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A new coordinate of Teichmüller space

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Abstract By using the theory of quadratic differentials, we give a new coordinate to the Teichmüller space as well as the trajectory structures of a special class of Jenkins-Strebel quadratic differentials.

Keywords: Teichmüller space, quadratic differentials, pants.

Let S_0 be a smooth 2-dimensional closed manifold, i.e. a compact surface without boundary. The famous Riemann moduli problem claimed that the equivalent classes of complex structures on the closed surface S_0 of genus $g \geqslant 1$ could be holomorphically parametrized by $3g-3$ complex parameters. Let $\tilde{T}(S_0)$ denote the space of all complex structures on S_0, and let Diff_0 be the group of diffeomorphisms isotopic to the identity on S_0, which acts by pulling back on $\tilde{T}(S_0)$. The Teichmüller space of S_0, $T(S_0)$, is defined as the quotient space $\tilde{T}(S_0)/\mathrm{Diff}_0$. Teichmüller defined the natural metric d_T (defined below) on $T(S_0)$ and proved that the space is homeomorphic to the unit ball in R^{6g-6} in the metric topology.

It is a well-known fact that besides the Teichmüller coordinate, there are many other global coordinates for $T(S_0)$, for instance, the Nielsen-Fenchel coordinate and the Fricke coordinate[1]. The main purpose of this paper is to give $T(S_0)$ a new global coordinate by using quadratic differentials on the Riemann surface. Also the topological structures for the trajectories of a certain class of Jenkins-Strebel quadratic differentials will be studied.

Unless otherwise stated, all surfaces considered in this paper will be assumed to be oriented and of genus $g>1$. All mappings between surfaces will be assumed to be bijective and orientation-preserving.

1 Background materials

The smooth compact surface S_0 may be given a complex structure S_σ by pulling back through a diffeomorphism $\sigma: S_0 \rightarrow S$, where S is the Riemann surface of the same genus as S_0. Let S_1 and S_2 be two Riemann surfaces of genus g, and let f be a quasiconformal mapping between S_1 and S_2. As usual, we denote by $\mu_f(z) = \dfrac{\partial_{\bar z} f \mathrm{d}\bar z}{\partial_z f \mathrm{d} z}$ the Beltrami coefficients of f between S_1 and S_2, and by $K[f] = \mathrm{esssup}_{z_0 \in S} \dfrac{1+|\mu_f(z_0)|}{1-|\mu_f(z_0)|}$ the maximal dilatation of f. The Teichmüller metric $d_T(\cdot,\cdot)$ is defined as $d_T([S_{\sigma_1}],[S_{\sigma_2}]) = \sup \log K[h]$, where the supremum is taken over all quasiconformal mappings $h \simeq id: S_{\sigma_1} \rightarrow S_{\sigma_2}$. It is well known that Teichmüller space $T(S_0)$ is a complete metric space in Teichmüller metric $d_T(\cdot,\cdot)$ (cf. ref. [2]).

We denote by $Q(S)$ the space of all holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ on S. It is a Banach space with L^1-norm:

$$\| \varphi \| = \iint_S |\varphi| dxdy.$$

As a conclusion from Riemann-Roch Theorem $Q(S)$ is a vector space of real dimension $6g-6$. Each nonzero $\varphi \in Q(S)$ induces a singular metric $ds = |\varphi(z)|^{1/2}|dz|$ on S; the φ-length of any curve $\gamma \subset S$ is defined as

$$l_\varphi(\gamma) = \int_\gamma |\varphi(z)|^{1/2}|dz|.$$

We call z_0 a critical point of φ if $\varphi(x)=0$, otherwise a regular point. At the regular point z_0, there is a natural parameter w with $dw^2 = \varphi(z)dz^2$, where $w(z) = u + iv = \int^z \sqrt{\varphi}$ (note that $\sqrt{\varphi}$ is a holomorphic 1-form). If z_0 is a zero of order p, there is a local chart w with $\varphi = w^p dw^2$ around E_0. The metric ds has a cone-like singularity with $(p+2)\pi$ degrees at z_0.

In this paper we will use the notion of the φ-height of a curve $\gamma \subset S$.

