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An extremality property of Jenkins-Strebel differentials

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Abstract Given a compact Riemann surface S with finitely many punctures, in this paper we obtain a new extremality property of a Jenkins-Strebel differential φ on S. As a consequence, we obtain the solutions of several kinds of moduli problems on S.

Keywords: Jenkins-Strebel differentials, extremality property, moduli.

0 Introduction

The idea of quadratic differentials goes back to Teichmüller. It plays an essential role in his work on the deformations of complex structures on Riemann surfaces. See, e.g., refs. [1,2].

For a non-zero quadratic differential on a Riemann surface, one can talk of its horizontal trajectories. Also the quadratic differential induces a singular metric on the Riemann surface, which is 'flat' away from the zeros of the quadratic differential.

In particular, Jenkins-Strebel differentials (i.e. their non-singular trajectories are closed) possess some interesting extremal properties, which are important in the study of quadratic differentials theory and Teichmüller theory. See, e.g., refs. [3–6].

In this paper we will give a new extremality property of those Jenkins-Strebel differentials with double poles. Furthermore, we obtain several results on the existence of Jenkins-Strebel differentials with the prescribed heights, circumferences or reduced moduli.

Let S be a compact Riemann surface with distinguished punctures Q_j, and let ζ_j be a local parameter around Q_j, where $1 \leqslant j \leqslant q$. Suppose that $\{\gamma_k\}_{1\leqslant k\leqslant p}$ is an admissible curves system on S. See sec. 1 for the related definitions.

Then we have

Theorem 0.1. Let φ be a Jenkins-Strebel differential on S. Suppose that it determines p ring annuli $\{R_k\} \subset S$ of type $\{\gamma_k\}$ and q punctured disks $\{D_j\}$ surrounding the distinguished punctures $Q_j,\ 1 \leqslant j \leqslant q$. Denote by m_k the conformal modulus of the annulus R_k, and denote by h_k its φ-height, where $1 \leqslant k \leqslant p$. Also denote by $a_j,\ 1 \leqslant j \leqslant q$, the circumference of the punctured disk D_j.

Suppose that $\{\widetilde{R}_k, \widetilde{D}_j\} \subset S$ is a system of non-overlapping annuli and punctured disks (around the punctures $\{Q_j\}$) of the same homotopy type of $\{R_k, D_j\}$. If in the local parameters ζ_j, the punctured disks D_j (resp. $\widetilde{D}_j$) have the reduced moduli m_j (resp. $\widetilde{m}_j$), then

$$\sum_k \frac{h_k^2}{m_k} - \sum_j a_j^2 m_j \leqslant \sum_k \frac{h_k^2}{\widetilde{m}_k} - \sum_j a_j^2 \widetilde{m}_j,$$

with an equality holds if and only if $R_k = \widetilde{R}_k,\ D_j = \widetilde{D}_j$ for $1 \leqslant k \leqslant p,\ 1 \leqslant j \leqslant q$.

In this paper we use the same conventions of ref. [5]. That is, annuli are denoted by the letter k and punctured disks are denoted by the letter j.

Once we prescribe the reduced moduli around the distinguished punctures $\{Q_j\}$, there are three kinds of existence theorems on Jenkins-Strebel differentials with double poles on S. For example, one can give the circumferences of the annuli, the heights or the ratio of their moduli.

Suppose that $(m_1, m_2, \cdots, m_q)$ is an admissible array associated with the punctures $\{Q_j\}$ on S (see sec. 1 for the definition). Then we have the following results which solve the existence problems on Jenkins-Strebel differentials with the second order poles.

Theorem 0.2. If $\{h_k > 0\}$ are numbers associated with the admissible system $\{\gamma_k\}$, then there is a Jenkins-Strebel differential φ on S with the following properties:

(1) The differential φ has p characteristic annuli $R_k,\ 1 \leqslant k \leqslant p$, of type γ_k, and q punctured disks $D_j,\ 1 \leqslant j \leqslant q$, surrounding the distinguished punctures Q_j.

