frac{2}{n}. \tag{1.3}$$

Taking $\lambda_k\equiv\lambda$ and $\lambda_k^k=\lambda^{2k}k!/(2k)!$ for $n=1$ and $n=2$, respectively, we then get the nonlinearity as in (1.2). We denote

$$F\big(|u|^2\big)=\frac{\mu}{2}\sum_{k\geqslant k(n)}\frac{\lambda_k^k}{(k+1)!}|u|^{2k+2}. \tag{1.4}$$

The solution of (1.1) with the nonlinearity as in (1.3) and initial value u_0 at $t=t_0$ formally satisfies the conservations of mass and Hamiltonian:

$$M(u):=\big\|u(t)\big\|^2_{L^2(\mathbb{R}^n)}=\|u_0\|^2_{L^2(\mathbb{R}^n)}, \tag{1.5}$$

$$H(u):=\frac{1}{2}\big\|\nabla u(t)\big\|^2_{L^2(\mathbb{R}^n)}+\int_{\mathbb{R}^n}F\big(\big|u(t,x)\big|^2\big)\,dx=H(u_0). \tag{1.6}$$

For convenience, we also write

$$E(u) := M(u) + H(u). \tag{1.7}$$

The scattering operator of NLS (1.1) in H^1 is defined as follows. Let $u_0^- \in H^1$, we look for a unique global solution u of NLS satisfying $u \in C(\mathbb{R}, H^1)$ and $\|u(t) - e^{it\Delta}u_0^-\|_{H^1} \to 0$ as $t \to -\infty$. Moreover, if there exists a unique $u_0^+ \in H^1$ satisfying $\|u(t) - e^{it\Delta}u_0^+\|_{H^1} \to 0$ as $t \to +\infty$, then we can define a mapping $S: u_0^- \to u_0^+$ and say that the scattering operator $S: H^1 \to H^1$.

For any $0 < E < \infty$, we denote

$$\mathcal{H}_E^1 = \{v\colon\ E(v) \leqslant E\}. \tag{1.8}$$

The following is our main result.

Theorem 1.1. *Let $n = 1, 2$, $0 < E < \infty$, $0 < \mu < \infty$. Let $f(u)$ be as in* (1.3) *with $\lambda_k \geqslant 0$ for $k \geqslant k(n)$ and*

$$\begin{cases} \sup_{k \geqslant k(n)} \lambda_k := \lambda < \infty, & n = 1, \\ \limsup_{k \to \infty} \lambda_k \leqslant c/E^2, & n = 2 \end{cases} \tag{1.9}$$

for some small constant c that is independent of E. Then the scattering operator $S: \mathcal{H}_E^1 \to \mathcal{H}_E^1$ is a homeomorphism.

It is easy to see that condition (1.9) covers the nonlinearity as in (1.2). Indeed, $\lambda_k \equiv \lambda$ for $n = 1$, and $\lambda_k = \lambda^2(k!/(2k)!)^{1/k} \to 0$ for $n = 2$. Hence, we have

Corollary 1.2. *Let $n = 1, 2$, $0 < E < \infty$, $0 < \lambda, \mu < \infty$. Let $f(u)$ be as in* (1.2). *Then the scattering operator $S: \mathcal{H}_E^1 \to \mathcal{H}_E^1$ is a homeomorphism.*

If $n = 2$, we see that Theorem 1.1 also contains $f(u) = \mu(e^{\lambda|u|^2} - 1 - \lambda|u|^2)u$, $0 < \lambda \leqslant c/E^2$, as a special case. In one spatial dimension, the growth of the nonlinearity in Theorem 1.1 is not optimal. In fact, Theorem 1.1 also holds for a class of more general functions and we have the following:[1]

Theorem 1.3. *Let $n = 1$, $0 < E < \infty$. Assume that $f(u) := h(|u|^2)u$ satisfies*

$$\begin{aligned} &F(|u|^2) := \int_0^{|u|^2} h(s)\,ds \gtrsim |u|^8, \\ &G(|u|^2) := h(|u|^2)|u|^2 - F(|u|^2) \gtrsim |u|^8, \\ &|f(u) - f(v)| \lesssim P(|u| \vee |v|)(|u| \vee |v|)^6 |u - v| \end{aligned} \tag{1.2a}$$

[1] The authors are greatly indebted to Professor M. Nakamura, who pointed out that Theorem 1.1 could most likely be generalized to the nonlinearity $f(u)$ with arbitrary growth at $u = \infty$.

for some continuous function $P:[0,\infty)\to[0,\infty)$. Then the scattering operator $S:\mathcal{H}_E^1\to\mathcal{H}_E^1$ is a homeomorphism.

It is easy to see that condition (1.2a) covers $f(u)=\mu(e^{\lambda|u|^m}-1)u$, $m\geqslant 6$, $\lambda,\mu>0$, as a special case.

Roughly speaking, the exponential nonlinearity in two spatial dimensions is critical in H^1, which corresponds to the limit case in Sobolev embedding; cf. [16]. But it seems necessary to make a delicate difference between the exponential growth orders for the nonlinearities. Let us compare the nonlinearity $f(u)$ in (1.2) with

$$\tilde f(u)=\sum_{2\leqslant k<N}|u|^{2k}u+\sum_{k\geqslant N}\frac{\lambda^k}{k!}|u|^{2k}u, \tag{1.2b}$$

which corresponds to the cases $\lambda_k^k=\lambda^{2k}k!/(2k)!$ and $\lambda_k\equiv\lambda$ $(k\gg 1)$ in (1.3), respectively. Due to $\lim_{k\to\infty}\lambda_k=0$ in the former case, we see that the growth of $f(u)$ as in (1.2) is slower than that of $\tilde f(u)$ as in (1.2b). In the latter case, by Theorem 1.1 we need $\lambda\leqslant c/E^2$ to guarantee the existence of the scattering operators. For the nonlinearity as in (1.2b), the Hamiltonian should be

$$\widetilde{H}(u)\sim\|\nabla u\|^2_{L^2(\mathbb{R}^n)}+\sum_{2\leqslant k<N}\|u\|^{2k+2}_{L^{2k+2}}+\sum_{k\geqslant N}\frac{\lambda^k}{(k+1)!}\|u\|^{2k+2}_{L^{2k+2}}. \tag{1.6a}$$

If the initial datum u_0 is only assumed to range over a bounded region in H^1, we could not get that the Hamiltonian $\widetilde{H}(u_0)$ is finite if λ is very large. Indeed, by Ozawa's critical Sobolev embedding (see (2.24)),

$$\|u_0\|^{2k+2}_{L^{2k+2}}\leqslant C^{2k+2}(2k+2)^{k+2}\|u_0\|^{2k+2}_{H^1},$$

and so, the Hamiltonian $\widetilde{H}(u_0)$ is controlled by

$$\widetilde{H}(u_0)\lesssim\|\nabla u_0\|_2^2+\sum_k\frac{\lambda^k}{(k+1)!}C^{2k+2}(2k+2)^{k+2}.$$

When λ is large, the above series is not convergent. This fact leads to condition (1.9) in our Theorem 1.1. On the other hand, let us recall Trudinger's inequality (cf. [20]): There exists $\lambda_0>0$ such that for any $0<\lambda\leqslant\lambda_0$, $\|u\|_{H^1}\lesssim 1$,

$$\int_{\mathbb{R}^n}\big(e^{\lambda|u(x)|^2}-1\big)\,dx\lesssim 1.$$

From Trudinger's inequality we also need λ suitably small to guarantee that the Hamiltonian $\widetilde{H}(u_0)$ is finite. We do not know what happens if $\lambda>0$ is sufficiently large. Further, if $f(u)$ grows faster than $\tilde f(u)$ as in (1.2b), we do not know how to obtain the existence of the scattering operators with large states even if $\lambda>0$ is small enough, see below, Remark 2.6.

It is convenient to use the integral version of Eq. (1.1). One can rewrite (1.1) with initial data u_0 at $t = t_0$ as the following equation:

$$u(t) = S(t)u_0 - \mathrm{i}\int_{t_0}^{t} S(t-\tau) f\big(u(\tau)\big)\,d\tau, \tag{1.10}$$

where $S(t) = e^{\mathrm{i}t\Delta}$. We have the following time–space estimates (cf. [4,10–13,27]).

Proposition 1.3 *(Strichartz inequalities). We have*

$$\big\| S(t)u_0 \big\|_{L^{\gamma(r)}(\mathbb{R},L^r)} \leqslant C\|u_0\|_2, \tag{1.11}$$

$$\left\| \int_{t_0}^{t} S(t-\tau) f(\tau)\,d\tau \right\|_{L^{\gamma(r)}(\mathbb{R},L^r)} \leqslant C\|f\|_{L^{2(n+2)/(n+4)}_{x,t\in\mathbb{R}}}, \tag{1.12}$$

where $2 \leqslant r \leqslant \infty$, $2/\gamma(r) = n(1/2 - 1/r)$, $\gamma(r) \in (2, \infty]$.

Notation 1.4. Throughout this paper, $c < 1$, $C > 1$ will stand for universal constants that can be changed from line to line, $C_{A,B,\ldots}$ means that the constant C depends only on $A, B, \ldots$. We denote by $A \lesssim B$ that $A \leqslant CB$ and by $A \sim B$ that $A \lesssim B$ and $B \lesssim A$. For any $p \in [1, \infty]$, we denote by p' the conjugate number of p, i.e., $1/p + 1/p' = 1$. Let $L^p := L^p(\mathbb{R}^n)$ be the Lebesgue space and the norm on L^p is denoted by $\|\cdot\|_p$. We denote by $\|f\|_{L^q(I,L^p)}$ the space–time norm $(\int_I (\int_{\mathbb{R}^n} |f(t,x)|^p\,dx)^{q/p}\,dt)^{1/q}$, and $L^p_{x,t\in I} = L^p(I, L^p)$. We write $D_x = (-\Delta)^{1/2}$. The Bessel (Riesz) potential spaces H^s_p ($\dot H^s_p$) are defined by $(I - \Delta)^{-s/2}L^p$ ($(-\Delta)^{-s/2}L^p$), $H^s = H^s_2$ ($\dot H^s = \dot H^s_2$). The Besov spaces $B^s_{p,q}$ can be defined as follows; cf. [1,24].