Definition 1. Let φ be a nonzero holomorphic quadratic differential on S. For any curve γ, the infimum $h_\varphi(\gamma) = \inf_{\tilde{\gamma} \sim \gamma} \int_{\tilde{\gamma}} |\Im \sqrt{\varphi}|$, where $\tilde{\gamma}$ varies over all rectifiable curves in the homotopy class of γ, is called the height of γ with respect to φ.

The arc on S is called a horizontal (vertical) line of φ if $\varphi > 0$ (<0) along it. The maximal horizontal (vertical) arc is called a horizontal (vertical) trajectory. A trajectory is said to be critical if it meets a singularity (named critical point) of φ when it is continued in either direction.

A system of finitely many smooth closed curves $\{\gamma_1, \gamma_2, \cdots, \gamma_p\} \subset S_0$ are called admissible, if none of the curves is homotopically trivial (homotopic zero) and any two curves γ_i and γ_j neither intersect nor are freely homotopic for $i \neq j$. On S_0, the maximal numbers of closed curves in an admissible system is $3g-3$. A ring domain $R_0 \subset S_0$ is said to be of homotopy type γ, if a simple closed curve $\gamma_0 \subset R_0$, as separating its two boundary components, is freely homotopic to γ. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_p\}$ be an admissible system, a set of non-overlapping ring domains $\{R_1, R_2, \cdots, R_p\}$ on S_0 is said to be of homotopy type $\{\gamma_1, \gamma_2, \cdots, \gamma_p\}$, if each R_i is of homotopy type γ_i.

Definition 2. A nonzero holomorphic quadratic differential $\varphi \in Q(S)$ is called a Jenkins-Strebel quadratic differential if all its non-critical trajectories are closed. The Jenkins-Strebel differential φ is said to be of homotopy type $\{\gamma_1, \gamma_2, \cdots, \gamma_p\}$, if its characteristic ring domains are of the type $\{\gamma_1, \gamma_2, \cdots, \gamma_p\}$.

It is well known that $\varphi \in Q(S)$ is a Jenkins-Strebel differential if and only if the set of its critical trajectories with their end points is compact. In other words, a Jenkins-Strebel differential on s divides the Riemann surfaces into several ring domains.

Jenkins proved an extremal length problems in Riemann surface theory by using this kind of quadratic differentials[3]. Later, Strebel studied the general theory of quadratic differentials deeply[5]. The following theorem proved simultaneously by Hubbard-Masur[4] and Renelt is crucial to this paper.

Theorem A[5]. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_p\}$ be an admissible system on a compact Riemann surface S. Then, for arbitrarily given numbers $b_i > 0$ $(i = 1, \cdots, p)$, there exists a Jenkins-Strebel quadratic differential φ on S, whose character ring domains R_i $(i = 1, 2, \cdots, p)$ have type γ_i and height b_i (metric by φ). Moreover, φ is uniquely determined.

2 Pants and its complex structure

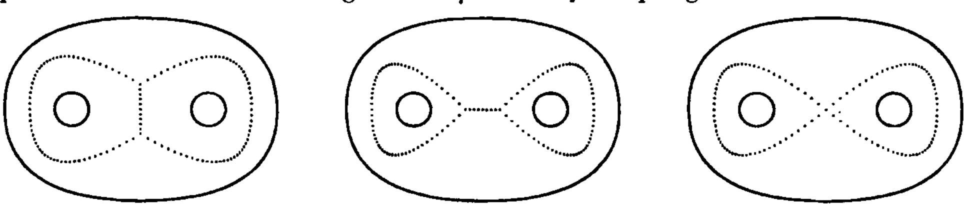
Let $\mathscr{P}$ be a surface of $(0,3)$ form, i.e. it is the resulting surface that cut three disks from the topological sphere, and label the border components of $\mathscr{P}$ by $\{\partial_1, \partial_2, \partial_3\}$. Let σ_P be a complex structure on $\mathscr{P}$. We call the resulting Riemann surface P a pant if none of its boundaries degenerates. Pants are blocks for all compact Riemann surfaces of genus greater than one. By the process of Schottky doubling P, we get a compact Riemann surface P^d of genus $g = 2$. The boundary curves $\{\partial_1, \partial_2, \partial_3\}$ of P are an admissible curves system on P^d. By Theorem A, there is a unique quadratic differential φ_{P^d} on P^d, which has the type $\{\partial_1, \partial_2, \partial_3\}$, and each character ring domain R_i $(i = 1, 2, 3)$ has height 2. We denote by φ_P the restriction of φ_{P^d} to P, and call φ_p the characteristic quadratic differential of P. From symmetry, it is obvious that the three boundary curves $\{\partial_1, \partial_2, \partial_3\}$ are the core curves of the three character domains.