(2) The annuli $\{R_k\}$ have φ-heights $\{h_k\}$. And in the local parameters $\zeta_j,\ 1 \leqslant j \leqslant q$, the punctured disks D_j have the reduced moduli m_j.

Moreover, the quadratic differential φ is uniquely determined.

Theorem 0.3. Let $a_k > 0,\ 1 \leqslant k \leqslant p$, denote a set of constants associated with the curves $\{\gamma_k\}$. Then there is a unique Jenkins-Strebel differential φ on S whose characteristic regions include q punctured disks $\{D_j\}$ surrounding the punctures $\{Q_j\}$, and the annuli (some may be degenerate) of type $\{\gamma_k\}$.

In the fixed local parameter system ζ_j, the punctured disk D_j has the reduced modulus m_j, where $1 \leqslant j \leqslant q$.

The non-collapsed annulus R_k has the circumference a_k. For these collapsed annulus (conformal moduli $M_k = 0$), their circumferences $\geqslant a_k$.

Theorem 0.4. If $m_k > 0,\ 1 \leqslant k \leqslant p$, then there exists a Jenkins-Strebel differential φ with the following properties:

(1) The differential φ has q non-degenerate punctured disks D_j around Q_j, and in the local parameter ζ_j, D_j has the reduced moduli m_j, $1 \leqslant j \leqslant q$.

(2) The characteristic annuli $\{R_k\}$ of φ are of homotopy type $\{\gamma_k\}$ with the conformal moduli $\{\lambda\, m_k\}$ for some $\lambda > 0$ independent of k.

Moreover, φ is uniquely determined up to a positive constant factor, i.e. with the normalization $||\varphi|| = 1$ it is uniquely determined. In particular, the annuli and punctured disks on S are uniquely determined.

1 Definitions and notations

For later use we give some notations and definitions in this section. Suppose that D is a punctured disk in the ζ-plane with a point-like boundary component ζ_0. For sufficiently small $r > 0$, denote

$$D(r) \equiv D \backslash \{\zeta : |\zeta - \zeta_0| \leqslant r\}.$$

If the 2-connected domain $D(r)$ has conformal modulus $m(r)$, then for any $0 < r' < r$, we have $m(r) + \frac{1}{2\pi} \log \frac{r}{r'} \leqslant m(r')$. That is,

$$m(r) + \frac{1}{2\pi} \log r \leqslant m(r') + \frac{1}{2\pi} \log r',$$

from which we deduce that $m(r) + \frac{1}{2\pi} \log r$ is increasing as $r \to 0^+$. It follows that the number $\lim\limits_{r \to 0^+} (m(r) + \frac{1}{2\pi} \log r)$ exists. It is called the reduced modulus of the punctured disk $D \subset \mathbb{C}$ with respect to the parameter ζ, see e.g. ref. [7].

We can extend the definition of the reduced modulus to the general Riemann surfaces.

Let Ω be a Riemann surface with a distinguished puncture Q. let $D \subset \Omega$ be a punctured disk around Q. If $\zeta : U \to \mathbb{C},\ \zeta(Q) = 0$ is a local patch at the neighborhood U of Q, for any sufficiently small $r > 0$ we denote $D(r) \equiv D \backslash \{|\zeta(q)| \leqslant r\}$.

Denote by $m(r)$ the conformal modulus of the region $D(r)$.

Definition 1.1. The limit $\lim_{r \to 0^+} \left(m(r) + \frac{1}{2\pi} \log r\right)$ is called the reduced modulus of the punctured disk D with respect to the local uniformizer ζ.

Let z be a holomorphic homeomorphism between the punctured disk D and the punctured disk $\{0 < |z| < \rho\}$ such that $z(Q) = 0,\ \frac{dz}{d\zeta}(0) = 1$. Then z is called the normalized local parameter of D, and the number ρ is called the mapping radius of D with respect to ζ.

Recall that S is an analytic finite Riemann surface with distinguished punctures Q_j, and ζ_j is the fixed local parameter at the neighborhood of Q_j, where $1 \leqslant j \leqslant q$.

