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A sharp estimate for the hexagonal circle packing constants

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Abstract In [6] it is shown that the hexagonal circle packing rigidity constants s_n satisfy

$$\lim_{n\to\infty} ns_n = \frac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)}.$$

In this paper we further prove that

$$s_n = \frac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)}\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Keywords Rigidity constant · Circle packing · Quasiconformal map

Mathematics Subject Classification (2000) 52C15 · 30C35 · 30C75

1 Introduction

Let $n \geq 2$ be an integer. Consider all circle packings H_n' in the complex plane $\mathbb{C}$ with the combinatorics of the n-generations regular hexagonal packing H_n. The hexagonal circle-packing rigidity constant s_n is defined to be the supremum over $\{(r_1/r_0) - 1\}$, where r_1 is the radius of a 1st generation circle in H_n', and r_0 is the radius of the center circle of H_n'.

The sequence $\{s_n\}$ contains valuable information. Thurston [20] conjectured that the Riemann mapping f from a simple connected region $\Omega \subsetneqq \mathbb{C}$ onto the unit disk $\mathbb{D}$ can

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be approximated by the correspondences $\{f_\epsilon\}$ between the circle packings with the same combinatorics, where ϵ is the size of the preimage circles. By showing $s_n \to 0$, Rodin and Sullivan [17] successfully proved Thurston's conjecture. In [7] the first author proved that $s_n = O(1/n)$. This estimate, together with some results in [5,16], shows that the circle packing solutions f_ϵ have first order derivatives (defined in an appropriate sense) locally uniformly converging to the first order derivatives of f. Further results on the approximations of f' and f'' in terms of f_ϵ were obtained in [8–10]. See [1] for an alternating proof of the result $s_n = O(1/n)$. Different approaches and related topics on circle packings were given in [2–4,13,18].

In addition to its important role in developing Thurston's idea of discrete version of the Riemann Mapping Theorem, the sequence $\{s_n\}$ is of interest in its own right. It was shown in [6] that

$$\lim_{n\to\infty} ns_n = \frac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)}.$$

In this paper we will prove the following result.

Theorem 1 $s_n = \dfrac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)}\dfrac{1}{n} + O\left(\dfrac{1}{n^2}\right).$

Theorem 1 suggests the following conjecture.

Conjecture 1 *There exist constants $\{a_k\}$ such that* $s_n = \sum_{k=1}^{\infty} \dfrac{a_k}{n^k}.$

The estimate of s_n is briefly sketched as follows. To obtain the upper bound let H_n' be any n-generations circle packing on $\mathbb{C}$. Then we construct a quasiconformal homeomorphism G_n between the polygonal regions of H_n and H_n', which are formed by the union of line segments joining the centers of pairs of tangent boundary circles of H_n and H_n'. The quasiconformal homeomorphism G_n has Beltrami differential μ_n. And G_n is conformal in interstices bounded by circles of H_n. Also we show that the integral of μ_n is bounded from above by $O(1/n)$. Let $z_0 = \frac{1}{2n} + \frac{i}{2\sqrt{3}n}$. Using the Bieberbach Theorem, we establish that $|G_n''(z_0)| \le 4/R_0 + O(1/n)$, where $R_0 = 3\sqrt[3]{4}\Gamma(\frac{2}{3})/\Gamma^2(\frac{1}{3})$. Therefore we obtain

$$s_n \le \frac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)}\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

To prove the converse, we use the Koebe packing $\mathcal{KH}_n$, as in [6]. Let $\mathcal{G}_n : P \to \mathcal{KP}_n$ be the quasiconformal homeomorphism between the polygons of H_n and $\mathcal{KH}_n$. Then we extend $\mathcal{G}_n$ to a quasiconformal homeomorphism from the regular hexagonal $P^* \supset P$ to the region $\mathbb{C}\backslash\{-\infty < z \le -R_0/4\}$, still denoted by $\mathcal{G}_n$.