Let $\psi : \mathbb{R}^n \to [0, 1]$ be a smooth radial bump function adapted to the ball $\{\xi \in \mathbb{R}^n\colon |\xi| \leqslant 2\}$, which equals 1 on the ball $\{\xi \in \mathbb{R}^n\colon |\xi| \leqslant 1\}$. Denote $\varphi(\xi) := \psi(\xi) - \psi(2\xi)$ and $\varphi_k(\xi) := \varphi(2^{-k}\xi)$, $k \in \mathbb{N}$. Assume that $\varphi_0 := 1 - \sum_{k=1}^{\infty}\varphi_k$ $(= \psi(\xi))$. One easily sees that $\operatorname{supp}\varphi_k \subset \{\xi \in \mathbb{R}^n\colon 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$ for $k \geqslant 1$ and $\operatorname{supp}\varphi_0 \subset \{\xi \in \mathbb{R}^n\colon |\xi| \leqslant 2\}$. Let $0 < p, q \leqslant \infty$. Denote $\varDelta_k := \mathscr{F}^{-1}\varphi_k\mathscr{F}$. The norm on Besov spaces is defined as follows:

$$\|f\|_{B^s_{p,q}} = \left(\sum_{k=0}^{\infty} 2^{ksq}\|\varDelta_k f\|_p^q\right)^{1/q}. \tag{1.13}$$

In the case $p = q = \infty$, we have a modification on the norm:

$$\|f\|_{B^s_{\infty,\infty}} = \sup_{k\geqslant 0} 2^{ks}\|\varDelta_k f\|_\infty. \tag{1.14}$$

2. Low mass implies the energy scattering

Using Nakamura and Ozawa's results as in [17], we see that the small Cauchy data in L^2 imply the energy scattering for NLS in one spatial dimension.

Theorem 2.1. *Let $n = 1$, $0 < E < \infty$, $u_0 \in H^1$. Assume that $f(u)$ satisfies*

$$f(0) = 0, \tag{2.1}$$

$$\big|f(u) - f(v)\big| \lesssim P\big(|u| \vee |v|\big)\big(|u| \vee |v|\big)^6 |u - v| \tag{2.2}$$

for some non-negative continuous function $P : \mathbb{R}_+ \to \mathbb{R}_+$. There exists $\eta := \eta(E) > 0$ such that if

$$\|u_0\|_2 \leqslant \eta, \tag{2.3}$$

then

$$\mathrm{i}u_t + u_{xx} - f(u) = 0, \qquad u(t_0) = u_0 \tag{2.4}$$

is globally well posed in $C(\mathbb{R}, H^1) \cap L^6(\mathbb{R}, H^1_6)$ and the solution u satisfies

$$\|u\|_{L^6_{x,t\in\mathbb{R}}} + \|\partial_x u\|_{L^6_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{2.5}$$

It is obvious that if $f(u)$ is given by (1.2), condition (2.2) holds for

$$P\big(|u|\big) \sim \sum_k \frac{C^k}{k!} |u|^{2k-6}. \tag{2.6}$$

Moreover, condition (2.2) also covers the nonlinearity

$$\mu\big(e^{\lambda |u|^m} - 1\big)u, \quad m \geqslant 6. \tag{2.7}$$

It is also known that the small Cauchy data in H^1 imply the existence of scattering operators in two spatial dimensions; cf. [16]. In this section we give an improved version in two spatial dimensions and we show the following theorem.

Theorem 2.2. *Let $n = 2$, $0 < \mu < \infty$, $0 < E < \infty$, $u_0 \in H^1$ and $E(u_0) \leqslant E$,*

$$f(u) = \mu \sum_{k\geqslant 1} \frac{\lambda_k^k}{k!} |u|^{2k} u, \tag{2.8}$$

where $\{\lambda_k\}$ satisfies condition (1.9). *Assume that $\|u_0\|_2 \leqslant \eta$, $0 < \eta := \eta(E) \ll 1$. Then*

$$\mathrm{i}u_t + \Delta u - f(u) = 0, \qquad u(t_0) = u_0 \tag{2.9}$$

is globally well posed in $C(\mathbb{R}, H^1) \cap L^4(\mathbb{R}, H^1_4)$ and the solution u satisfies

$$\|u\|_{L^4_{x,t\in\mathbb{R}}} + \|D_x u\|_{L^4_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{2.10}$$

For the proof of Theorem 2.2, our idea is to use the energy and $\|u\|_{L^4_{x,t\in\mathbb{R}}}$ to estimate the nonlinearity. We resort to the interpolation inequality

$$\|u\|_{L^q(\mathbb{R}^2)} \leqslant C\|u\|^{\theta}_{L^p(\mathbb{R}^2)}\|u\|^{1-\theta}_{\dot H^1(\mathbb{R}^2)}, \quad \text{where} \tag{2.11}$$

$$\frac{1}{q} = \frac{\theta}{p} + (1-\theta)\left(\frac{1}{2} - \frac{1}{n}\right), \tag{2.12}$$

$$0 < \theta < 1, \qquad 1 < q < \infty. \tag{2.13}$$

If we treat $q \in [p,\infty)$ as a variable parameter (p is a fixed number), the constant C in (2.11) is increasing as q tends to ∞. For our purpose it seems necessary to give a delicate value of $C := C_q$ in (2.11). We will mainly use a critical embedding inequality which is due to Ozawa [20], see (2.24).

Proposition 2.3. *Let $1 < r < p < \infty$ be fixed indices. Then for any $q \in [p,\infty)$,*

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,r} q^{1/r' + p/rq}\|u\|^{p/q}_{L^p(\mathbb{R}^n)}\|u\|^{1-p/q}_{\dot H^{n/r}_r(\mathbb{R}^n)}, \tag{2.14}$$

where $C_{p,r}$ depends only on p, r and n.

In order to show Proposition 2.3, we need the following proposition.

Proposition 2.4. *Let $1 < r < p < \infty$, $0 < s < \infty$ be fixed indices. Let $p_1 \in [r,p]$ be a variable parameter. Suppose that $0 < \theta < 1$ satisfying*

$$\frac{1}{p_1} = \frac{\theta}{p} + \frac{1-\theta}{r}, \tag{2.15}$$

$$s_1 = \theta \cdot 0 + (1-\theta)s. \tag{2.16}$$

Then

$$\|u\|_{\dot H^{s_1}_{p_1}(\mathbb{R}^n)} \leqslant C_{p,r,s}\|u\|^{\theta}_{L^p(\mathbb{R}^n)}\|u\|^{1-\theta}_{\dot H^s_r(\mathbb{R}^n)}, \tag{2.17}$$

where $C_{p,r,s}$ depends only on p, r, s and n.

Proof. Proposition 2.4 is essentially known; cf. [8,10]. Since we need to fix $C_{p,r,s}$ in (2.17), we sketch its proof. We write (see Triebel [24])

$$\|u\|_{\dot F^{s_1}_{p_1,2}} = \left\| \left(\sum_{k\in\mathbb{Z}} \left(2^{ks_1}|\Delta_k u|\right)^2 \right)^{1/2} \right\|_{L^{p_1}(\mathbb{R}^n)}, \tag{2.18}$$

where Δ_k is the same one as in (1.13) if $k \geqslant 1$; $\Delta_k = \mathscr{F}^{-1}\varphi(2^{-k}\cdot)\mathscr{F}$ if $k \leqslant 0$. It is known that (see Stein [22], Triebel [24])

$$A_{p_1}\|u\|_{\dot F^0_{p_1,2}} \leqslant \|u\|_{L^{p_1}} \leqslant B_{p_1}\|u\|_{\dot F^0_{p_1,2}}. \tag{2.19}$$

Since $p_1 \in [r, q]$, by complex interpolation we have

$$C_{p,r} \leqslant A_{p_1} \leqslant B_{p_1} \leqslant C_{p,r}. \tag{2.20}$$

Hence,

$$C_{p,r} \|u\|_{\dot F^0_{p_1,2}} \leqslant \|u\|_{L^{p_1}} \leqslant C_{p,r} \|u\|_{\dot F^0_{p_1,2}}. \tag{2.21}$$

On the other hand, using Hörmander's multiplier theorem (cf. Triebel [24, p. 88]) and by complex interpolation, we see that if $s_1 \in [0, s]$, $p_1 \in [r, q]$, then

$$C_{p,r,s} \|u\|_{\dot F^{s_1}_{p_1,2}} \leqslant \|u\|_{\dot H^{s_1}_{p_1}} \leqslant C_{p,r,s} \|u\|_{\dot F^{s_1}_{p_1,2}}. \tag{2.22}$$

By (2.22) and Hölder's inequality

$$\|u\|_{\dot H^{s_1}_{p_1}} \leqslant C_{p,r,s} \|u\|_{\dot F^{s_1}_{p_1,2}} \leqslant C_{p,r,s} \|u\|^{\theta}_{\dot F^0_{p,2}} \|u\|^{1-\theta}_{\dot F^s_{r,2}} \leqslant C_{p,r,s} \|u\|^{\theta}_{L^p} \|u\|^{1-\theta}_{\dot H^s_r}, \tag{2.23}$$

which implies the result, as desired. □

Proof of Proposition 2.3. Let us recall that in [20], Ozawa established the following embedding inequality:

$$\|u\|_q \leqslant C_{p_1} q^{1/p_1'} \|u\|_{\dot H^{n/p_1 - n/q}_{p_1}}, \tag{2.24}$$

where $p_1 \in [r, p]$. Since p_1 ranges over a compact interval, we know that the constant C_{p_1} has an upper bound $C_{p,r}$; cf. [16], that is,

$$\|u\|_q \leqslant C_{p,r} q^{1/p_1'} \|u\|_{\dot H^{n/p_1 - n/q}_{p_1}}. \tag{2.25}$$

Taking

$$\frac{1}{p_1} = \frac{1}{q} + \frac{1 - p/q}{r}, \tag{2.26}$$

we see that for $\theta = p/q$,

$$\frac{1}{p_1} = \frac{\theta}{p} + \frac{1-\theta}{r}, \qquad \frac{n}{p_1} - \frac{n}{q} = \theta \cdot 0 + (1-\theta)\frac{n}{r}. \tag{2.27}$$

Hence, by Proposition 2.4,

$$\|u\|_{\dot H^{n/p_1 - n/q}_{p_1}} \leqslant C_{p,r} \|u\|_p^{p/q} \|u\|^{1-p/q}_{\dot H^{n/r}_r}. \tag{2.28}$$

Noticing that $1/p_1' = 1/r' + (p/r - 1)/q$, by (2.25) and (2.28) we immediately have the result, as desired. □

Proof of Theorem 2.2. We can assume that $\mu = 1$. Recall that

$$f(u) = \sum_{k=1}^{\infty} \frac{\lambda_k^k}{k!} |u|^{2k} u, \tag{2.29}$$

$$Df(u) = \sum_{k=1}^{\infty} \frac{\lambda_k^k}{k!} \big((k+1)|u|^{2k} Du + ku^2 |u|^{2k-2} D\bar{u}\big), \tag{2.30}$$

where $D = \partial/\partial x_1$, or $D = \partial/\partial x_2$. We have for any interval $I \subset \mathbb{R}$,

$$\big\| f(u)\big\|_{L^{4/3}_{x,t\in I}} \leqslant \sum_{k=1}^{\infty} \frac{\lambda_k^k}{k!} \big\| |u|^{2k} u\big\|_{L^{4/3}_{x,t\in I}} \leqslant \sum_{k=1}^{\infty} \frac{\lambda_k^k}{k!} \|u\|^{2k}_{L^{4k}_{x,t\in I}} \|u\|_{L^4_{x,t\in I}}. \tag{2.31}$$