The following two theorems show that the complex structure equivalence classes on $\mathscr{P}$ can be uniquely determined by $l_1 = l_{\varphi_p}(\partial_1)$, $l_2 = l_{\varphi_p}(\partial_2)$ and $l_3 = l_{\varphi_p}(\partial_3)$, where φ_p is the characteristic differential on P.

Theorem 1. Suppose that σ is a complex structure on $\mathscr{P}$, and it determines a Riemann surface P. Then the complex structure of P is uniquely determined up to conformal mapping which is homotopic to the identity by the triple (l_1, l_2, l_3) given above.

Proof. For convenience, we discuss the general condition first.

Suppose that $\{\partial_1, \partial_2, \partial_3\}$ are the border components of P and φ is any Jenkins-Strebel quadratic differential on P^d which has the type $\{\partial_1, \partial_2, \partial_3\}$. Because the genus of P^d is 2, by Riemann-Roch Theorem, the total orders of the critical points of φ are 4. Clearly, there is no critical point of φ on the three boundaries of P. Thus the total orders of the critical points of φ in P are 2, i.e. φ has a critical point of order 2 or two critical points of order 1 on P. The critical graph $\mathscr{C}$ (the dotted curves in fig. 1) of φ has only 3 topological structures.



(a) (b) (c)

Fig. 1. (a) φ has two critical points of order 1 in P; none of the end points of one critical trajectory is the same. $\mathscr{C}$ includes three critical trajectories, each of which joins the two critical points of φ. The length of any boundary trajectory is less than the sum of the other two. (b) φ has two critical points of order 1 in P and the critical graph $\mathscr{C}$ includes critical trajectory, whose two end points are the same critical point of φ. In this case, $\mathscr{C}$ includes two close curves and the third trajectory joins the two critical points. The length of one boundary trajectory is greater than the sum of the other two. (c) φ has only one critical point O of order 2, and there exist 4 critical horizontal trajectories that end at O. The only topological structure for the critical graph $\mathscr{C}$ of φ in P is a closed curve transversing itself at O. The length of one boundary trajectory is equal to the sum of the other two.

In the pant P, we assume $l_i = l_{\varphi_p}(\partial_i)$, and set $L = (l_1, l_2, l_3)$. If we have another triple

$\tilde{L} = (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$ with respect to another pant $\tilde{P}$ and its characteristic differential $\varphi_{\tilde{P}}$, and if $L = \tilde{L}$, then the critical graphs of φ_P and $\varphi_{\tilde{P}}$ are topologically equivalent on $\mathscr{P}$. Without losing generality, we suppose that their critical graphs meet case (a). For $i = 1,2,3$, the ring domains $R_i \subset P$ and $\tilde{R}_i \subset \tilde{P}$ have the same height 1 (in the respective characteristic differential singular metrics) and $l_{\varphi_P}(\partial_i) = l_{\varphi_{\tilde{P}}}(\partial_i)$; thus R_i and $\tilde{R}_i$ have the same module. We can construct a holomorphic homeomorphism h_i between R_i and $\tilde{R}_i$ satisfying $h_i(O_1) = \tilde{O}_1$ and $h_i(O_2) = \tilde{O}_2$, where $O_i(\tilde{O}_i)$ are the two critical points of $\varphi_P(\varphi_{\tilde{P}})$. The holomorphic mappings $h_i (i = 1,2,3)$ can be wedded into a holomorphic mapping h between P and $\tilde{P}$ meeting $h(\partial_i) = \tilde{\partial}_i$. Furthermore, $h \simeq id: \mathscr{P} \to \mathscr{P}$. Thus we can reach the theorem.

Theorem 2. For any positive numbers triple $L = (l_1, l_2, l_3)$, there is a complex structure σ on $\mathscr{P}$, so the resultant pant P satisfies the condition: with respect to the characteristic quadratic differential φ_P on P, we have $l_i = l_{\varphi_P}(\partial_i)$, $i = 1,2,3$ (from Theorem 1, it is uniquely determined).