Let μ_n be the Beltrami differential of $\mathcal{G}_n$. After showing that the integral of μ_n in P^* is bounded from above by $O(1/n)$, we obtain $|\mathcal{G}_n''(z_0)| \ge 4/R_0 + O(1/n)$. It implies that

$$s_n \ge \frac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)}\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Notational conventions. Through this paper, for a ring domain R, we denote by $M(R)$ the conformal modulus of R. Also we denote by C or $C_j,\ j = 1, 2, \ldots$ some universal constants independent of n.

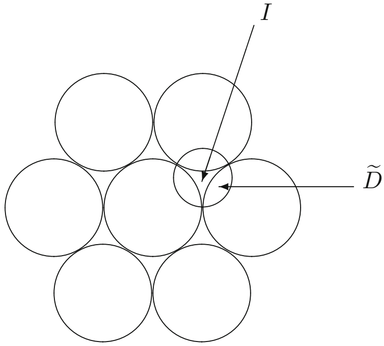

Fig. 1 An interstice and the corresponding dual disk

2 Preliminary results

We start by presenting some key results and notations along the lines as presented in [6,7]. A *circle packing* in the complex plane $\mathbb{C}$ is a collection of circles in $\mathbb{C}$ with disjoint interiors. An n-generations hexagonal circle packing H_n' is defined to be a circle packing combinatorially equivalent to the n-generations regular hexagonal packing H_n.

Let $c_k \in H_n$, $k = 1, 2, \ldots, 6$, be the first generation circles tangent to the center circle c_0 and let $c_1', c_2', \ldots, c_6'$ be the corresponding first generation circles in H_n'. We define

$$s_n = \sup_{\{(H_n', c_0')\}} \max_{1 \le k \le 6} \left(\frac{\text{radius}(c_k')}{\text{radius}(c_0')} - 1 \right),$$

where $\{(H_n', c_0')\}$ runs over all n-generations hexagonal circle packings in $\mathbb{C}$.

The main known results about the rigidity constants s_n are summarized in the following.

Theorem A ([17])$s_n \to 0$ *as* $n \to \infty$.

Theorem B ([7])$s_n \le C/n$ *for some constant* C *independent of* n.

Theorem C ([6])$\lim_{n\to\infty} n s_n = 2\sqrt[3]{2}\Gamma^2(\frac{1}{3})/3\Gamma(\frac{2}{3})$.

In the remainder of the paper H_n and H_n' are normalized as follows.

(1) The center circle c_0 or c_0' is centered at $0 \in \mathbb{C}$ and has radius of $1/2n$;
(2) $1/2n \in \mathbb{C}$ is the tangent point between the center circle and a circle of generation 1.

The closed bounded region bounded by three mutually tangent circles is called an *interstice*; the closed disk whose boundary circle is orthogonal to all three circles will be referred to as a *dual disk*, see Fig. 1. In this paper we will denote by I an interstice, and denote by $\widetilde{D}$ a dual disk.

Let H_{n+1}' be a hexagonal circle packing in $\mathbb{C}$ combinatorially equivalent to H_{n+1}. We denote by $c \mapsto c'$ the correspondence of circles under the combinatorial isomorphism $H_{n+1} \to H_{n+1}'$.

Let I be an interstice bounded by circles in H_{n+1}. There is a unique Möbius transformation $M_I : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $M_I|_I$ maps I to the interstice bounded by the corresponding circles in H_{n+1}'. The Möbius transformation M_I is uniquely determined by the position of 3 tangency points. By gluing together all these conformal maps $M_I|_I$, we obtain a conformal mapping from the union of interstices bounded by circles of H_{n+1} to the union of interstices bounded by circles of H_{n+1}'. The resulting conformal map is denoted by Φ_{n+1}.

In addition Φ_{n+1} maps each circle of the subpacking H_n to the corresponding circle of the subpacking H'_n. Also it is C_1-bi-Lipschitz. We can extend Φ_{n+1} radially on each disk bounded by circles of H_n. The Ring Lemma[1] [17] implies that the resulting map is quasiconformal inside each circle of H_n. Hence it is a quasiconformal in the carrier[2] of H_n.