Using Proposition 2.3, we have

$$\|u\|_{L^{4k}_x} \leqslant C(4k)^{1/2+1/2k} \|u\|^{1/k}_{L^4_x} \|u\|^{1-1/k}_{\dot{H}^1_x}. \tag{2.32}$$

It follows from (2.32) that

$$\|u\|_{L^{4k}_{x,t\in I}} \leqslant C(4k)^{1/2+1/2k} \|u\|^{1/k}_{L^4_{x,t\in I}} \|u\|^{1-1/k}_{L^\infty(I,\dot{H}^1_x)} \leqslant CE(4k)^{1/2+1/2k} \|u\|^{1/k}_{L^4_{x,t\in I}}. \tag{2.33}$$

Inserting (2.33) into (2.31), one has that

$$\big\| f(u)\big\|_{L^{4/3}_{x,t\in I}} \leqslant \sum_{k=1}^{\infty} \frac{(\lambda_k^{1/2} CE)^{2k}}{k!} (4k)^{k+1} \|u\|^3_{L^4_{x,t\in I}}. \tag{2.34}$$

Similarly,

$$\begin{aligned} \big\| Df(u)\big\|_{L^{4/3}_{x,t\in I}} &\leqslant \sum_{k=1}^{\infty} \frac{(2k+1)\lambda_k^k}{k!} \|u\|^{2k}_{L^{4k}_{x,t\in I}} \|Du\|_{L^4_{x,t\in I}} \\ &\leqslant \sum_{k=1}^{\infty} \frac{(2k+1)}{(k-1)!} \big(4\lambda_k (CE)^2 k\big)^k \|u\|^2_{L^4_{x,t\in I}} \|Du\|_{L^4_{x,t\in I}}. \end{aligned} \tag{2.35}$$

It is easy to see that if condition (1.9) holds, then

$$\lambda_k < \frac{c}{E^2} := \frac{1}{8e(CE)^2}, \quad k \gg 1, \tag{2.36}$$

which implies that the series

$$\sum_{k=1}^{\infty} \frac{(2k+1)}{(k-1)!} \big(4\lambda_k (CE)^2 k\big)^k \leqslant C_E < \infty \tag{2.37}$$

is convergent. Hence, by (2.34), (2.35) and (2.37), one has that

$$\|f(u)\|_{L^{4/3}_{x,t\in I}} \leqslant C_E \|u\|^3_{L^4_{x,t\in I}}, \tag{2.38}$$

$$\|Df(u)\|_{L^{4/3}_{x,t\in I}} \leqslant C_E \|u\|^2_{L^4_{x,t\in I}} \|Du\|_{L^4_{x,t\in I}}. \tag{2.39}$$

In view of the Strichartz estimates, (2.38) and (2.39) imply that

$$\|u\|_{L^4_{x,t\in I}} \leqslant C\|u_0\|_2 + C_E \|u\|^3_{L^4_{x,t\in I}}, \tag{2.40}$$

$$\|Du\|_{L^4_{x,t\in I}} \leqslant CE + C_E \|u\|^2_{L^4_{x,t\in I}} \|Du\|_{L^4_{x,t\in I}}. \tag{2.41}$$

Since $\|u_0\|_2 \leqslant \eta$ and η is sufficiently small, in view of the standard continuity method we have from (2.40) that

$$\|u\|_{L^4_{x,t\in I}} \leqslant 2C\eta. \tag{2.42}$$

It follows from (2.41) and (2.42) that

$$\|Du\|_{L^4_{x,t\in I}} \leqslant 2CE. \tag{2.43}$$

Taking $I = \mathbb{R}$, we obtain the result, as desired. □

Remark 2.5. Following the proof of Theorem 2.2, in view of the Strichartz inequalities (1.11) and (1.12), we see that for any $2 \leqslant r < \infty$, the Strichartz norms $\|u\|_{L^{\gamma(r)}(I,L^r)}$ and $\|D_x u\|_{L^{\gamma(r)}(I,L^r)}$ have the same upper bounds as those of $\|u\|_{L^4_{x,t\in I}}$ and $\|D_x u\|_{L^4_{x,t\in I}}$, respectively. So, by a standard method as in [3,5,7,18], we conclude that (2.10) has implied the existence of scattering operators in energy spaces. Similarly, in one spatial dimension, (2.5) also implies the existence of scattering operators in energy spaces.

Remark 2.6. Unfortunately, our method for two-dimensional NLS in this section is invalid for the nonlinearity that grows faster than that of (1.2b), say $f(u) = (e^{\lambda|u|^m} - 1)u$, $m > 2$. Indeed, if $f(u)$ takes such a form, similar to (2.31)–(2.34), we have

$$\begin{aligned}\big\|\big(e^{\lambda|u|^m} - 1\big)u\big\|_{L^{4/3}_{x,t\in I}} &\leqslant \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \|u\|^{mk}_{L^{2mk}_{x,t\in I}} \|u\|_{L^4_{x,t\in I}} \\ &\leqslant \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} (CE)^{mk} k^{mk/2+1} \|u\|^3_{L^4_{x,t\in I}}. \end{aligned} \tag{2.44}$$

We need to treat the series $\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} (CE)^{mk} k^{mk/2+1}$, which is divergent for any $0 < \lambda \ll 1$. In this case, we cannot obtain an analogous result to Theorem 2.2 for NLS (1.1) with $f(u) = (e^{\lambda|u|^m} - 1)u$, $m > 2$, even if λ is small enough. Moreover, using the method above, it seems also difficult to get the local well-posedness in $\mathcal{H}^1_E$, since the nonlinearity is out of the control of the Strichartz norms and the H^1 norms.

3. $\|u\|_{L^{(n+2)^2/n}_{x,t\in\mathbb{R}}} < \infty$ implies scattering

In this section we show that the finiteness of $\|u\|_{L^{(n+2)^2/n}_{x,t\in\mathbb{R}}}$ implies the existence of scattering operators. In view of Remark 2.5, it suffices to get an upper bound of $\|u\|_{L^{2+4/n}_{x,t\in\mathbb{R}}} + \|D_x u\|_{L^{2+4/n}_{x,t\in\mathbb{R}}}$.

Lemma 3.1. *Let $n = 1, 2$, $0 < E < \infty$ and $f(u)$ be as in* (1.3). *Assume that condition* (1.9) *is satisfied. Let u be the energy solution of* (1.1) *with $E(u) \leqslant E$. If there exists $C_E > 0$ such that*

$$\|u\|_{L^{(n+2)^2/n}_{x,t\in\mathbb{R}}} \leqslant C_E < \infty, \tag{3.1}$$

then we have

$$\|u\|_{L^{2+4/n}_{x,t\in\mathbb{R}}} + \|D_x u\|_{L^{2+4/n}_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{3.2}$$

Proof. We divide the proof into the following two cases. First, we consider the case $n = 1$. We need to prove that $\|u\|_{L^9_{x,t\in\mathbb{R}}} \leqslant C_E$ implies that $\|u\|_{L^6_{x,t\in\mathbb{R}}} + \|\partial_x u\|_{L^6_{x,t\in\mathbb{R}}} \leqslant C_E$. One has that

$$\big\|f(u)\big\|_{L^{6/5}_{x,t\in I}} \leqslant \sum_{k=3}^{\infty} \frac{\lambda_k^k}{k!}(CE)^{2k}\|u\|^6_{L^9_{x,t\in I}}\|u\|_{L^6_{x,t\in I}} \leqslant C_E\|u\|^6_{L^9_{x,t\in I}}\|u\|_{L^6_{x,t\in I}}. \tag{3.3}$$

Similar to (3.3), we also have

$$\begin{aligned}\big\|\partial_x f(u)\big\|_{L^{6/5}_{x,t\in I}} &\leqslant \sum_{k=3}^{\infty} \frac{\lambda_k^k}{k!}(CE)^{2k}(2k+1)\|u\|^6_{L^9_{x,t\in I}}\|\partial_x u\|_{L^6_{x,t\in I}} \\ &\leqslant C_E\|u\|^6_{L^9_{x,t\in I}}\|\partial_x u\|_{L^6_{x,t\in I}}.\end{aligned} \tag{3.4}$$

Using the same way as in Bourgain [2,3], one can split $\mathbb{R}$ into finite many pairwise disjoint intervals,

$$\mathbb{R} = \bigcup_{j=1}^{J} I_j; \qquad \|u\|_{L^9_{x,t\in I_j}} \leqslant \eta, \qquad C_E\eta^6 \leqslant 1/2. \tag{3.5}$$

By the Strichartz inequalities, and (3.3) and (3.4),

$$\big\|\partial_x^k u\big\|_{L^6_{x,t\in I_j}} \leqslant CE + C_E\eta^6\big\|\partial_x^k u\big\|_{L^6_{x,t\in I_j}}, \quad k = 0, 1,\ j = 1, \ldots, J. \tag{3.6}$$

Hence, by (3.5) and (3.6)

$$\big\|\partial_x^k u\big\|_{L^6_{x,t\in I_j}} \leqslant 2CE, \quad k = 0, 1,\ j = 1, \ldots, J, \tag{3.7}$$

which implies the result, as desired.

Next, we consider the case $n=2$. One needs to show that if

$$\|u\|_{L^8_{x,t\in\mathbb{R}}} \leqslant C_E < \infty, \tag{3.8}$$

then

$$\|u\|_{L^4_{x,t\in\mathbb{R}}} + \|D_x u\|_{L^4_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{3.9}$$

Let $D = \partial/\partial x_1$ or $D = \partial/\partial x_2$. Using the same way as in (2.31), (2.33) and (2.35), we have

$$\big\|f(u)\big\|_{L^{4/3}_{x,t\in I}} \leqslant C_E \|u\|^4_{L^8_{x,t\in I}} \|u\|_{L^4_{x,t\in I}}, \tag{3.10}$$

$$\big\|Df(u)\big\|_{L^{4/3}_{x,t\in I}} \leqslant C_E \|u\|^4_{L^8_{x,t\in I}} \|Du\|_{L^4_{x,t\in I}}. \tag{3.11}$$

In fact,

$$\big\|Df(u)\big\|_{L^{4/3}_{x,t\in I}} \leqslant \sum_{k=2}^{\infty} \frac{\lambda_k^k}{k!}(2k+1)\|u\|^{2k}_{L^{4k}_{x,t\in I}} \|Du\|_{L^4_{x,t\in I}}. \tag{3.12}$$

Interpolating L^{4k}_x between L^8_x and $\dot H^1_x$, by (2.14) we have

$$\|u\|_{L^{4k}_x} \leqslant C(4k)^{1/2+1/k} \|u\|^{2/k}_{L^8_x} \|u\|^{1-2/k}_{\dot H^1_x}, \tag{3.13}$$

$$\|u\|_{L^{4k}_{x,t\in I}} \leqslant C(4k)^{1/2+1/k} \|u\|^{2/k}_{L^8_{x,t\in I}} \|u\|^{1-2/k}_{L^\infty(I,\dot H^1_x)}. \tag{3.14}$$

By (3.12) and (3.14), we obtain that

$$\big\|Df(u)\big\|_{L^{4/3}_{x,t\in I}} \leqslant \sum_{k=2}^{\infty} \frac{\lambda_k^k}{k!}(2k+1)(4k)^{k+2}(CE)^{2k-4} \|u\|^4_{L^8_{x,t\in I}} \|Du\|_{L^4_{x,t\in I}}. \tag{3.15}$$

Noticing that $\lambda_k < c/E^2$ for $k \gg 1$, (3.15) implies (3.11).