Definition 1.2. A vector $(m_1, m_2, \cdots, m_q) \in \mathbb{R}^q$ is called admissible on S, if there exists a system of non-overlapping punctured disks $\{\widetilde{D}_j\} \subset S$ surrounding $\{Q_j\}$ such that in the local parameters ζ_j their reduced moduli $\widetilde{m}_j$ satisfy that

$$m_j < \widetilde{m}_j,\ 1 \leqslant j \leqslant q.$$

We refer to ref. [5] for the basic materials on the quadratic differentials and the related topics.

A holomorphic $(2,0)$ form $\varphi = \varphi(z)dz^2$ on S is called a quadratic differential. A non-zero quadratic differential φ induces a metric $ds = \sqrt{|\varphi(z)|}|dz|$. Given a piecewise smooth curve $\gamma \subset S$, φ induces the φ-height of the curve γ

$$h = \inf_{\tilde{\gamma}\sim\gamma} \int_{\tilde{\gamma}} |\Im\sqrt{\varphi}|,$$

where $\tilde{\gamma}$ varies over all piecewise smooth curves in the homotopy class of γ.

A holomorphic quadratic differential $\varphi \neq 0$ on S is said to have the closed trajectories, if its non-closed trajectories cover a set of measure zero. A quadratic differential with the closed trajectories is called a Jenkins-strebel differential.

A Jenkins-Strebel differential with the second order poles decomposes the Riemann surface into the characteristic annuli and the characteristic punctured disks, which are swept out by the closed trajectories.

A system of finitely many piecewise smooth closed curves $\{\gamma_1, \gamma_2, \cdots, \gamma_p\} \subset S$ is called admissible, if none of γ_k is homotopically trivial (homotopic to zero or homotopic to the distinguished punctures), and if two curves $\gamma_k \neq \gamma_l$ neither intersect, nor are freely homotopic.

Then we have the following results.

Lemma 1.3. Let $\varphi \neq 0$ be a non-zero Jenkins-Strebel differential on S with type $\{\gamma_k\}_{1\leqslant k\leqslant p}$, and its characteristic annuli $\{R_k\}$ have the heights $\{h_k\}$. If $\{\widetilde{R}_k\}$ is a system of non-overlapping annuli on S with the homotopy type $\{\gamma_k\}$, then their conformal moduli $\widetilde{M}_k \equiv M(R_k)$ satisfy that

$$\sum_k \frac{h_k^2}{\widetilde{M}_k} \geqslant \sum_k \frac{h_k^2}{M_k},$$

with an equality holds if and only if $\widetilde{R}_k = R_k$ for each k.

Lemma 1.4. For an arbitrary $h_k > 0$, $1 \leqslant k \leqslant p$, and $a_j > 0\,, 1 \leqslant j \leqslant q$, there is a unique Jenkins-Strebel differential φ on S with the following properties:

(1) The differential φ has p characteristic annuli $\{R_k\}$ with type $\{\gamma_k\}$ and φ-heights $\{h_k\}$.

(2) φ has q punctured disks $\{D_j\}$ which are swept out by the closed trajectories around the punctures $\{Q_j\}$. The closed horizontal trajectories in D_j have the same φ-length a_j. Equivalently, φ has a second order pole at Q_j with a leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$, where $1 \leqslant j \leqslant q$.

Lemma 1.5. Let $\{a_k > 0\}$ denote a set of constants associated with the curves system $\{\gamma_k\}$. Let $\{a_j > 0\}$ be constants associated with the distinguished punctures $\{Q_j\}$.

Then there is a unique Jenkins-Strebel differential whose characteristic domains include q non-degenerate punctured disks with the circumferences $\{a_j\}$, and the characteristic annuli (some may be degenerate) of type $\{\gamma_k\}$.

For these annuli R_k which are not collapsed (the conformal modulus $\neq 0$), their circumferences are a_k. Moreover, for those collapsed annuli (they have the conformal moduli $M_k = 0$), their circumferences $\geqslant a_k$.

For the complete proofs of these results, we refer to refs. [6,8,9] and footnote 1).