By the classical quasiconformal extension theorem (see, e.g., [12]), the restriction of the above quasiconformal homeomorphism to the carrier of $H_{[n/2]}$ can be extended to a quasiconformal homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$. For each circle $c \in H_{[n/2]}$ and the corresponding circle $c' \in H'_{[n/2]}$, let γ_c (resp. $\gamma_{c'}$) denote the inversion on circle c (resp. c'). To further increase the region on which ϕ is conformal, we replace ϕ restricted to the disk bounded by the circle c by $\gamma_{c'} \circ \phi \circ \gamma_c$. The resulting map, denoted by ϕ_n^1, is conformal not only on the union I of interstices bounded by circles of $H_{[n/2]}$ but also on $\cup_{c \in H_{[n/2]}} \gamma_c(I)$. Next we further modify ϕ_n^1 in the disks bounded by circles $\gamma_{c_1}(c_2)$, where $c_1, c_2 \in H_{[n/2]}$, $c_1 \neq c_2$, by using inversions on the circles $\gamma_{c_1}(c_2)$ and $\gamma_{c'_1}(c'_2)$. The resulting map is denoted by ϕ_n^2. Similarly we may modify ϕ_n^2 in the disks bounded by circles $\gamma_{c_1}(\gamma_{c_2}(c_3))$, where $c_1, c_2, c_3 \in H_{[n/2]}$ and $c_1 \neq c_2 \neq c_3$, and obtain ϕ_n^3. Continuing in this way, we can find for each k a C-quasiconformal homeomorphism $\phi_n^k : \mathbb{C} \to \mathbb{C}$. Then ϕ_n^k converges to some quasi-conformal mapping $\phi_n^\infty : \mathbb{C} \to \mathbb{C}$. It is conformal on the union of interstices I bounded by circles of $H_{[n/2]}$ under the elements of the Schottky group generated by inversions of circles in $H_{[n/2]}$, see e.g., [7].

Let $I_1, I_2, \ldots, I_6$ be the chain of interstices adjacent to the center circle c_0 of H_n so that I_1 is the interstice which has vertices $\{1/(2n),\ e^{i\pi/3}/(2n),\ (1+e^{i\pi/3})/(2n)\}$.

Let M_{I_j} be the Möbius transformations which satisfies that $M_{I_j}|_{I_j} = \Phi_{n+1}|_{I_j}$, $j = 1, 2, \ldots, 6$. Suppose that D_0 and D'_0 are the disks bounded by the center circles c_0 and c'_0, respectively. In [7] the first author proved that the area of the subset of D_0 where $\phi_n^\infty : D_0 \to D'_0$ fails to be conformal is bounded from above by $O\left(1/n^2\right) \cdot \mathrm{Area}(D_0)$. By using the Area-Length method he also proved that, for $j = 1, 2, \ldots, 6$,

$$|\phi_n^\infty(z) - M_{I_j}(z)| \leq O(1/n) \cdot r(D_0), \quad \forall z \in D_0, \tag{2.1}$$

where $r(D_0)$ denotes the radius of D_0. If, in addition, z is in the boundary c_0, then there is a better estimate (see Lemma 1.5 in [6])

$$|\phi_n^\infty(z) - M_{I_j}(z)| \leq O(1/n^2) \cdot r(D_0), \quad \forall z \in c_0, \quad j = 1, 2, \ldots, 6. \tag{2.2}$$

Recall that the subpackings H_n and H'_n are normalized. It follows from (2.2) that

$$|M_{I_i}(z) - M_{I_j}(z)| \leq O(1/n^3), \quad \forall z \in c_0, \quad 1 \leq i, j \leq 6.$$

Note that the radii of the dual disks $\widetilde{D}_j$ $(1 \leq j \leq 6)$ are bounded from above and from below by $O(1/n)$. By the C^1 convergence of circle packings to Riemann map (see, e.g., [5,6]), we deduce $\mathrm{d}M_{I_i}(z)/\mathrm{d}z$ is uniformly bounded from above and from below in D_0 independent of n. This implies

$$|M_{I_i}^{-1} \circ M_{I_j}(z) - z| \leq O(1/n^3), \quad \forall z \in c_0, \quad 1 \leq i, j \leq 6. \tag{2.3}$$

In this paper we will use another quasiconformal homeomorphism G_n, which is similar to ϕ_n^∞.