Proof. On the complex z-plane, we have the ring domain $R_i = \left\{ z:\ 1 < |z| < \exp\left(\frac{2\pi}{l_i}\right) \right\}$ and the quadratic differential $\varphi_i = \left(\frac{l_i}{2\pi}\right)^2 \left(\frac{\mathrm{d}z}{z}\right)^2$ on $R_i (i = 1,2,3)$. In terms of the parameter z, the horizontal trajectories of φ_i are the circle $\{z:\ |z| = r\}$, where $1 < r < \exp\left(\frac{2\pi}{l_i}\right)$. The φ_i-lengths of the two boundaries of R_i are l_i and the φ_i-height of the ring domain R_i is 1.

There is a critical graph $\mathscr{C}$ on $\mathscr{P}$ with respect to the triple (l_1, l_2, l_3). We can wed the three ring domains R_i into a pant P. At the same time, three quadratic differentials φ_i are wedded into a quadratic differential φ_P on P with type $\{\partial_1, \partial_2, \partial_3\}$ and height $b_i = 1$. The horizontal trajectories of φ_P are just taken from that of φ_i on the respective ring domains R_i (cf. ref. [5]). Therefore the resulting pant P has the propositions states as Theorem 2.

Now, let $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$ be a fixed admissible curves system on S_0. Then Γ divides S_0 into $2g - 2$ surfaces of $(0,3)$ form, which are labeled by $\{\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_{2g-2}\}$. Associated to each complex structure σ on S_0, a unique Jenkins-Strebel differential $\varphi_\Gamma = \varphi(z)\mathrm{d}z^2$ has type Γ, and height $b_i = 2$ $(i = 1,2,\cdots,3g-3)$ appears. We call φ_r the characteristic quadratic differential of S_σ.

Here and hereinafter, we set $\mathbb{R}_+ = \{x \in R \mid x > 0\}$. To each $[\sigma] \in T(S_0)$, we set $l_\sigma(\gamma_i) = l_{\varphi_\Gamma}(\tilde{\gamma}_i)$, where $\tilde{\gamma}_i$ is the core curve of the character domain R_i of φ_Γ. Consider the mapping

$$L:\ T(S_0) \to \mathbb{R}_+^{3g-3},$$
$$[S_\sigma] \mapsto (l_\sigma(\gamma_1), l_\sigma(\gamma_2), \cdots, l_\sigma(\gamma_{3g-3})).$$

Proposition 1. $L:\ T(S_0) \to \mathbb{R}_+^{3g-3}$ is a well defined function on the Teichmüller space $T(S_0)$.

Proof. This is the straightforward consequence of the definition of the Teichmüller space $T(S_0)$.

Proposition 2. $L:\ T(S_0) \to \mathbb{R}_+^{3g-3}$ is onto.

Proof. In order to prove the above proposition, we introduce the notion of cubic graphs[6].

A cubic graph is a finite 3-regular connected graph, which is the combinatorial skeleton for the pasting of pairs of pants. For our purposes it is convenient to view each edge of the cubic graph as the union of two half-edges, and each half-edge as emanating from one of the two connected vertices. A graph $\mathscr{G}$ is called 3-regular, if every vertex has three emanating edges. In the construction of the compact Riemann surface, each pant with its three boundary horizontal trajectories will be interpreted as a vertex with its three half-edges (fig. 2). Two 3-graph $\mathscr{G} \subset S_0$ and $\tilde{\mathscr{G}} \subset S_0$ are called equivalent if and only if there is a homeomorphism $f \simeq id: S_0 \to S_0$ satisfying $f(\mathscr{G}) = \tilde{\mathscr{G}}$.