2 Extremality properties

Now we can begin to prove Theorem 0.1. It gives an extremality property on characteristic annuli and punctured disks of a Jenkins-Strebel differential with double poles on S.

Proof of Theorem 0.1. If the degenerate punctured disks $\widetilde{D}_j$ (namely $\widetilde{m}_j = -\infty$) or the degenerate ring domains $\widetilde{R}_k$ ($\widetilde{m}_k = 0$) occur, then there is nothing to do.

Let z_j be the normalized local parameter near the puncture Q_j. Then $\frac{dz_j}{d\zeta_j}(0) = 1$. In terms of the local parameter z_j we have

$$\varphi = -\left(\frac{a_j}{2\pi}\right)^2 \frac{dz_j^2}{z_j^2},$$

and $D_j = \{0 < |z_j| < r_j\}$, $1 \leqslant j \leqslant q$, where r_j is the mapping radius of D_j with respect to the local parameter ζ_j. Therefore the reduced moduli of D_j is $m_j \equiv \frac{1}{2\pi}\log r_j$.

Choose a sufficient small ρ such that the mapping radius $r_j \geqslant \rho$ for each $1 \leqslant j \leqslant q$. Denote by $S(\rho)$ the bordered Riemann surface $S\backslash \cup\{|z_j| < \rho\}$. Let $m_j(\rho)$ (resp. $\widetilde{m}_j(\rho)$) be the conformal moduli of the annuli $R_j(\rho) = D_j\backslash \cup \{|z_j| < \rho\}$(resp. $\widetilde{R}_j(\rho) = \widetilde{D}_j\backslash \cup \{|z_j| < \rho\}$). Then in the bordered Riemann surface $S(\rho)$, by using Lemma 1.3 we have

$$\sum_k \frac{h_k^2}{m_k} + \sum_j \frac{(\frac{a_j}{2\pi}\log\frac{r_j}{\rho})^2}{m_j(\rho)} \leqslant \sum_k \frac{h_k^2}{\widetilde{m}_k} + \sum_j \frac{(\frac{a_j}{2\pi}\log\frac{r_j}{\rho})^2}{\widetilde{m}_j(\rho)}. \tag{2.1}$$

From the definitions it follows that

$$m_j(\rho) = \frac{1}{2\pi}\log\frac{r_j}{\rho} \quad \text{and} \quad \widetilde{m}_j(\rho) + \frac{1}{2\pi}\log\rho \to \widetilde{m}j = \frac{1}{2\pi}\log\tilde{r}_j, \quad 1 \leqslant j \leqslant q,$$

as $\rho \to 0^+$, where $\widetilde{m}_j$ is the reduced modulus of $\widetilde{D}_j$ and $\tilde{r}_j$ is the mapping radius of $\widetilde{D}_j$ under the local parameter ζ_j. Adding the terms $\sum_j \frac{a_j^2}{2\pi}\log\rho$ to both sides of (2.1) and letting $\rho \to 0$ give

$$\sum_k \frac{h_k^2}{m_k} + \sum_j a_j^2 m_j \leqslant \sum_k \frac{h_k^2}{\widetilde{m}_k} + \sum_j a_j^2\,(2m_j - \widetilde{m}_j).$$

1) Liu J. Jenkins-Strebel differentials with poles. Comment Math Helv. (in press)

Therefore we have the desired result

$$\sum_k \frac{h_k^2}{m_k} - \sum_j a_j^2 m_j \leqslant \sum_k \frac{h_k^2}{\widetilde{m}_k} - \sum_j a_j^2 \widetilde{m}_j. \tag{2.2}$$

To discuss the equality sign, we have to deal with the estimates (2.2) more accurate. Map the ring domain $\widetilde{R}_j(\rho)$, after cutting it radially, onto a horizontal rectangle $\tilde{S}_j(\rho)$ in the $z = x + iy$-plane with the heights $\frac{a_j}{2\pi} \log \frac{r_j}{\rho}$ and the lengths

$$\tilde{a}_j(\rho) = \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right) \Big/ \widetilde{m}_j(\rho), \quad 1 \leqslant j \leqslant q.$$

