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EXCERPT OF DISSERTATION

Studies on Schrödinger-Poisson Systems

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Abstract The bipolar (defocusing nonlinear) Schrödinger-Poisson system and quasi-linear Schrödinger-Poisson equations are studied. The wellposedness, large time behavior and modified scattering theory is established for the Cauchy problem to the bipolar (defocusing nonlinear) Schrödinger-Poisson systems. The initial-(Dirichlet) boundary problem for a high field version of the Schrödinger-Poisson equations, quasi-linear Schrödinger-Poisson equations, which include a nonlinear term in the Poisson equation corresponding to a field-dependent dielectric constant and an effective potential in the Schrödinger equations on the unit cube are also discussed.

Key words the bipolar defocusing nonlinear Schrödinger-Poisson system, the quasi-linear Schrödinger-Poisson systems, wellposedness of Cauchy problems, initial boundary value problem

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1 Introduction

The Schrödinger-Poisson system is used to simulate the transport of charged particles in semiconductor science and plasma physics. We mainly discuss the following (pure) bipolar defocusing nonlinear Schrödinger-Poisson system (BDNLSP)

$$i\varepsilon\dot{\psi}_j = -\frac{\varepsilon^2}{2}\Delta\psi_j + (q_jV + h_j(|\psi_j|^2))\psi_j, \quad j = 1,2 \tag{1}$$

$$-\lambda^2\Delta V = |\psi_1|^2 - |\psi_2|^2, \tag{2}$$

with the initial data

$$\psi_j(0,\cdot) = \varphi_j, \quad j = 1,2, \tag{3}$$

where Δ denotes the Laplacian on $\mathbf{R}^d$ and $\dot{\psi}_j = \partial\psi_j/\partial t$, the wave functions $\psi_j = \psi_j(t,x): \mathbf{R}^{1+d} \to \mathbf{C}, j = 1,2$, describe the state of the particle in the position space under the action of the electrostatic potential $V = V(t,x)$ at every instant t. The nonlinear self-interacting potential $h_j(s)$ is assumed to be given by

$$h_j(s) = a_j^2 s^{\gamma_j}, \text{for } s \geq 0$$

and some

$$a_j > 0, \frac{2}{d} < \gamma_j < \alpha(d),$$

where $\alpha(d) = 2/(d-2)$ if $d \geq 3$ and $\alpha(d) = \infty$ if $d = 1,2$. The charges of the particles described by the wave functions ψ_j are defined by $q_1 = 1$ and $q_2 = -1$, respectively. ε is the scaled Planck constant and λ is the scaled Debye length.

A large amount of interesting work has been devoted to the study of the time-dependent[1~11] and time-independent[12~17] Schrödinger-Poisson systems. By applying the estimates of a modulated energy functional and the Wigner measure method, Jüngel and Wang[9] discussed the combined semi-classical and quasineutral limit of the (BDNLSP) with the initial data (3) in the whole space where $a_1 = a_2$ and $\gamma_1 = \gamma_2$ provided the solution of (1) ~ (3) exists. And Castella[4] proved the global existence and the asymptotic behavior of solutions in the function space L^2 for the mixed-state unipolar Schrödinger-Poisson systems without the defocusing nonlinearity.

2 Initial value problem without nonlinearity

We first study the global existence and uniqueness of solutions for the initial value problem to the following (pure state) bipolar Schrödinger-Poisson equations (BSP) without the defocusing nonlinearity

$$\mathrm{i}\partial_t\psi = -\Delta\psi + V\psi, \tag{4}$$

$$\mathrm{i}\partial_t\phi = -\Delta\phi - V\phi, \tag{5}$$

$$-\Delta V = |\psi|^2 - |\phi|^2, \tag{6}$$

$$\psi(0,x) = \psi_0, \phi(0,x) = \phi_0, \tag{7}$$

By using the dual space-time estimates, Strichartz' estimates and the properties of the Besov space, etc., we can obtain the following wellposedness theorem.

Theorem 2.1(Ref. [18]) Let $s\in\mathbf{R}, s\geqslant 0$. Let $a\in[2,18/7]$. Assume that $\psi_0, \phi_0 \in H^s(\mathbf{R}^3)$. Then, there exists a unique solution of the IVP (4) ~ (7) for which it holds

$$\psi,\phi \in C(\mathbf{R};H^s(\mathbf{R}^3)) \cap L^{\gamma(a)}_{loc}(\mathbf{R};B^s_{a,2}(\mathbf{R}^3))$$

where $2/\gamma(a) = 3(1/2 - 1/a)$.

Moreover, when s is an integer, the result also holds with the Besov space $B^s_{a,2}$ replaced by H^s_a.