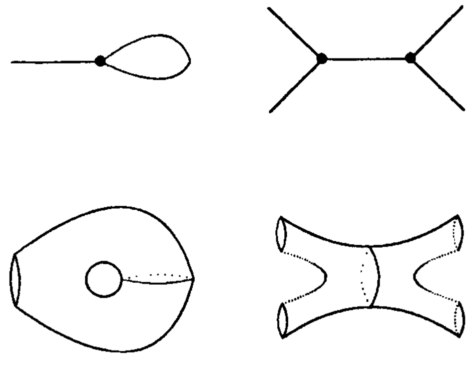


Fig. 2

Associated to the maximal curves system Γ, an equivalent class of cubic graphs $\mathscr{G}$ on S_0 occurs. For $i = 1, 2, \cdots, 2g-2$, we denote the three border components of $\mathscr{P}_i$ by ∂_{ij}, $j = 1, 2, 3$. To each $(l_1, l_2, \cdots, l_{3g-3}) \in \mathbb{R}_+^{3g-3}$, we construct a unique pant P_i meeting the condition that the φ_{P_i}-length of its boundary curves ∂_{ij} is l_{ij}, and each ring domain has φ_{P_i}-height 1. It is possible to glue these pants together now. We can determine the local relationship of all the $P_i (i = 1, 2, \cdots, 2g-2)$ via the cubic graph $\mathscr{G}$, and then we need to identify the pants' edges. It could be done simply by picking two pants arbitrarily and then going on with the characteristic quadratic differential length of arc as a parameter towards the positive direction of the loops. With this construction, one can get a Riemann surface S_σ and the characteristic quadratic differential φ_Γ on S_σ, with $L([S_\sigma]) = (l_1, l_2, \cdots, l_{3g-3})$.

Since the map L is onto, in order to study $T(S_0)$, we have to consider the fiber of L. Because the φ_P-lengths of the three border edges uniquely determine the complex structure P on $\mathscr{P}$, we must study how many different complex structures can be obtained by gluing these $2g-2$ pants. The only freedom in the construction is to twist a certain angle along the edges before gluing it to the other, so other $3g-3$ real parameters corresponding to the angles have to be taken into account. Obviously, with this method, one can obtain all the possible complex structures on S_0 from the fixed value of L, and it turns out that these complex structures are pairwise non-equivalent. Thus $T(S_0)$ can be parametrized by $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$.

On S, we suppose the border components of the pant P_i to be $\partial_{ij} (j = 1, 2, 3)$; for convenience, we set $\partial_{i4} = \partial_{i1}$. Let $\tilde{r}_{ij}$ be a simple path joining ∂_{ij} to $\partial_{i(j+1)}$. Doubling the pant P_i, we can obtain the surface P_i^d of genus 2 endowed with the characteristic quadratic differential φ_{P_i}, which is the same as the characteristic differential φ_Γ restriction to P_i. Moreover, $\tilde{r}_{ij}$ produces a non-trivial simple loop $\tilde{r}'_{ij}$ on P_i.

Now we consider the φ_{P_i}-geodesics in the isotopy class of $\tilde{r}'_{ij}$. Considering the lifting of P_i^d and φ_{P_i} to the universal covering space $\mathbb{D}$, there are two possibilities by Theorem 14.3 of ref. [5]:

(i) there exists only one φ_{P_i}-geodesic r_{ij} in the isotopy class of $\tilde{r}'_{ij}$;

(ii) there exist infinite φ_{P_i}-geodesics in the isotopy class of $\tilde{r}'_{ij}$. In this case, any two of them bind a ring domain which is swept out by parallel ϑ-trajectories isotopic to $\tilde{r}'_{ij}$. We denote by $\mathscr{R}$ the maximal ring domain on P_i^d which is swept out by all parallel ϑ-trajectories isotopic to $\tilde{r}'_{ij}$, and we denote the core curve of $\mathscr{R}$ by r_{ij}. Obviously, we have $\vartheta = \pi/2$ from the symmetry.

Let ζ_{ij} denote the intersecting point of r_{ij} with ∂_{ij}. ζ_{ij} is a unique basepoint on ∂_{ij}. Suppose that the pants P_{i_1} and P_{i_2} share the same boundary γ_i (note that P_{i_1} and P_{i_2} may be the same), with the above process, we get the unique basepoint ζ_i on γ_i from P_{i_1}, and another point ζ_i' on γ_i from P_{i_2}. We consider the left twist between the two points ζ_i and ζ'_i, and denote it by $\phi_i(S_\sigma)$ (note that the notion of left twist depends only on the orientation of S_σ and no orientation of γ_i is involved). We set $\theta_i(S_\sigma) = 2\pi * \dfrac{\phi_i(S_\sigma)}{l_i}$, and call $\theta_i(S_\sigma)$ the angle parameters.

Remark. $\{l_i, \theta_i\}$ $(i = 1, 2, \cdots, 3g-3)$ are the global real coordinates on $T(S_0)$, and $\{\theta_i\}$ are unique up to the choice of the basepoint.