Using $z = x + iy$ as a parameter, and integrating along any complete horizontal line in the rectangle $\widetilde{S}_j(\rho)$, we have

$$a_j \leqslant \int |\varphi(z)|^{\frac{1}{2}} dx.$$

Assume that for some j one of the horizontal $y =$ constant (in the rectangle $\tilde{S}_j(\rho)$) is not a closed horizontal trajectory of φ. Then there exist positive numbers ϵ and δ which are independent of ρ, such that

$$a_j + \epsilon \leqslant \int |\varphi(z)|^{\frac{1}{2}} dx, \tag{2.3}$$

holds for all y in the δ-neighborhood of some y_0. By integrating (2.3) over y, we deduce that

$$a_j \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right) + \epsilon \cdot \delta \leqslant \iint_{\widetilde{R}_j(\rho)} |\varphi(z)|^{\frac{1}{2}} dxdy.$$

Doing the same thing in the annuli $\widetilde{R}_k, 1 \leqslant k \leqslant p$, and $\tilde{R}_{j'}(\rho),\ j' \neq j$, we have the similar inequalities (without the term $\epsilon \cdot \delta$). Summing over all j and k, we get

$$\sum_k a_k h_k + \sum_j a_j \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right) + \epsilon \cdot \delta \leqslant \iint_{S \setminus \cup_j \{|z_j| \leqslant \rho\}} |\varphi(z)|^{\frac{1}{2}} dx,$$

where a_k is the length of the horizontal trajectories in $R_k,\ 1 \leqslant k \leqslant p$. Applying the Schwartz inequality, we get

$$\begin{aligned}
&\left(\sum_k a_k h_k + \sum_j a_j \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right) + \epsilon \cdot \delta\right)^2 \\
\leqslant\ &\left(\sum_k \tilde{a}_k h_k + \sum_j \tilde{a}_j(\rho) \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right)\right) \cdot \left(\iint_{S \setminus \cup_j \{|z_j| \leqslant \rho\}} |\varphi(z)| dxdy\right) \\
=\ &\left(\sum_k \tilde{a}_k h_k + \sum_j \tilde{a}_j(\rho) \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right)\right) \cdot \left(\sum_k a_k h_k + \sum_j a_j \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right)\right),
\end{aligned}$$

where $\tilde{a}_k = h_k / \widetilde{m}_k,\ 1 \leqslant k \leqslant p$, and $\tilde{a}_j(\rho) = \frac{a_j}{2\pi} \log \frac{r_j}{\rho} / \widetilde{m}_j(\rho),\ 1 \leqslant j \leqslant q$.

This proves that

$$\sum_k a_k h_k + \sum_j a_j \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right) + \epsilon \cdot \delta \leqslant \sum_k \tilde{a}_k h_k + \sum_j \tilde{a}_j(\rho) \cdot \left(\frac{a_j}{2\pi} \log \frac{r_j}{\rho}\right).$$

That is

$$\sum_k \frac{h_k^2}{m_k} + \sum_j \frac{(\frac{a_j}{2\pi} \log \frac{r_j}{\rho})^2}{m_j(\rho)} + \epsilon \cdot \delta \leqslant \sum_k \frac{h_k^2}{\widetilde{m}_k} + \sum_j \frac{(\frac{a_j}{2\pi} \log \frac{r_j}{\rho})^2}{\widetilde{m}_j(\rho)}. \tag{2.4}$$

Adding $\sum_j \frac{a_j^2}{2\pi} \log \rho$ onto the both sides of (2.4) and letting $\rho \to 0$, we conclude that

$$\sum_k \frac{h_k^2}{m_k} - \sum_j a_j^2 m_j + \epsilon \cdot \delta \leqslant \sum_k \frac{h_k^2}{\widetilde{m}_k} - \sum_j a_j^2 \widetilde{m}_j.$$