In view of the Strichartz estimates, together with (3.10) and (3.11), and using the same way as in the case $n=1$, we can get (3.9). The details are omitted. □

Remark 3.2. If $n=1$, one can easily generalize the argument above to the case that $f(u)$ is given by (1.2a) and the details are omitted.

4. Mass concentration phenomenon

In this section, we show that if the space–time $L^{(n+2)^2/n}$ norm of the solution in a time interval I is not small, say $\|u\|_{L^{(n+2)^2/n}_{x,t\in I}} \sim \eta$, then there exist $t_0 \in I$ and $x_0 \in \mathbb{R}$ such that the mass at t_0 will have a concentration phenomenon in a spatial ball with center at x_0, and the size of such a ball depends only on η and is independent of the length of I. We write

$$p := \frac{(n+2)^2}{n} = \begin{cases} 9, & n=1, \\ 8, & n=2, \end{cases} \qquad \alpha := 2 + \frac{4}{n} = \begin{cases} 6, & n=1, \\ 4, & n=2. \end{cases} \tag{4.1}$$

In the sequel we will always assume that p and α are as in (4.1) if there is no explanation. Let $\eta > 0$ be a small number, say $0 < \eta < \eta_0$,

$$C_E \eta_0^{2+4/n} = 1/2. \tag{4.2}$$

Lemma 4.1. *Let $n = 1, 2$. Let u be the energy solution of* (1.1) *and* (1.2), $0 < E < \infty$, $E(u) \leqslant E$, $\|u\|_{L^p_{x,t\in I}} = \eta$. *Then there exist $C_\eta > 0$, which depends only on E and η, and $t_0 \in I$, $x_0 \in \mathbb{R}^n$ such that*

$$\big\|u(t_0)\big\|_{L^2(|x-x_0|\leqslant C_\eta)} \geqslant c_E \eta^{\alpha/2}. \tag{4.3}$$

In order to show (4.3), we use an interpolation lemma, which is established in Wang [28] by applying Bourgain's idea in [3].

Proposition 4.2. *Let $1 \leqslant r_0 < r < \infty$, $-\infty < s_1 < s < s_0 < \infty$, $0 < \theta < 1$ and*

$$\frac{1}{r} = \frac{\theta}{r_0} + \frac{1-\theta}{\infty}, \qquad s = \theta s_0 + (1-\theta) s_1. \tag{4.4}$$

Then we have

$$\|u\|_{H^s_r(\mathbb{R}^n)} \leqslant C \|u\|^{1-\theta}_{B^{s_1}_{\infty,\infty}} \|u\|^{\theta}_{B^{s_0}_{r_0,r_0}}. \tag{4.5}$$

Proof of Lemma 4.1. In view of (3.3), (3.4), (3.10) and (3.11), we have

$$\big\|f(u)\big\|_{L^{(2+4/n)'}(I,H^1_{(2+4/n)'})} \leqslant C_E \|u\|^{\alpha}_{L^p_{x,t\in I}} \|u\|_{L^{2+4/n}(I,H^1_{2+4/n})}. \tag{4.6}$$

By the Strichartz inequalities and (4.6),

$$\|u\|_{L^{2+4/n}(I,H^1_{2+4/n})} \leqslant CE + C_E \eta^{\alpha} \|u\|_{L^{2+4/n}(I,H^1_{2+4/n})}. \tag{4.7}$$

(4.7) and (4.2) imply that

$$\|u\|_{L^{2+4/n}(I,H^1_{2+4/n})} \leqslant 2CE. \tag{4.8}$$

Let $s_0 = n(n\alpha - 4)/8$ and $s_1 = -n/2$. It is easy to see that $s_1(1-4/n\alpha) + s_0(4/n\alpha) = 0$. Noticing that $H^{s_0}_{2+4/n} \subset B^{s_0}_{2+4/n,2+4/n}$, by Proposition 4.2 we have

$$\eta = \|u\|_{L^p_{x,t\in I}} \leqslant C \|u\|^{1-4/n\alpha}_{L^\infty(I,B^{s_1}_{\infty,\infty})} \|u\|^{4/n\alpha}_{L^{2+4/n}(I,H^{s_0}_{2+4/n})}. \tag{4.9}$$

Hence, by (4.8) and (4.9)

$$\|u\|_{L^\infty(I,B^{s_1}_{\infty,\infty})} \geqslant c_E \eta^{\alpha/2}. \tag{4.10}$$

So, we get some $t_0 \in I$, $x_0 \in \mathbb{R}^n$, $j \in \mathbb{N} \cup \{0\}$ satisfying (we can assume that $j \geqslant 1$, since $j = 0$ is similar to the case $j \geqslant 1$)

$$2^{js_1} \big| \big(\Delta_j u(t_0)\big)(x_0)\big| \geqslant c_E \eta^{\alpha/2}, \tag{4.11}$$

which implies that

$$2^{js_1} \big| \big[\big(\mathcal{F}^{-1}\varphi_j\big) * u(t_0)\big](x_0)\big| \geqslant c_E \eta^{\alpha/2}, \tag{4.12}$$

where φ_j is as in (1.13), that is

$$2^{(n+s_1)j} \left| \int\limits_{\mathbb{R}^n} \big(\mathcal{F}^{-1}\varphi\big)\big(2^j y\big) u(t_0, x_0 - y)\, dy \right| \geqslant c_E \eta^{\alpha/2}. \tag{4.13}$$

Notice that

$$\big\| \mathcal{F}^{-1}\varphi\big(2^j \cdot\big)\big\|_{L^2(|x| \geqslant C_\eta)} = 2^{-jn/2} \big\|\mathcal{F}^{-1}\varphi\big\|_{L^2(|x| > 2^j C_\eta)}. \tag{4.14}$$

We can take C_η large enough ($j \geqslant 1$) verifying

$$\big\|\mathcal{F}^{-1}\varphi\big\|_{L^2(|x| > C_\eta)} \big\|u(t_0)\big\|_2 \leqslant c_E \eta^{\alpha/2}/2. \tag{4.15}$$

Collecting (4.13)–(4.15), we have

$$\big\|\mathcal{F}^{-1}\varphi\big\|_{L^2} \|u\|_{L^2(|x - x_0| < C_\eta)} \geqslant c_E \eta^{\alpha/2}. \tag{4.16}$$

Hence, (4.3) is deduced. □

5. Length estimate of I with $\|u\|_{L^p_{x,t\in I}} = \eta$

Let u be the energy solution. Let $E, M > 0$ and $M \leqslant E$. Now we assume

$$\sup\big\{\|u\|_{L^p_{x,t\in[A,B]}}:\ M(u) \leqslant M,\ E(u) \leqslant E\big\} \geqslant G \gg C_E \tag{5.0}$$

for some interval $[A, B] \subset \mathbb{R}$. It is easy to see that $B - A \gg C_E$. Considering the decomposition $[A, B] = \bigcup_{j\in\Lambda} I_j$,

$$\|u\|_{L^p_{x,t\in I_j}} = \eta, \quad j \in \Lambda, \tag{5.1}$$

we have from Lemma 4.1 that there exists $t_j \in I_j$ and $x_j \in \mathbb{R}^n$ satisfying

$$\big\|u(t_j)\big\|_{L^2(|x - x_j| \leqslant C_\eta)} \geqslant c_E \eta^{\alpha/2}, \tag{5.2}$$

where $C_\eta > 1$ is a fixed constant depending only on η, E. Our goal of this section is to show that if G is sufficiently large, then

$$\sup_{j\in\Lambda} |I_j| \gg C_{E,\eta}. \tag{5.3}$$

We will use the ideas of Bourgain [3] and Nakanishi [18].

Lemma 5.1. [3,18] *Let u be the energy solution of* (1.1) *and* (1.2). *Let $B \subset \mathbb{R}^n$ be a compact set. $B(Q) := \{x \in \mathbb{R}^n\colon\ \exists y \in B,\ |x-y| \leqslant Q\}$. Then we have*

$$\|u(t_1)\|_{L^2(B(Q))} \geqslant \|u(t_0)\|_{L^2(B)} - C_E|t_1-t_0|/Q. \tag{5.4}$$

In particular,

$$\|u(t_1)\|_{L^2(|x-x_0|\leqslant C_\eta+Q|t_1-t_0|)} \geqslant \|u(t_0)\|_{L^2(|x-x_0|\leqslant C_\eta)} - C_E/Q. \tag{5.5}$$

The following generalized Morawetz-type estimate is also due to Nakanishi [18].

Lemma 5.2. *Let u be the energy solution of* (1.1) *and* (1.2). *We have*

$$\iint\limits_{\mathbb{R}^{1+n}} \frac{\langle t\rangle^2 G(|u|^2)}{\langle t\rangle^3 + |x|^3}\,dx\,dt \leqslant C_E, \tag{5.6}$$

where $\langle t\rangle = \sqrt{1+t^2}$, $G(|u|^2) = f(u)\bar{u} - F(|u|^2) \geqslant c|u|^{2+\alpha}$.

In (5.5), we can take $Q := C_\eta > C_E\eta^{-4}$, it follows from (5.2) and (5.5) that ($\eta^4 \ll \eta^{\alpha/2}$)

$$\|u(t)\|_{L^2(|x-x_j|\leqslant C_\eta(1+|t-t_j|))} \geqslant c_E\eta^{\alpha/2}. \tag{5.7}$$

Hence, if $|t-t_j| \leqslant C_\eta$, we have from Hölder's inequality,

$$\|u(t)\|_{L^{2+\alpha}(|x-x_j|\leqslant C_\eta(1+|t-t_j|))} \geqslant c_{E,\eta}. \tag{5.8}$$

By (5.1), we have

$$\eta = \|u\|_{L^p_{x,t\in I_j}} \leqslant |I_j|^{1/p}\|u\|_{L^\infty(I_j,L^p)} \leqslant CE|I_j|^{1/p}, \tag{5.9}$$

which yields $|I_j| \geqslant c_E\eta^p$.

Therefore, for any $j \in \Lambda$, there exists $J_j \subset I_j$ with $|J_j| \geqslant c_\eta$ such that (5.7) and (5.8) hold for any $t \in J_j$. We may assume that t_j is the left end-point of J_j.