3 Wellposedness to the BDNLSP system

Now we turn to the initial value problem in d-dimensional spatial space for (BDNLSP) system with assuming $\lambda = 1$ for simplicity.

We assume that the initial data

$$\varphi_j(x) \in \sum(\mathbf{R}^d) := \{u \in H^1(\mathbf{R}^d): |x|u \in L^2(\mathbf{R}^d)\} \ (j = 1,2),$$

with the norm

$$\|\psi_j\|_{\sum} = \|\psi_j\|_{H^1} + \||x|\psi_j\|_{L^2}.$$

We can get the following conservation laws.

Theorem 3.1 (Conservation laws Ref. [19]) Let $d\in\mathbf{N}$, $\{\psi_j\}$ be a solution to (BDNLSP) with the initial value $\varphi_j(x) \in \sum(\mathbf{R}^d)$. Then, we have the following conservation laws for all $t\in\mathbf{R}$

(i) L^2-norm law:

$$\|\psi_j(t)\|_2 = \|\varphi_j\|_2 \quad \text{for} \quad j = 1,2;$$

(ii) Energy conservation law:

$$\varepsilon^2\sum_{j=1}^{2}\|\nabla\psi_j(t)\|_2^2 + \lambda^2\|\nabla V\|_2^2 + 2\sum_{j=1}^{2}\frac{a_j^2}{\gamma_j+1}\|\psi_j(t)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} = \text{const};$$

(iii) Pseudo-conformal conservation law

$$\sum_{j=1}^{2}\|x\psi_j + \mathrm{i}\varepsilon t\nabla\psi_j\|_2^2 + \lambda^2t^2\|\nabla V\|_2^2 + 2t^2\sum_{j=1}^{2}\frac{a_j^2}{\gamma_j+1}\|\psi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)}$$

$$+2\sum_{j=1}^{2}\frac{a_j^2(d\gamma_j-2)}{\gamma_j+1}\int_0^t\tau\,\|\psi_j(\tau)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)}d\tau$$

$$=\sum_{j=1}^{2}\|\,|x|\varphi_j\|_2^2+(4-d)\lambda^2\int_0^t\tau\,\|\nabla V(\tau)\|_2^2d\tau.$$

By the above conservation laws, the L^p-L^q dual estimates and using the properties of the Galilei-type operator $J(t)=x+\mathrm{i}\varepsilon t\nabla$, we are able to prove the global existence of the smooth solutions.

Theorem 3.2 (Existence and uniqueness Ref. [19]) Let $\varphi_j\in\sum(\mathbf{R}^3)$. Assume that $\rho\in[2,6)$. Then, there exists a unique solution to the (BDNLSP) system (1) - (2) with the initial data (3) for which it holds

$$\psi_j\in C(\mathbf{R};\sum(\mathbf{R}^3))\cap L^\infty(\mathbf{R};H^1(\mathbf{R}^3))\cap L_{loc}^{\gamma(\rho)}(\mathbf{R};H_\rho^1(\mathbf{R}^3)),\text{for } j=1,2.$$

Moreover, we have the following large time behavior for the solution constructed in Theorem 3.2.

Theorem 3.3 (Large time behavior Ref. [19]) Let (ψ_1,ψ_2,V) and ρ be as in Theorem 3.2. Then, there exist constants C depending only on $\|\varphi_j\|_{H^1}$ and $\|\,|x|\varphi_j\|_2$ such that

$$\|\psi_j\|_\rho\leqslant C|t|^{-\frac{1}{\gamma(\rho)}},\ \forall\rho\in[2,6),\ \forall\,|t|\geqslant1,$$

$$\|\nabla V(t)\|_\rho\leqslant C|t|^{-(1-\frac{3}{2\rho})},\ \forall\rho\in(\frac{3}{2},\infty),\ \forall\,|t|\geqslant1,$$

$$\|V(t)\|_\rho\leqslant C|t|^{-\frac{1}{2}(1-\frac{3}{\rho})},\ \forall\rho\in(3,\infty),\ \forall\,|t|\geqslant1.$$

4 Modified scattering theory

To study the asymptotic behavior in time and the existence of the modified scattering operator of the solutions to the (BDNLSP) system in the spatial space $\mathbf{R}^3$, we rewrite it as the following:

$$\mathrm{i}\dot\psi_j=-\frac{1}{2}\Delta\psi_j+(q_jV(\psi_1,\psi_2)+a_j^2|\psi_j|^p)\psi_j,\text{for } j=1,2, \tag{8}$$

$$V=\frac{1}{4\pi|x|}*(|\psi_1|^2-|\psi_2|^2), \tag{9}$$

$$\psi_j(0,x)=\phi_j(x),x\in\mathbf{R}^3. \tag{10}$$

We also assume that $4/3<p<4$ in the nonlinear self-interacting potential $a_j^2|\psi_j|^p$ where $a_j\in\mathbf{R}$.