If the equality

$$\sum_k \frac{h_k^2}{m_k} - \sum_j a_j^2 m_j = \sum_k \frac{h_k^2}{\widetilde{m}_k} - \sum_j a_j^2 \widetilde{m}_j \tag{2.5}$$

holds, the above argument implies that each horizontal $y = c \ \ (0 < c < \frac{a_j}{2\pi} \log \frac{r_j}{\rho})$ is actually a closed trajectory of φ. In particular it follows that

$$\widetilde{D}_j \subset D_j, \ \ 1 \leqslant j \leqslant q. \tag{2.6}$$

Repeating the same argument, from the equality (2.5) it follows that

$$\widetilde{R}_k \subset R_k, \ \ 1 \leqslant k \leqslant p. \tag{2.7}$$

Therefore, from (2.5), (2.6) and (2.7) we deduce that $R_k = \widetilde{R}_k, \ \widetilde{D}_j = D_j$ for all $k, \ j$, which is the desired result. □

3 Proofs of the results

Now we can begin the proofs of the theorems.

Proof of Theorem 0.2. Its proof is similar to that of Theorem 4.6 in footnote 1). Hence it is only outlined in this paper.

Denote by $U \subset \mathbb{R}^q$ the subset consisting of all admissible vectors $(m_1, m_2, \cdots, m_q) \in \mathbb{R}^q$ on the Riemann surface S.

For any $C = (c_1, c_2, \cdots, c_q) \in \mathbb{R}_+^q$, by using Lemma 1.4, it follows that there is a Jenkins-Strebel differential on S whose characteristic domains include q non-degenerate punctured disks with the circumferences $\{c_j\}$, and p characteristic annuli $\{R_k\}$ of type $\{\gamma_k\}$. In the φ-metric, the annuli R_k, $1 \leqslant k \leqslant p$, have the heights h_k.

Denote $H \equiv (h_1, h_2, \cdots, h_p)$. Then we have a map

$$F_H : \mathbb{R}_+^q \to U$$

by sending $F_H(C) = (m_1, m_2, \cdots, m_q)$, where m_j, $1 \leqslant j \leqslant q$, are the reduced moduli of the punctured disks D_j with respect to the local parameters $\{\zeta_j\}$. Obviously F_H is well defined.

To prove Theorem 0.2, it is sufficient to show that $F_H : \mathbb{R}_+^q \to U$ is a homeomorphism.

By Theorem 24.7 in ref. [5], it follows that F_H is continuous. Theorem 0.1 implies that $F_H : \mathbb{R}_+^q \to U$ is injective. Finally, the properness of F_H is obtained by using the similar way as in ref. [8] without any substantial change.

1) See footnote 1) on p. 1098.

Therefore $F_H : \mathbb{R}^q_+ \to U$ is a homeomorphism, as desired. $\square$

Proof of Theorem 0.3. The uniqueness of the proof follows from Lemma 1.5.

Since the proof of the existence part is similar to that of Theorem 0.2, we omit it here. $\square$

Proof of Theorem 0.4. Let φ and φ' be two solutions. Then the conformal moduli of their characteristic annuli $M_k = \lambda\, m_k$ and $M'_k = \lambda'\, m_k$ for all k. And the punctured disks $D_j,\ D'_j$ have the same reduced moduli m_j with respect to the local parameter ζ_j. It follows from Theorem 0.1 that

$$\min_k \left\{ \frac{M'_k}{M_k} \right\} = \frac{\lambda'}{\lambda} \leqslant 1.$$

Similarly, by starting with the quadratic differential φ', we obtain $\lambda/\lambda' \leqslant 1$.

Therefore $\lambda = \lambda'$, which implies that $R_k = R'_k,\ D_j = D'_j$ for all $k,\ j$. Hence φ and φ' must have the same closed trajectories, which implies that φ is uniquely determined up to a positive constant factor.

Now we can begin the proof of the existence part.