Lemma 5.3. *Let u be the energy solution of* (1.1) *and* (1.2). *Let I_j be as in* (5.1). *If G in* (5.0) *is sufficiently large, then we have* $\sup_{j\in\Lambda}|I_j| \gg C_{E,\eta}$.

Proof. We imitate Nakanishi's proof in [18] (some earlier ideas related this issue can be found in [7,14,15]). Denote $k_1 = 1$, by induction we define

$$k_\ell = \min\bigl\{j \in \Lambda\colon\ |x_{k_i} - x_j| > 2C_\eta + C_\eta|t_{k_i} - t_j|,\ i = 1,\dots,\ell-1\bigr\}. \tag{5.10}$$

Then we define

$$\mathbb{N}_\ell = \bigl\{j \in \Lambda\colon\ j \geqslant k_\ell,\ |x_{k_\ell} - x_j| \leqslant 2C_\eta + C_\eta|t_{k_\ell} - t_j|\bigr\}. \tag{5.11}$$

It is easy to see that $\Lambda = \bigcup_\ell \mathbb{N}_\ell$. First, we show the above $k_1, \dots, k_\ell$ are finitely many. We easily see that $B(x_{k_\ell}, C_\eta) \cap B(x_{k_i}, C_\eta(1 + |t_{k_\ell} - t_{k_i}|)) = \emptyset$, $i = 1, \dots, \ell - 1$. Hence, by Lemma 5.1,

$$\begin{aligned} E &\geqslant \left\| u(t_{k_\ell}) \right\|_{L^2(\bigcup_{i=1}^{\ell} B(x_{k_i}, C_\eta(1+|t_{k_\ell} - t_{k_i}|)))} \\ &\geqslant \left\| u(t_{k_\ell}) \right\|_{L^2(B(x_{k_\ell}, C_\eta))} + \left\| u(t_{k_{\ell-1}}) \right\|_{L^2(\bigcup_{i=1}^{\ell-1} B(x_{k_i}, C_\eta(1+|t_{k_{\ell-1}} - t_{k_i}|)))} - c_E \eta^{\alpha/2}/2 \\ &\geqslant \cdots \geqslant \sum_{i=1}^{\ell} \left\| u(t_{k_i}) \right\|_{L^2(B(x_{k_i}, C_\eta))} - \ell c_E \eta^{\alpha/2}/2 \geqslant \ell c_E \eta^{\alpha/2}/2. \end{aligned} \tag{5.12}$$

Hence, $\ell \leqslant C_{E,\eta}$. By Lemma 5.2 and (5.8),

$$\begin{aligned} C_E &\geqslant \iint_{\mathbb{R}^{n+1}} \frac{\langle t - t_{k_1} \rangle^2 G(|u|^2)}{\langle t - t_{k_1} \rangle^3 + |x - x_{k_1}|^3} \, dx \, dt \\ &\geqslant c_\eta \iint_{|x - x_{k_1}| \leqslant C_\eta(1+|t - t_{k_1}|)} \frac{|u|^{2+\alpha}}{\langle t - t_{k_1} \rangle} \, dx \, dt \\ &\geqslant c_\eta \sum_{j \in \mathbb{N}_1} \int_{J_j} \frac{dt}{\langle t - t_{k_1} \rangle} \left\| u(t) \right\|^{2+\alpha}_{L^{2+\alpha}(|x - x_{k_1}| \leqslant C_\eta(1+|t - t_{k_1}|))} \\ &\geqslant c_{E,\eta} \sum_{j \in \mathbb{N}_1} \frac{1}{1 + |t_j - t_{k_1}|} \geqslant c_{E,\eta} \sum_{j \in \mathbb{N}_1} \frac{1}{1 + |t_j|}. \end{aligned} \tag{5.13}$$

Replacing (t_{k_1}, x_{k_1}) by (t_{k_i}, x_{k_i}) in (5.13), and noticing that $k_1, \dots, k_\ell$ are finitely many, we immediately obtain that

$$C_{E,\eta} \geqslant \sum_{j \in \Lambda} \frac{1}{1 + |t_j|}. \tag{5.14}$$

Hence, (5.3) follows. □

Lemma 5.4. *Let $E > 0$, $M > 0$, $M \leqslant E$ be fixed constants. Let $\eta > 0$ be a small number, say $C_E \eta^\alpha \leqslant 1/2$. Let $L > 1$ be an arbitrarily large number. If the energy solution u of* (1.1) *and* (1.2) *satisfies $M(u) \leqslant M$, $E(u) \leqslant E$ and*

$$\|u\|_{L^p_{x,t \in [A,B]}} \gg C_{L,E}$$

for some interval $[A, B] \subset \mathbb{R}$, then there exist $[t_0, d] \subset [A, B]$ (or $[d, t_0] \subset [A, B]$), $R > 1$, $|d - t_0| > LR$, $x_0 \in \mathbb{R}^n$ satisfying the following properties:

$$\|u\|_{L^p_{x,t \in [t_0,d]}} \leqslant \eta, \qquad \|u\|_{L^{2+4/n}_{x,t \in [t_0,d]}} + \|D_x u\|_{L^{2+4/n}_{x,t \in [t_0,d]}} \leqslant \eta,$$

$$\left\| u(t_0)/\langle x - x_0 \rangle \right\|_{L^2(\mathbb{R}^n)} \leqslant \eta^5, \qquad \left\| u(t_0) \right\|_{L^2(|x - x_0| \leqslant R)} \geqslant c_E \eta^{\alpha/2}.$$

Proof. By Lemma 5.3, if G is sufficiently large, then there exists an $I_j := [b', b]$, $|I_j| > C_{E,\eta} L^{C_{E,\eta}}$. Assume without loss of generality that

$$|b - t_j| \geqslant \frac{1}{2}|I_j| > C_{E,\eta} L^{C_{E,\eta}}. \tag{5.15}$$

One can divide $[t_j, b]$ into

$$[t_j, b] = \bigcup_{a \in \Lambda_j} I_j^a, \tag{5.16}$$

such that

$$\|u\|_{L^{2+4/n}_{x,t\in I_j^a}} + \|D_x u\|_{L^{2+4/n}_{x,t\in I_j^a}} = \eta. \tag{5.17}$$

Due to $\|u\|_{L^{2+4/n}_{x,t\in I_j}} + \|D_x u\|_{L^{2+4/n}_{x,t\in I_j}} \leqslant 2CE$, we see that Λ_j has at most $O(C_E/\eta^{2+4/n})$ elements. Assume that

$$I_j^a = \left[t_j^a, t_j^{a+1}\right], \qquad t_j^0 = t_j. \tag{5.18}$$

Using a decaying estimate (cf. [18, Lemma 5.3]),

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} \frac{|u(t,x)|^2}{\langle x - x_j \rangle} \, dx \right)^{\alpha+2} \frac{dt}{\langle t - t_j^a \rangle} \leqslant C_E, \tag{5.19}$$

we see that there exists $T_j^a > t_j^a$ verifying

$$\left\| u\left(T_j^a\right) / \langle x - x_j \rangle \right\|_{L^2(\mathbb{R})} < \eta^5, \qquad T_j^a - t_j^a < e^{C_E/\eta^{5(2+\alpha)}} := C_{E,\eta}. \tag{5.20}$$

We show that there exists at least an $a \in \Lambda_j$ satisfying

$$t_j^{a+1} - T_j^a > LC_\eta\left(1 + T_j^a - t_j\right). \tag{5.21}$$

Assume for the contrary that (5.21) does not hold:

$$t_j^{a+1} - T_j^a \leqslant LC_\eta\left(1 + T_j^a - t_j\right), \quad \forall a \in \Lambda_j, \tag{5.22}$$

which implies that for all $a \in \Lambda_j$,

$$t_j^{a+1} - t_j^a \leqslant LC_{E,\eta} + LC_\eta\left(T_j^a - t_j\right), \tag{5.23}$$

by induction,

$$t_j^{a+1} - t_j^a \leqslant L^a C_{E,\eta}. \tag{5.24}$$

Hence,

$$b - t_j \leqslant L^{C_{E,\eta}} C_{E,\eta}, \tag{5.25}$$

which contradicts the choice of I_j.

Therefore, we have found a subinterval $J_j := [T_j^a, t_j^{a+1}] := [T_j, d] \subset I_j$ and $R := C_\eta(1 + T_j - t_j) > 1$, $|J_j| > LR$ satisfying

$$\begin{cases} \|u\|_{L^{2+4/n}_{x,t\in J_j}} + \|D_x u\|_{L^{2+4/n}_{x,t\in J_j}} \leqslant \eta, \\ \|u(T_j)/\langle x - x_j\rangle\|_{L^2(\mathbb{R}^n)} \leqslant \eta^5, \\ \|u(T_j)\|_{L^2(|x-x_j|\leqslant R)} \geqslant c_E \eta^{\alpha/2}, \end{cases} \tag{5.26}$$

which implies the result, as desired. □

6. Induction on $M(u)$

Let $M > 0$, $E > 0$, $M \leqslant E$. Let u be the energy solution of (1.1) and (1.2) with initial data u_0 at $t = t_0$. In Section 2 we have shown that

$$M(u_0) \leqslant \eta^{\alpha+1/2} \quad \text{and} \quad E(u_0) \leqslant E \quad \Longrightarrow \quad \|u\|_{L^p_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{6.1}$$

By induction hypothesis, we assume that the following claim holds:

$$M(u_0) \leqslant M - \eta^{\alpha+1/2} \quad \text{and} \quad E(u_0) \leqslant E \quad \Longrightarrow \quad \|u\|_{L^p_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{6.2}$$

Our aim of this and the next sections is to show the following conclusion:

Lemma 6.1. *Assume that $M(u_0) \leqslant M$ and $E(u_0) \leqslant E$. Then we have*

$$\|u\|_{L^p_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{6.3}$$

Recalling that $p = (n+2)^2/n$, from Lemmas 3.1 and 6.1 one can easily deduce the result of Theorem 1.1.

Assume for the contrary that (6.3) does not hold, that is,

$$\sup\bigl\{\|u\|_{L^p_{x,t\in\mathbb{R}}}:\ M(u_0) \leqslant M,\ E(u_0) \leqslant E\bigr\} = \infty, \tag{6.4}$$

where u is taken over all energy solutions of (1.1) and (1.2).

Now we connect our discussions with Lemma 5.4. Let E, L be as in Lemma 5.4 and assume that u is a solution satisfying $\|u\|_{L^p_{x,t\in\mathbb{R}}} \gg C_{E,L}$. One easily sees that there exists a decomposition of $\mathbb{R}$, $\mathbb{R} = (-\infty, A] \cup [A, B] \cup [B, \infty)$ verifying $\|u\|_{L^p_{x,t\in\mathcal{J}}} \gg C_{E,L}$ for $\mathcal{J} = (-\infty, A]$, $[A, B]$ and $[B, \infty)$.