3 Main result

The goal of this section is to give the proof of the main theorem in this paper.

The basic tool used in the proof of the main theorem is the theory of measure foliations. First of all, we recall that a measured foliation $\mathscr{F}$ on a surface is a foliation equipped with a measure on the space of the leaves. For every regular point z on the leaves of $\mathscr{F}$, we have a neighborhood and a chart ϕ: $U \to R^2$, which sends the leaves of $\mathscr{F}$ to horizontal line of R^2. If two neighborhoods U_i and U_j overlap, there is a translation function ϕ_{ij} defined on $\phi_j(U_j)$ such that $\phi_{ij} \circ \phi_j = \phi_i$ on $U_i \cap U_j$, with the property that ϕ_{ij} is of the form $\phi_{ij}(x, y) = (f(x, y), c \pm y)$, where c is a constant. The measure defined on leaves of $\mathscr{F}$ is $|dy|$ for each local chart ϕ: $U \to R^2$. Obviously, this definition of measure is independent of the choice of the local charts. Letting Σ be all non-trivial homotopy classes of simple closed curves on S_0, for each $\gamma \in \Sigma$ and the measure foliation $\mathscr{F}$, we define the intersection number of $\mathscr{F}$ and γ as $i(\mathscr{F}, \sigma) = \inf\limits_{\tilde{\gamma} \simeq \gamma} \int_{\tilde{\gamma}} \mathscr{F}$. For example, on any Riemann surface S, a nonzero quadratic differential $\varphi = \varphi(z)dz^2 \in Q(S)$ induces the measure foliation on S, whose leaves are the horizontal trajectories and the measure $|dy| = \left|\Im \sqrt{\varphi(z)dz^2}\right|$. It is clear that $i(\varphi, \gamma) = h_\varphi(\gamma)$.

Two measure foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ on S_0 are called measure equivalent if and only if for each $\gamma \in \Sigma$, $i(\mathscr{F}_1, \gamma) = i(\mathscr{F}_2, \gamma)$, and we denote by $\mathscr{MF}$ the measure foliation space, i.e. the measure equivalent classes of measure foliations. Then we have the mapping

$$i_*: \mathscr{MF} \to R^\Sigma,$$
$$i_*(\mathscr{F})(\gamma) = i(\mathscr{F}, \gamma).$$

By definition, i_* is an injection, so we can view $\mathscr{MF}$ as a subspace of R^Σ. The product topological structure on R^Σ induces the topological structure on $\mathscr{MF}$. Then $\lim\limits_{i \to +\infty} \mathscr{F}_i = \mathscr{F}$ if and only if for each $\gamma \in \Sigma$, $i(\mathscr{F}_i, \gamma) \to i(\mathscr{F}, \gamma)$.

Although the following two results will not be used in the context, we cite it here for the completeness. The first theorem is due to Thurston.

Theorem 3[8]. The space $\mathscr{MF}$ is homeomorphic to $\mathbb{R}^{6g-6}$.

As noted earlier, the horizontal trajectories of each nonzero quadratic differential φ on S induces a measure foliation. Conversely, Kerckhoff[7], Hubbard and Masur[4] have proved the following perfect theorem.

Theorem 4[4,7]. For any compact Riemann surface S and the measure foliation $\mathscr{F} \in \mathscr{MF}$, there is exactly one differential $\varphi \in Q(S)$, whose horizontal measure foliation is measure equivalent to $\mathscr{F}$.

Consider p: $Q \to T(S_0)$, whose fiber over a point S_σ is the quadratic differential space $Q(S_\sigma)$. The union of these spaces Q can yield the structure of a vector bundle——the cotangent bundle of the Teichmüller space. Let $\Gamma = \{\gamma_i: i = 1, 2, \cdots, 3g-3\}$ be the fixed admissible system on S_0, and let $\mathscr{E}_\Gamma \subset Q$ be the space of all the Jenkins-Strebel differentials whose associate system of curves is homotopic to Γ. Denote p_Γ: $\mathscr{E}_\Gamma \to T(S_0) \times \mathbb{R}_+^{3g-3}$, the mapping whose first factor is the canonical projection p restricted to $\mathscr{E}_\Gamma$ and whose second factor gives the heights of the cylinders with respect to curve γ_i.

Hubbard and Masur obtained the following result[4].