Denote $m \equiv (m_1, m_2, \cdots, m_q)$ the admissible array. Denote by $\mathcal{C}_{\Updownarrow}$ the set consisting of all $c > 0$ such that there exists a system of the disjointed annuli $\{\widetilde{R}_k\}$ with the conformal moduli $\{cm_k\}$ and the punctured disks $\{\widetilde{D}_j\}$ with the reduced moduli $\{m_j\}$ on S. Then it follows from Theorem 0.2 or 0.3 that $\mathcal{C}_{\Updownarrow} \neq \emptyset$.

Obviously each $c \in \mathcal{C}_{\mathcal{M}}$ is bounded from the above. Using the normal family argument, there exists a system of the non-overlapping annuli $\{R_k\}$ and the punctured disks $\{D_j\}$ on S attaining the maximal value of $c \in \mathcal{C}_{\mathcal{M}}$.

Let z_j be the normalized local parameters near the punctures Q_j. Then

$$D_j = \{z_j : 0 < |z_j| < r_j\}, \quad 1 \leqslant j \leqslant q,$$

where $\frac{dz_j}{d\zeta_j}(0) = 1$ and r_j is the mapping radius of D_j with respect to the local parameter ζ_j. Suppose that $\rho > 0$ is a sufficiently small number with $0 < \rho < r_j$, for all $1 \leqslant j \leqslant q$. Denote

$$D_j(\rho) \equiv \{z_j : \rho < |z_j| < r_j\}, \quad 1 \leqslant j \leqslant q.$$

Then we claim the annuli $\{R_k\ \ D_j(\rho)\}$ are associated with a Jenkins-Strebel differential φ_ρ on the bordered Riemann surface $S(\rho) \equiv S \backslash \cup_j \{z_j : |z_j| < \rho\}$.

Otherwise, we would have a system of the annuli $\{\widetilde{R}_k\}$ and $\{\widetilde{D}_j(\rho)\}$ on $S(\rho)$ with the conformal moduli $(1+\epsilon)\, m_k$ and $(1+\epsilon)\, \frac{1}{2\pi} \log \frac{r_j}{\rho}$, respectively. Adding the punctured disks $\{z_j : 0 < |z_j| < \rho\}$ to $S(\rho)$, we obtain a system of the ring annuli with the conformal moduli $\{(1+\epsilon)\, m_k\}$ and the punctured disks with the reduced moduli (in the local parameter ζ_j)

$$m'_j \geqslant (1+\epsilon)\, \frac{1}{2\pi} \log \frac{r_j}{\rho} + \frac{1}{2\pi} \log \rho > \frac{1}{2\pi} \log r_j = m_j, \quad 1 \leqslant j \leqslant q.$$

It contradicts the assumption that $c \in \mathcal{C}_{\mathcal{M}}$ attains the maximal value.

Thus the annuli $\{R_k, D_j(\rho)\} \subset S(\rho)$ are associated with a Jenkins-Strebel differential φ_ρ on the bordered Riemann surface $S(\rho)$. Moreover, the boundary curves of $S(\rho)$ are the closed trajectories of φ_ρ. Therefore φ is the desired Jenkins-Strebel differential on S. $\square$

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References

1 Teichmüller O. Untersuchungenuber konforme und quasikonforme Abbildungen. Deutsche Math, 1938, 3: 621–678
2 Teichmüller O. Extremale quasikonforme Abbildungen und quadratische Differentiale. Abh Preuss Akad Wiss math-naturw KI, 1939, 22: 1–197
3 Jenkins J A. On the existence of certain general extremal metrics. Ann Math, 1957, 66: 440–453
4 Liu J. A new coordinate of Teichmüller space. Sci China Ser A-Math, 2001, 44 (12): 1523–1530
5 Strebel K. Quadratic Differentials. Berlin and New York: Springer-Verlag, 1984
6 Zwiebach B. How covariant closed string theory solves a minimal area problem. Commun Math Phys, 1991, 136: 83–118
7 Jenkins J A. Univalent functions and conformal mapping. Berlin and New York: Springer-Verlag, 1958
8 Liu J. On the existence of Jenkins-Strebel differentials. Bull London Math Soc, 2004, 36: 365–377
9 Strebel K. On quadratic differentials with closed trajectories and second order poles. J D'analyse Math, 1967, 19: 373–382