Let $\zeta : \mathbb{R}^n \to [0, 1]$ be a smooth radial bump function, say

$$\zeta = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| > 2. \end{cases} \tag{6.5}$$

Let R be as in Lemma 5.4. Put $\zeta_R = \zeta((x - x_0)/R)$,

$$v(t_0) = u(t_0)\zeta_R. \tag{6.6}$$

Considering the free Cauchy problem

$$\mathrm{i}v_t + \Delta v = 0, \qquad v|_{t=t_0} = v(t_0). \tag{6.7}$$

Lemma 6.2. *Let u be the solution of* (1.1) *and* (1.2) *as in Lemma* 5.4*. Then for the solution v of* (6.7)*, we have*

$$\|v\|_{L^{2+4/n}_{x,t\in J}} + \|D_x v\|_{L^{2+4/n}_{x,t\in J}} \leqslant C_\zeta \eta, \tag{6.8}$$

where η is the same one as in Lemma 5.4.

Proof. We follow Bourgain [3]. By Strichartz's inequality, (3.4), (3.11) and Lemma 5.4,

$$\begin{aligned}\big\|S(t-t_0)u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} &\leqslant \left\| u - \mathrm{i}\int_{t_0}^{t} S(t-\tau) f\big(u(\tau)\big)\, d\tau \right\|_{L^{2+4/n}_{x,t\in J}} \\ &\leqslant \|u\|_{L^{2+4/n}_{x,t\in J}} + C\big\|f(u)\big\|_{L^{(2+4/n)'}_{x,t\in J}} \leqslant \eta + C_E \eta^{2+\alpha} \leqslant 2\eta. \end{aligned}\tag{6.9}$$

We have

$$\begin{aligned}\big|e^{\mathrm{i}(t-t_0)\Delta} v(t_0)\big| &= \big|\mathcal{F}^{-1}\big[e^{\mathrm{i}(t-t_0)|\xi|^2}\big(\hat{\zeta}_R * u(t_0)(\xi)\big)\big]\big| \\ &= \left| \int_{\mathbb{R}^n} \hat{\zeta}_R(\xi_1)\, d\xi_1 \int_{\mathbb{R}^n} e^{\mathrm{i}(t-t_0)|\xi|^2 + \mathrm{i}x\xi} \hat{u}(\xi - \xi_1)\, d\xi \right|. \end{aligned}\tag{6.10}$$

Using Minkowski's inequality, we have

$$\begin{aligned}\big\|S(t-t_0)v(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} &\leqslant \int_{\mathbb{R}^n} \big|\hat{\zeta}_R(\xi_1)\big| \left\| \int_{\mathbb{R}^n} e^{\mathrm{i}(t-t_0)|\xi|^2+\mathrm{i}x\xi} \hat{u}(\xi-\xi_1)\, d\xi \right\|_{L^{2+4/n}_{x,t\in J}} d\xi_1 \\ &\leqslant \big\|\hat{\zeta}_R\big\|_1 \sup_{\xi_1} \left\| \int_{\mathbb{R}^n} e^{\mathrm{i}(t-t_0)|\xi|^2+\mathrm{i}x\xi} \hat{u}(\xi-\xi_1)\, d\xi \right\|_{L^{2+4/n}_{x,t\in J}}. \end{aligned}\tag{6.11}$$

Taking notice of $\|\hat{\zeta}_R\|_{L^1} = \|\hat{\zeta}\|_{L^1}$ and

$$\left| \int_{\mathbb{R}^n} e^{\mathrm{i}(t-t_0)|\xi|^2+\mathrm{i}x\xi} \hat{u}(\xi-\xi_1)\, d\xi \right| = \big|\big(e^{\mathrm{i}(t-t_0)\Delta} u(t_0)\big)\big(x + 2(t-t_0)\xi_1\big)\big|, \tag{6.12}$$

we immediately have

$$\big\|S(t-t_0)v(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} \leqslant C_\zeta \big\|S(t-t_0)u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} \leqslant 2C_\zeta \eta. \tag{6.13}$$

Similarly, we can estimate $\|D_x v\|_{L^{2+4/n}_{x,t\in J}}$. Indeed,

$$\big\|D_x S(t-t_0)v(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} \leqslant \big\|S(t-t_0)(D_x\zeta_R)u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} + \big\|S(t-t_0)\zeta_R\big(D_x u(t_0)\big)\big\|_{L^{2+4/n}_{x,t\in J}}. \tag{6.14}$$

Analogous to (6.9), we have

$$\big\|D_x S(t-t_0)u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} \leqslant \|D_x u\|_{L^{2+4/n}_{x,t\in J}} + \big\|D_x f(u)\big\|_{L^{(2+4/n)'}_{x,t\in J}} \leqslant \eta + C_E\eta^{2+\alpha} \leqslant 2\eta. \tag{6.15}$$

Using the same way as in (6.13),

$$\big\|S(t-t_0)\zeta_R D_x u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} \leqslant \|\zeta_R\|_{L^1(\mathbb{R}^n)}\big\|S(t-t_0)D_x u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}}. \tag{6.16}$$

Due to $\partial_x\zeta_R = \zeta'((x-x_0)/R)/R$ for $n=1$, $\partial_{x_i}\zeta_R = \zeta'((x-x_0)/R)/R$ for $n=2$, we see that

$$\big\|S(t-t_0)(D_x\zeta_R)u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} \leqslant \frac{C_\zeta}{R}\big\|\hat{\zeta}'_R\big\|_{L^1}\big\|S(t-t_0)u(t_0)\big\|_{L^{2+4/n}_{x,t\in J}} \leqslant \frac{C_\zeta\eta}{R}. \tag{6.17}$$

Since $R>1$, it follows from (6.14), (6.16) and (6.17) that $\|D_x v\|_{L^{2+4/n}_{x,t\in J}} \leqslant C_\zeta\eta$. $\square$

Now let u and v be the solution of (1.1) in Lemma 5.4 and (6.17), respectively. Let $w=u-v$. One has that

$$\begin{cases} \mathrm{i}w_t + \Delta w - f(v+w) = 0, \\ w|_{t=t_0} = (1-\zeta_R)u(t_0). \end{cases} \tag{6.18}$$

Recall that $J=[t_0,d]$ (or $J=[d,t_0]$). We now estimate $\|w(d)\|_{L^2(\mathbb{R}^n)}$. By (6.18), we have

$$\big\|w(d)\big\|_2^2 = \big\|w(t_0)\big\|_2^2 + 2\int_{t_0}^{d}\int_{\mathbb{R}^n}\Im f(v+w)\overline{w}\,dx\,dt. \tag{6.19}$$

It is easy to see that $\Im f(v+w)\overline{w} = \Im f(v+w)\bar{v}$. For $n=1$, by Lemmas 5.4 and 6.2, we have

$$\begin{aligned} \left|\int_{t_0}^{d}\int_{\mathbb{R}}\Im f(v+w)\bar{v}\,dx\,dt\right| &\leqslant \sum_{k=3}^{\infty}\int_{t_0}^{d}\int_{\mathbb{R}}\frac{\lambda_k^k}{k!}|u|^{2k+1}|v|\,dx\,dt \\ &\leqslant C_E\|u\|^6_{L^9_{x,t\in J}}\|u\|_{L^6_{x,t\in J}}\|v\|_{L^6_{x,t\in J}} \leqslant C_E\eta^8. \end{aligned} \tag{6.20}$$

For $n=2$, in a similar way as in (3.15), one has that

$$\left|\int_{t_0}^{d}\int_{\mathbb{R}^2}\Im f(v+w)\bar{v}\,dx\,dt\right| \leqslant \sum_{k=2}^{\infty}\frac{\lambda_k^k}{k!}\|u\|_{L^{4k}_{x,t\in J}}^{2k}\|u\|_{L^4_{x,t\in J}}\|v\|_{L^4_{x,t\in J}}$$
$$\leqslant C_E\|u\|^4_{L^8_{x,t\in J}}\|u\|_{L^4_{x,t\in J}}\|v\|_{L^4_{x,t\in J}}. \tag{6.21}$$

Invoking Lemmas 5.4 and 6.2, we have from (6.21) that

$$\left|\int_{t_0}^{d}\int_{\mathbb{R}^2}\Im f(v+w)\bar{v}\,dx\,dt\right| \leqslant C_E\eta^6. \tag{6.22}$$

Summarizing (6.20) and (6.22), we have

$$\left|\int_{t_0}^{d}\int_{\mathbb{R}^n}\Im f(v+w)\overline{w}\,dx\,dt\right| \leqslant C_E\eta^{2+\alpha}. \tag{6.23}$$

Since

$$\|w(t_0)\|_2^2 \leqslant \|u(t_0)\|_2^2 - \|u(t_0)\|^2_{L^2(|x-x_0|<R)} \leqslant M - C_E\eta^\alpha, \tag{6.24}$$

from (6.19), (6.23) and (6.24) we have

$$\|w(d)\|_2^2 \leqslant M - C_E\eta^\alpha + C_E\eta^{2+\alpha} \leqslant M - \eta^{\alpha+1/2}. \tag{6.25}$$

We now estimate $H(w(t_0))$:

$$H\big(w(t_0)\big) \leqslant \frac{1}{2}\big\|\big(\nabla u(t_0)\big)(1-\zeta_R)\big\|_2^2 + \frac{1}{2}\|u(t_0)\nabla\zeta_R\|_2^2 + \int_{\mathbb{R}^n} F\big(|w(t_0,x)|^2\big)\,dx$$
$$\leqslant \frac{1}{2}\|\nabla u(t_0)\|_2^2 + C_\zeta\|u(t_0)/\langle x-x_0\rangle\|_2^2 + \int_{\mathbb{R}^n} F\big(|w(t_0,x)|^2\big)\,dx$$
$$\leqslant H\big(u(t_0)\big) + C_\zeta\eta^{10}. \tag{6.26}$$

Since w satisfies (6.18), we see that

$$H\big(w(d)\big) - H\big(w(t_0)\big) = \int_{t_0}^{d} 2\Re\big(\mathrm{i}w_t + \Delta w - f(w), w_t\big)\,dt$$
$$= \int_{t_0}^{d}\big[2\Re\big(f(u)-f(w), \mathrm{i}\Delta w\big) + 2\Re\big(f(w), -\mathrm{i}f(u)\big)\big]\,dt. \tag{6.27}$$

Integrating by part and then using the same ways as in (6.20) and (6.21), we conclude that

$$\left|\int_{t_0}^{d} 2\Re\big(f(u), \mathrm{i}\Delta w\big)\,dt\right| \leqslant C_E \|u\|^{\alpha}_{L^p_{x,t\in J}} \|\nabla u\|_{L^{2+4/n}_{x,t\in J}} \|\nabla w\|_{L^{2+4/n}_{x,t\in J}}. \tag{6.28}$$

Noticing that $w = u - v$, by Lemmas 5.4 and 6.2, it follows from (6.28) that

$$\left|\int_{t_0}^{d} 2\Re\big(f(u), \mathrm{i}\Delta w\big)\,dt\right| \leqslant C_E \eta^{2+\alpha}. \tag{6.29}$$