We consider the Cauchy problem under the following condition on initial data

$$\phi_j\in H^{\gamma,0}\cap H^{0,\gamma},$$

with

$$\gamma>3/2,j=1,2$$

and the norm $\sum_{j=1,2}\|\phi_j\|_{\gamma,0}+\|\phi_j\|_{0,\gamma}$ is sufficiently small, where the space $H^{\gamma,\nu}$ is the usual weighted Sobolev space defined by

$$H^{\gamma,\nu}:=\{u\in L^2:\|u\|_{\gamma,\nu}=\|(1+|x|^2)^{\nu/2}(1-\Delta)^{\gamma/2}u\|_2<\infty\},\quad\gamma,\nu\in\mathbf{R}.$$

We can get the following theorems on the existence and scattering theory.

Theorem 4.1 (Global existence Ref. [20]) We assume that $\phi_j\in H^{\gamma,0}\cap H^{0,\gamma}$ and $\sum_{j=1,2}[\|\phi_j\|_{\gamma,0}+\|\phi_j\|_{0,\gamma}]=:\varepsilon_1\leqslant\varepsilon$, where ε is sufficiently small and $3/2<\gamma\leqslant5/3$. Then there exists a unique global solution (ψ_1,ψ_2,V) to the above Cauchy problem such that for $j=1,2$

$$\psi_j\in C(\mathbf{R};H^{\gamma,0}\cap H^{0,\gamma}),$$

$$\|\psi_j(t)\|_\infty\leqslant C\varepsilon_1(1+|t|)^{-3/2},$$

$$\|V(t)\|_\infty\leqslant C\varepsilon_1(|t|^{-\alpha+C(\varepsilon_1^2+\varepsilon_1^p)}+|t|^{1-\frac{3p}{2}+C(\varepsilon_1^2+\varepsilon_1^p)}),t\in\mathbf{R}.$$

where $C\varepsilon < \alpha < 1, 4/3 < p < 4$.

Theorem 4.2 (Asymptotic behavior Ref.[20]) Let (ψ_1, ψ_2) be the solution obtained in Theorem 4.1. Then for any $\phi_j \in H^{\gamma,0} \cap H^{0,\gamma}$, $j = 1,2$, there exist a unique pair of functions $(\mathscr{W}_1, \mathscr{W}_2)$ with $\mathscr{W}_j \in L^\infty$, $j = 1,2$, and a real-valued function $\Lambda \in L^\infty$ such that for all $|t| \geqslant 1$

$$\| \mathscr{F}(S(-t)\psi_j(t)) e^{-iq_j\int_{\Lambda(t)}^{V(t)} V(\hat{\psi}_1, \hat{\psi}_2)\frac{d\tau}{|\tau|}} - \mathscr{W}_j \|_\infty$$
$$\leqslant C\varepsilon_1(|t|^{-\alpha + C(\varepsilon_1^2 + \varepsilon_1^p)} + |t|^{1-\frac{3p}{2} + C(\varepsilon_1^2 + \varepsilon_1^p)}),$$

and

$$\| \int_{\Lambda(t)}^{V(t)} V(\hat{\psi}_1, \hat{\psi}_2) |\tau|^{-1} d\tau - V(\mathscr{W}_1, \mathscr{W}_2) \ln |t| - \Lambda \|_\infty$$
$$\leqslant C\varepsilon_1(|t|^{-\alpha + C(\varepsilon_1^2 + \varepsilon_1^p)} + |t|^{1-\frac{3p}{2} + C(\varepsilon_1^2 + \varepsilon_1^p)})^\theta,$$

where $\Lambda(t) = \begin{cases} 1, t \geqslant 1 \\ t, t \leqslant -1 \end{cases}$, $V(t) = \begin{cases} t, t \geqslant 1 \\ -1, t \leqslant -1 \end{cases}$, $0 < \theta < 2/3$, $C\varepsilon < \alpha < 1$ and $\gamma > 3/2 + 2\alpha$. We recall that ε_1 is defined in Theorem 4.1. Furthermore, we have the estimate for $|t| \geqslant 1$ that

$$\| \mathscr{F}(S(-t)\psi_j) - \mathscr{W}_j e^{iq_j(V(\mathscr{W}_1, \mathscr{W}_2)\ln|t| + \Lambda)} \|_\infty$$
$$\leqslant C\varepsilon_1(|t|^{-\alpha + C(\varepsilon_1^2 + \varepsilon_1^p)} + |t|^{1-\frac{3p}{2} + C(\varepsilon_1^2 + \varepsilon_1^p)})^\theta.$$