Theorem B[4]. The mapping p_Γ: $\mathscr{E}_\Gamma \to T(S_0) \times \mathbb{R}_+^{3g-3}$ is a homeomorphism.

From the result in sec. 2, we have the mapping as follows:

$$\tilde{L}: T(S_0) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3},$$

$$\tilde{L}([S_\sigma]) = (L(S_\sigma), \theta_1(S_\sigma), \cdots, \theta_{3g-3}(S_\sigma)).$$

Main Theorem. The mapping $\tilde{L}$: $T(S_0) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ is a homeomorphism.

According to Main Theorem, $\tilde{L}$ gives another global coordinate system on $T(S_0)$, which is different from the previous Fenchel-Nielsen coordinate of $T(S_0)$, etc.

Proof of Main Theorem. The proof of the theorem proceeds in three steps:

(1) $\tilde{L}$ is a continuous mapping;

(2) $\tilde{L}$ is surjective;

(3) $\tilde{L}$ is injective.

Let $(S_k, k = 1, 2, \cdots) \subset T(S_0)$ be a sequence satisfying $S_k \to S$ in the Teichmüller metric as $k \to \infty$. We set $\mathbb{R}_+^{3g-3} = \{2, 2, \cdots, 2\}$ in Theorem B. Since the aforementioned mapping is a homeomorphism, the differentials $-\varphi_k$ must be close to $-\varphi$ (in $\mathscr{MF}$) as $k \to \infty$, where $\varphi_k(\varphi)$ are the characteristic differentials of $S_k(S)$. So we have $l_i(S_k) = h_{-\varphi_k}(\gamma_i) \to l_i(S) = h_{-\varphi}(\gamma_i)$, $i = 1, 2, \cdots, 3g-3$. For each $\gamma_i \in \Gamma$, we assume that $\mathscr{P}_{i_1}$ and $\mathscr{P}_{i_2}$ share γ_i as the same boundary component on S_0 and set $\mathscr{W} = \mathscr{P}_{i_1} \cup \gamma_i \cup \mathscr{P}_{i_2}$. Furthermore, fix a simple closed curve $\delta \subset \mathscr{W}$ which intersects γ_i. Then $i(-\varphi_k, \delta) \to i(-\varphi, \delta)$. Using Theorem 25.4 in ref. [5], we have $\phi_i(S_k) \to \phi_i(S)$; thus $\theta_i(S_k) = 2\pi \dfrac{\phi_i(S_k)}{l_i(S_k)} \to \theta_i(S) = 2\pi \dfrac{\phi_i(S)}{l_i(S)}$. As a consequence we prove (1), i.e. $\tilde{L}$ is continuous.

From the construction of the Riemann surface S_σ, we have already proved that $\tilde{L}$ is a surjective.

Finally we have to show that $\tilde{L}$ is injective. Suppose that $[S]$ and $[\tilde{S}]$ are any two points in $T(S_0)$, and $\tilde{L}(S) = \tilde{L}(\tilde{S})$. Then $L(S) = L(\tilde{S})$. The corresponding pants $P_i \subset S$ and $\tilde{P}_i \subset \tilde{S}$ have the same length of boundary edges in the respective characteristic quadratic differentials. According to Theorem 1, we can construct a holomorphic homeomorphism h_i between P_i and $\tilde{P}_i$ such that h_i maps the boundary edges of P_i onto the corresponding boundary edges of $\tilde{P}_i$. Since $\theta_j(S) = \theta_j(\tilde{S})(j = 1, 2, \cdots, 3g-3)$, by Painlevé's theorem and the cubic graph $\mathscr{G}$, all the holomorphic homeomorphism h_i between P_i and $\tilde{P}_i$ can be glued together into a holomorphic homeomorphism h between S and $\tilde{S}$, which is isotopic to the identity mapping, we have $[S] = [\tilde{S}]$, which implies that $\tilde{L}$ is injective. According to Brouwer's theorem on the invariance of domain,

$$\tilde{L}: T(S_0) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

is a homeomorphism.

Remark. We can directly prove that $\tilde{L}^{-1}: \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \to T(S_0)$ is continuous. If we use quasiconformal mapping instead of using Brouwer Theorem, then we can give a new proof to the case where the Teichmüller space $T(S_0)$ is homeomorphic to the unit ball in the $6g-6$-dimensional Euclidean space.

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