Analogous to (6.29), we have

$$\left|\int_{t_0}^{d} 2\Re\big(f(w), \mathrm{i}\Delta w\big)\,dt\right| \leqslant C_E \eta^{2+\alpha}. \tag{6.30}$$

We now estimate the last term in the RHS of (6.27). For $n = 1$,

$$\begin{aligned}\int_{t_0}^{d}\int_{\mathbb{R}} \big|f(w)f(u)\big|\,dx\,dt &\leqslant \sum_{k,\ell\geqslant 3} \frac{\lambda_k^k}{k!}\frac{\lambda_\ell^\ell}{\ell!} \|u\|^{2k-6}_{L^\infty_{x,t\in J}} \|w\|^{2\ell}_{L^\infty_{x,t\in J}} \|u\|^6_{L^9_{x,t\in J}} \|w\|_{L^6_{x,t\in J}} \|u\|_{L^6_{x,t\in J}} \\ &\leqslant \sum_{k,\ell\geqslant 3} \frac{\lambda_k^k}{k!}\frac{\lambda_\ell^\ell}{\ell!} (CE)^{2k}(CE)^{2\ell}\eta^8 \leqslant C_E\eta^8. \end{aligned} \tag{6.31}$$

For $n = 2$,

$$\begin{aligned}\int_{t_0}^{d}\int_{\mathbb{R}^2} \big|f(w)f(u)\big|\,dx\,dt &\leqslant \sum_{k,\ell\geqslant 2} \frac{\lambda_k^k}{k!}\frac{\lambda_\ell^\ell}{\ell!} \|u\|^{2k}_{L^{8k}_{x,t\in J}} \|w\|^{2\ell}_{L^{8\ell}_{x,t\in J}} \|u\|_{L^4_{x,t\in J}} \|w\|_{L^4_{x,t\in J}} \\ &\leqslant C\eta^2 \sum_{k,\ell\geqslant 2} \frac{\lambda_k^k}{k!}\frac{\lambda_\ell^\ell}{\ell!} (8k)^{k+1}(8\ell)^{\ell+1} C^{2k}C^{2\ell} \\ &\quad \times \|u\|^{2k-2}_{L^\infty(J,H^1)} \|w\|^{2\ell-2}_{L^\infty(J,H^1)} \|u\|^2_{L^8_{x,t\in J}} \|w\|^2_{L^8_{x,t\in J}}. \end{aligned} \tag{6.32}$$

Noticing that $\lambda_k < c/E^2$ as $k \gg 1$, we see that

$$\int_{t_0}^{d}\int_{\mathbb{R}^2} \big|f(w)f(u)\big|\,dx\,dt \leqslant C_E\eta^6. \tag{6.33}$$

By (6.31) and (6.33), one has that

$$\left| \int_{t_0}^{d} 2\Re\big(f(w), \mathrm{i} f(u)\big)\, dt \right| \leqslant C_E \eta^{2+\alpha}. \tag{6.34}$$

Collecting (6.26), (6.27), (6.29) and (6.34), we have

$$H\big(w(d)\big) \leqslant H\big(w(t_0)\big) + C_\zeta \eta^{10} + C_E \eta^{2+\alpha}. \tag{6.35}$$

Hence,

$$E\big(w(d)\big) \leqslant E\big(u(t_0)\big) - C_E \eta^{\alpha} + C_\zeta \eta^{10} + C_E \eta^{2+\alpha} \leqslant E. \tag{6.36}$$

By (6.25) and (6.36), and the induction hypothesis, we get:

Lemma 6.3. *The solution $\widetilde{w}$ of the following problem*

$$\mathrm{i}\widetilde{w}_t + \Delta \widetilde{w} - f(\widetilde{w}) = 0, \qquad \widetilde{w}|_{t=d} = w(d) \tag{6.37}$$

satisfies a uniform estimate

$$\|\widetilde{w}\|_{L^p_{x,t\in\mathbb{R}}} \leqslant C_E < \infty. \tag{6.38}$$

7. Perturbation analysis

Our goal of this section is to show that (6.4) will lead to a contradiction, which implies the result as in Lemma 6.1. The main technique is to use Bourgain's perturbation analysis argument as in [2].

Proof of Lemma 6.1. Let u, v and $\widetilde{w}$ be the solutions of (1.1), (6.7) and (6.37), respectively. Denote

$$\delta = u - v - \widetilde{w}. \tag{7.1}$$

It is easy to see that δ satisfies

$$\mathrm{i}\delta_t + \Delta\delta - f(u) + f(\widetilde{w}) = 0, \qquad \delta|_{t=d} = 0. \tag{7.2}$$

We will show that if κ is small enough, then $\|\delta\|_{L^p_{x,t\in[d,\infty)}}$ is also small, whence $\|u\|_{L^p_{x,t\in[d,\infty)}} \leqslant C_E$. But this contradicts (6.4).

One can rewrite (7.2) as an integral equation:

$$\delta(t) = \mathrm{i} \int_{d}^{t} S(t-\tau)\Big(f\big(u(\tau)\big) - f\big(\widetilde{w}(\tau)\big)\Big)\, d\tau. \tag{7.3}$$

Recall that

$$\big\|S(t-t_0)v(t_0)\big\|_{L^{2+4/n}_{x,t\in[d,\infty)}} \leqslant \big\|v(t_0)\big\|_2 \leqslant CE, \tag{7.4}$$

$$\big\|S(t-t_0)v(t_0)\big\|_\infty \leqslant |t-t_0|^{-n/2}R^{n/2}\big\|v(t_0)\big\|_2. \tag{7.5}$$

It follows from (7.5) and $|J| = d - t_0 > LR$ that

$$\big\|S(t-t_0)v(t_0)\big\|_{L^\infty_{x,t\in[d,\infty)}} \leqslant C_E L^{-n/2}. \tag{7.6}$$

Interpolating $L^p_{x,t\in[d,\infty)}$ between $L^\infty_{x,t\in[d,\infty)}$ and $L^{2+4/n}_{x,t\in[d,\infty)}$, we get

$$\big\|S(t-t_0)v(t_0)\big\|_{L^p_{x,t\in[d,\infty)}} \leqslant C_E L^{-n/\alpha} := \kappa. \tag{7.7}$$

By Lemmas 6.3 and 3.1, we can split $[d,\infty)$ into consecutive intervals

$$[d,\infty) = \bigcup_{j=1}^{K} I_j \tag{7.8}$$

such that

$$\|\widetilde{w}\|_{L^{2+4/n}_{x,t\in I_j}} + \|\widetilde{w}\|_{L^p_{x,t\in I_j}} \leqslant \gamma < 1, \qquad C_E\gamma^{\alpha/2} \leqslant 1/4. \tag{7.9}$$

Assume that

$$I_j = [d_j, d_{j+1}], \qquad d_0 = d, \quad j = 1,\dots,K. \tag{7.10}$$

Let us rewrite (7.3) as

$$\delta(t) = S(t-d_j)\delta(d_j) + \mathrm{i}\int_{d_j}^{t} S(t-\tau)\big(f\big(u(\tau)\big) - f\big(\widetilde{w}(\tau)\big)\big)\,d\tau. \tag{7.11}$$

Using the Strichartz estimates, we have

$$\|\delta\|_{L^{2+4/n}_{x,t\in I_j}\cap L^\infty(I_j,L^2)} \leqslant C\big\|\delta(d_j)\big\|_2 + C\big\|f(u) - f(\widetilde{w})\big\|_{L^{(2+4/n)'}_{x,t\in I_j}}. \tag{7.12}$$

Let us observe the following identity

$$|a|^{2k}a - |b|^{2k}b = a^{k+1}\big(\bar{a}^k - \bar{b}^k\big) + \big(a^{k+1} - b^{k+1}\big)\bar{b}^k, \tag{7.13}$$

$$\big(\bar{a}^k - \bar{b}^k\big) = \big(\bar{a} - \bar{b}\big)\sum_{i=0}^{k-1}\bar{a}^{k-1-i}b^i. \tag{7.14}$$

It follows from (7.13) and (7.14) that

$$|u|^{2k}u-|\widetilde{w}|^{2k}\widetilde{w}=(v+\widetilde{w}+\delta)^k\big(\bar{v}+\bar{\delta}\big)\sum_{i=0}^{k-1}\big(\bar{v}+\overline{\widetilde{w}}+\bar{\delta}\big)^{k-1-i}\,\overline{\widetilde{w}}^{\,i} + \overline{\widetilde{w}}^{\,k}(v+\delta)\sum_{i=0}^{k}(v+\widetilde{w}+\delta)^{k-i}\,\widetilde{w}^{\,i}. \tag{7.15}$$

By (7.15) we have

$$\big||u|^{2k}u-|\widetilde{w}|^{2k}\widetilde{w}\big|\leqslant 3^{2k+2}\big(|v|^{2k}+|\widetilde{w}|^{2k}+|\delta|^{2k}\big)\big(|v|+|\delta|\big). \tag{7.16}$$

In view of (7.16) one has that

$$\begin{aligned}\big\|f(u)-f(\widetilde{w})\big\|_{L^{(2+4/n)'}_{x,t\in I_j}} &= \sum_{k\geqslant k(n)}\frac{\lambda_k^k}{k!}3^{2k+2}\Big(\big\|\big(|v|^{2k}+|\widetilde{w}|^{2k}+|\delta|^{2k}\big)v\big\|_{L^{(2+4/n)'}_{x,t\in I_j}} \\ &\quad + \big\|\big(|v|^{2k}+|\widetilde{w}|^{2k}+|\delta|^{2k}\big)\delta\big\|_{L^{(2+4/n)'}_{x,t\in I_j}}\Big) \\ &:= \sum_{k\geqslant k(n)}\frac{\lambda_k^k}{k!}3^{2k+2}\big(\Gamma_k^1+\Gamma_k^2\big). \end{aligned} \tag{7.17}$$

First, we consider the case $n=1$.

$$\begin{aligned}\Gamma_k^1 &\leqslant \|\widetilde{w}\|^{2k-6}_{L^\infty_{x,t\in I_j}}\|\widetilde{w}\|^5_{L^9_{x,t\in I_j}}\|\widetilde{w}\|_{L^6_{x,t\in I_j}}\|v\|_{L^9_{x,t\in I_j}}+\|v\|^{2k-6}_{L^\infty_{x,t\in I_j}}\|v\|^6_{L^9_{x,t\in I_j}}\|v\|_{L^6_{x,t\in I_j}} \\ &\quad+\|\delta\|^{2k-4}_{L^\infty_{x,t\in I_j}}\|\delta\|^4_{L^6_{x,t\in I_j}}\|v\|_{L^6_{x,t\in I_j}}, \end{aligned} \tag{7.18}$$

$$\Gamma_k^2 \leqslant \|v\|^{2k-6}_{L^\infty_{x,t\in I_j}}\|v\|^6_{L^9_{x,t\in I_j}}\|\delta\|_{L^6_{x,t\in I_j}}+\|\widetilde{w}\|^{2k-6}_{L^\infty_{x,t\in I_j}}\|\widetilde{w}\|^6_{L^9_{x,t\in I_j}}\|\delta\|_{L^6_{x,t\in I_j}}+\|\delta\|^{2k-4}_{L^\infty_{x,t\in I_j}}\|\delta\|^5_{L^6_{x,t\in I_j}}. \tag{7.19}$$