5 Initial-boundary value problems for quasi-linear Schrödinger-Poisson equations

In this section, we consider the self-consistent quasi-linear Schrödinger-Poisson system (QSP) on the unit cube $\Omega := (0,1)^d$

$$i\partial_t \psi_m = -\frac{1}{2}\Delta\psi_m + V\psi_m, m \in \mathbf{N} \tag{11}$$

$$-\nabla \cdot ((\varepsilon_0 + \varepsilon_1 |\nabla V|^2)\nabla V) = n - n^*, \tag{12}$$

$$n(x,t) = \sum_{m=1}^\infty \lambda_m |\psi_m(x,t)|^2, \tag{13}$$

with the following initial and boundary conditions

$$\psi_m(x,0) = \phi_m(x), \tag{14}$$

$$\psi_m(x,t) = 0, \text{ on } \partial\Omega, \tag{15}$$

$$V(x,t) = 0, \text{ on } \partial\Omega, \tag{16}$$

where $d \in \mathbf{N}, d \leqslant 3, t \in \mathbf{R}$ and $\varepsilon_0, \varepsilon_1 > 0$. $\{\psi_m(x,t)\}_{m\in\mathbf{N}}$ is a sequence of complex valued wave functions. The electrostatic potential $V(x,t)$ is a real valued function. $\{\lambda_m\}_{m\in\mathbf{N}}$ is a specified sequence of probabilities, with $\sum_{m\in\mathbf{N}} \lambda_m = 1$. n^* is a given time-independent dopant density which may be represented as

$$n^* = n_D^+ - n_A^-,$$

where n_D^+ is the density of donors and n_A^- is the density of acceptors. We always look forward to seeking a solution satisfying the following charge neutrality:

$$\int_\Omega (n - n^*) dx = 0.$$

We introduce the following spaces

$$X := \{\Psi = (\psi_m)_{m\in\mathbf{N}} : \psi_m \in L^2(\Omega),$$
$$\|\Psi\|_X = (\sum_{m\in\mathbf{N}} \lambda_m \|\psi_m\|_{L^2(\Omega)}^2)^{1/2} < \infty\},$$

$$X^1 := \{\Psi = (\psi_m)_{m\in\mathbf{N}} : \psi_m \in H_0^1(\Omega), \quad \|\Psi\|_{X^1} = (\sum_{m\in\mathbf{N}} \lambda_m \|\psi_m\|_{H^1(\Omega)}^2)^{1/2} < \infty\},$$

and

$$X^2 := \{\Psi = (\psi_m)_{m\in\mathbf{N}} : \psi_m \in H^2(\Omega) \cap H_0^1(\Omega), \quad \|\Psi\|_{X^2} = (\sum_{m\in\mathbf{N}} \lambda_m \|\psi_m\|_{H^2(\Omega)}^2)^{1/2} < \infty\}.$$

Resorting to the techniques of quasi-linear elliptic PDE (*cf.* Ref. [21, 22]), the Sobolev embedding theorem and the Schauder fixed point theorem, we obtain the following existence theorem.

Theorem 5.1 (Ref. [23]) Let $\Phi = (\phi_m)_{m\in\mathbf{N}} \in X^2$ and $n^* \in C^1(\bar{\Omega})$. Then there is a unique solution (Ψ, V) such that

$$\Psi \in C^1(\mathbf{R}; X) \cap C(\mathbf{R}; X^2),$$
$$V \in C(\mathbf{R}; X^2),$$

with the conserved quantities

(i) $\|\Psi(\cdot, t)\|_X = \|\Phi(\cdot)\|_X$,

(ii) $\|\nabla\Psi(\cdot, t)\|_X^2 + \varepsilon_0 \|\nabla V(\cdot, t)\|_{L_2(\Omega)}^2 + \frac{3}{2}\varepsilon_1 \|\nabla V(\cdot, t)\|_{L^4(\Omega)}^4 = \text{constant}.$

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关于 Schrödinger-Poisson 系统的研究

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摘 要 研究了双极(非线性) Schrödinger-Poisson 系统和拟线性 Schrödinger-Poisson 方程,得到了双极 Schrödinger-Poisson 系统的整体适定性及其修正散射理论,以及单位方体上的具有 Dirichlet 边值条件的拟线性 Schrödinger-Poisson 方程的初边值问题整体解的存在唯一性.

关键词 双极 Schrödinger-Poisson 系统,拟线性 Schrödinger-Poisson 方程组,初值问题适定性,初边值问题

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