We write

$$\varepsilon_j=\|\delta\|_{L^{2+4/n}_{x,t\in I_j}\cap C(I_j,L^2_x)}. \tag{7.20}$$

Using the following facts:

$$\begin{aligned}&\|v\|_{L^\infty_{x,t\in I_j}}\leqslant\kappa^3,\qquad \|v\|_{L^9_{x,t\in I_j}}\leqslant\kappa,\qquad \|v\|_{L^6_{x,t\in I_j}}\leqslant CE,\\ &\|\delta\|_{L^\infty_{x,t\in I_j}}\leqslant\|u-v-\widetilde{w}\|_{L^\infty_{x,t\in I_j}}\leqslant CE,\end{aligned} \tag{7.21}$$

together with (7.9), we have

$$\Gamma_k^1 \leqslant \kappa^6 + (CE)^{2k}\gamma^6\kappa + (CE)^{2k}\varepsilon_j^4, \tag{7.22}$$

$$\Gamma_k^2 \leqslant \kappa^6\varepsilon_j + (CE)^{2k}\gamma^6\varepsilon_j + (CE)^{2k}\varepsilon_j^5. \tag{7.23}$$

Collecting (7.17), (7.22) and (7.23), we obtain that

$$\big\| f(u) - f(\widetilde{w})\big\|_{L^{6/5}_{x,t\in I_j}} \leqslant C_E\big(\kappa^6 + \gamma^6\kappa + \varepsilon_j^4 + \big(\kappa^6 + \gamma^6 + \varepsilon_j^4\big)\varepsilon_j\big). \tag{7.24}$$

Hence, by (7.12) and (7.24) ($C_E\kappa^5 \leqslant 1/4$),

$$\varepsilon_j \leqslant 2C\varepsilon_{j-1} + \kappa + C_E\big(\varepsilon_j^4 + \varepsilon_j^5\big), \qquad \varepsilon_0 = 0, \quad j = 1, \dots, K. \tag{7.25}$$

By choosing κ is sufficiently small, say $(4C)^{K+1}\kappa \ll 1$, we get that $\varepsilon_j \leqslant (4C)^j\kappa$, $j = 1, \dots, K$. Hence,

$$\|\delta\|_{L^6_{x,t\in[d,\infty)}} < 1. \tag{7.26}$$

By (7.21) and (7.26), we also have

$$\|\delta\|_{L^p_{x,t\in[d,\infty)}} < C_E. \tag{7.27}$$

Hence, we have from $u = v + \delta + \widetilde{w}$ that

$$\|u\|_{L^p_{x,t\in[d,\infty)}} \leqslant C_E. \tag{7.28}$$

But this contradicts the fact $\|u\|_{L^p_{x,t\in[B,\infty)}} \gg C_E$.

Next, we consider the case $n = 2$. We have from (7.6), (7.7) and (7.4) that

$$\|v\|_{L^\infty_{x,t\in I_j}} \leqslant \kappa^2, \qquad \|v\|_{L^8_{x,t\in I_j}} \leqslant \kappa, \qquad \|v\|_{L^4_{x,t\in I_j}} \leqslant CE,$$
$$\big\|\delta(t)\big\|_{H^1} \leqslant \big\|(u - v - \widetilde{w})(t)\big\|_{H^1} \leqslant CE, \quad t \geqslant d. \tag{7.29}$$

Let ε_j be as in (7.20). Using Hölder's inequality and Proposition 2.3,

$$\begin{aligned}\Gamma_k^1 &\leqslant \|\widetilde{w}\|^{2k-1}_{L^{8(2k-1)/3}_{x,t\in I_j}}\|\widetilde{w}\|_{L^4_{x,t\in I_j}}\|v\|_{L^8_{x,t\in I_j}} + \|v\|^{2k-4}_{L^\infty_{x,t\in I_j}}\|v\|^4_{L^8_{x,t\in I_j}}\|v\|_{L^4_{x,t\in I_j}} + \|\delta\|^{2k}_{L^{4k}_{x,t\in I_j}}\|v\|_{L^4_{x,t\in I_j}}\\ &\leqslant \kappa^3 + (6k)^{k+1}(CE)^{2k}\gamma^4\kappa + (4k)^{k+1}(CE)^{2k+1}\varepsilon_j^2,\end{aligned} \tag{7.30}$$

$$\begin{aligned}\Gamma_k^2 &\leqslant \big(\|v\|^{2k-4}_{L^\infty_{x,t\in I_j}}\|v\|^4_{L^8_{x,t\in I_j}} + \|\widetilde{w}\|^{2k}_{L^{4k}_{x,t\in I_j}} + \|\delta\|^{2k}_{L^{4k}_{x,t\in I_j}}\big)\|\delta\|_{L^4_{x,t\in I_j}}\\ &\leqslant \kappa^4\varepsilon_j + (4k)^{k+1}(CE)^{2k}\big(\gamma^2\varepsilon_j + \varepsilon_j^3\big).\end{aligned} \tag{7.31}$$

Inserting (7.30) and (7.31) into (7.17) and noticing that $\lambda_k < c/E^2$, we have

$$\left\| f(u) - f(\widetilde{w}) \right\|_{L^{4/3}_{x,t\in I_j}} \leqslant C_E\left(\kappa^3 + \gamma^4\kappa + \varepsilon_j^2 + \left(\kappa^4 + \gamma^2 + \varepsilon_j^2\right)\varepsilon_j\right). \tag{7.32}$$

Hence ($C_E\kappa^2 \leqslant 1/4$),

$$\varepsilon_j \leqslant 2C\varepsilon_{j-1} + \kappa + C_E\left(\varepsilon_j^2 + \varepsilon_j^3\right), \qquad \varepsilon_0 = 0, \quad j = 1, \ldots, K. \tag{7.33}$$

By the same reason as above, we have

$$\|u\|_{L^8_{x,t\in[d,\infty)}} \leqslant C_E. \tag{7.34}$$

But this contradicts the fact $\|u\|_{L^8_{x,t\in[B,\infty)}} \gg C_E$. This finishes the proof of Lemma 6.1. □

As indicated in Section 6, it follows from Lemma 6.1 that Theorem 1.1 holds true. It is easy to see that the arguments in Sections 4–7 can be developed to the nonlinearity as in (1.2a), which implies that Theorem 1.3 holds.

Final remark. The idea of this paper can also be developed to the modified sinh-Gordon equation $u_{tt} - \Delta u + (\sinh u - u^3/3! - u^5/5!) = 0$.

Acknowledgments

Special thanks are due to the referee for his/her valuable suggestions and comments, which are helpful for us correcting some vague expressions in the original manuscript. This work was partially supported by the National Science Foundation of China, grant 10571004.

References

[1] J. Bergh, J. Löfström, Interpolation Spaces, Springer-Verlag, 1976.
[2] J. Bourgain, Global Solutions of Nonlinear Schrödinger Equations, Amer. Math. Soc. Colloq. Publ., vol. 46, Amer. Math. Soc., Providence, RI, 1999.
[3] J. Bourgain, Global well-posedness of defocussing critical nonlinear Schrödinger equation in the radial case, J. Amer. Math. Soc. 12 (1999) 145–171.
[4] T. Cazenave, F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Anal. 14 (1990) 807–836.
[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Ann. of Math., in press.
[6] J.E. Colliander, J.M. Delort, C.E. Kenig, G. Staffilani, Bilinear estimates and applications to 2D NLS, Trans. Amer. Math. Soc. 353 (2001) 3307–3325.
[7] J. Ginibre, G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, J. Math. Pures Appl. 64 (1985) 363–401.
[8] J. Ginibre, G. Velo, Time decay of finite energy solutions of the nonlinear Klein–Gordon and Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. 43 (1985) 399–442.
[9] M. Grillakis, On nonlinear Schrödinger equations, Comm. Partial Differential Equations 25 (2000) 1827–1844.
[10] B.L. Guo, B.X. Wang, The Cauchy problem for the Davey–Stewartson systems, Comm. Pure Appl. Math. 52 (1999) 1477–1490.
[11] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. 46 (1986) 113–129.
[12] M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998) 955–980.

[13] C.E. Kenig, G. Ponce, L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J. 40 (1991) 33–69.

[14] J.E. Lin, W. Strauss, Decay and scattering of solutions of a nonlinear Schrödinger equation, J. Funct. Anal. 30 (1978) 245–263.

[15] C. Morawetz, W. Strauss, Decay and scattering of solutions of a nonlinear relativistic wave equation, Comm. Pure Appl. Math. 25 (1972) 1–31; 26 (1973) 47–54.

[16] M. Nakamura, T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order, J. Funct. Anal. 155 (2) (1998) 364–380.

[17] M. Nakamura, T. Ozawa, Small solutions to nonlinear Schrödinger equations in the Sobolev spaces, J. Anal. Math. 81 (2000) 305–329.

[18] K. Nakanishi, Energy scattering for nonlinear Klein–Gordon and Schrödinger equations in spatial dimensions 1 and 2, J. Funct. Anal. 169 (1999) 201–225.

[19] K. Nakanishi, Remark on the energy scattering for nonlinear Klein–Gordon and Schrödinger equations, Tohoku Math. J. 53 (2001) 285–303.

[20] T. Ozawa, On the critical cases of Sobolev's inequality, J. Funct. Anal. 127 (1995) 259–269.

[21] E. Ryckman, M. Visan, Global well-posedness and scattering for the defocussing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, Amer. J. Math., in press.

[22] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, NJ, 1970.

[23] W. Strauss, Nonlinear Wave Equations, CBMS Lecture Notes, vol. 73, Amer. Math. Soc., Providence, RI, 1989.

[24] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.

[25] Y. Tsutsumi, Scattering problem for nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. 43 (1985) 321–347.

[26] M. Visan, The defocussing energy-critical nonlinear Schrödinger equation in higher dimensions, preprint, math.AP/0508298.

[27] B.X. Wang, The limit behavior of solutions for the Cauchy problem of the complex Ginzburg–Landau equation, Comm. Pure Appl. Math. 55 (2002) 481–508.

[28] B.X. Wang, Concentration phenomenon for the L^2 super critical nonlinear Schrödinger equation in energy spaces, Commun. Contemp. Math. 8 (2006) 309–330.

[29] B.X. Wang, Nonlinear scattering theory for a class of wave equations in H^s, J. Math. Anal. Appl. 296 (2004) 74–96.

[30] B.X. Wang, The smoothness of scattering operators for sinh-Gordon and nonlinear Schrödinger equations, Acta Math. Sin. (Engl. Ser.) 18 (2002) 549–564.