[1] The Ring Lemma says that there is a universal lower bound for the ratio of radii of two neighbor circles in H'_n.

[2] The carrier of a circle packing is by definition the union of all closed disks bounded by circles and all interstices bounded by circles of the packing.

Recall that Φ_{n+1} is a conformal mapping from the union of interstices bounded by circles of H_{n+1} to the union of interstices bounded by circles of H'_{n+1}. We let $G_n \equiv \Phi_{n+1}$ on the union of interstices bounded by circles of H_n. By radial extension we can define G_n in the interior of each circle $c \in H_n \backslash H_{n-1}$. For each disk D bounded by circle $c \in H_{n-1}$, let I_D be one of the six interstices adjacent to D. Also we denote by M_D the transformation which satisfies that $M_D|_{I_D} = \Phi_{n+1}|_{I_D}$. We define $G_n|_D \equiv M_D \circ F_D$, where F_D is the radial extension of the map $M_D^{-1} \circ \Phi_{n+1}|_c$.

Let $P \equiv P_n$ (resp. P'_n) be the Jordan region bounded by the polygon formed by the union of line segments joining centers of pairs of tangent boundary circles of H_n (resp. H'_n). Therefore we obtain a quasiconformal homeomorphism

$$G_n : P \to P'_n, \quad \text{with} \ \ G_n(0) = O(1/n^2). \tag{2.4}$$

Suppose that μ_n is the Beltrami differential of the quasiconformal homeomorphism G_n. Let

$$F_n : P \to P \tag{2.5}$$

be the quasiconformal homeomorphism with the Beltrami differential μ_n. If we further require that $F_n^{-1}(0) = G_n^{-1}(0)$ and $F_n(1/2) > 0$, then F_n is uniquely determined. The following results play important roles in our estimates.

Lemma 2.1 *For $|z| \le 3/4$, we have $|\mu_n(z)| = O(1/n^2)$ a.e. Moreover on the hexagonal P we have*

$$\iint\limits_P |\mu_n(z)| \,\mathrm{d}x\mathrm{d}y \le O\left(\frac{1}{n}\right).$$

Lemma 2.2 *The moduli of the regions $P\backslash\{|z| \le 1/2n\}$ and $F_n(P\backslash\{|z| \le 1/2n\})$ satisfy*

$$M(F_n(P\backslash\{|z| \le 1/2n\})) = M(P\backslash\{|z| \le 1/2n\}) + O(1/n).$$

Lemma 2.3 *With respect to the quasiconformal map F_n we have*

$$|F_n(z) - z| = O(1/n), \quad \forall\ |z| < 3/4.$$

Furthermore

$$|F'_n(z_0) - 1| \le O(1/n), \quad |F''_n(z_0)| \le O(1/n),$$

where $z_0 = \frac{1}{2n} + \frac{i}{2\sqrt{3}n}$ is an interior point of the interstice I_1.

The proofs of the above lemmas will be postponed to Sect. 5.

3 Estimate of the upper bound of s_n

Recall that I_1 is the interstices which has vertices $\{1/(2n),\ e^{i\pi/3}/(2n),\ (1+e^{i\pi/3})/(2n)\}$. For notation simplicity we denote $M_n \equiv M_{I_1}$. Note that the circles c_0, c'_0 both have radii $1/(2n)$. Since the Möbius transformation M_n satisfies that $M_n(c_0) = c'_0$, we obtain $M_n(z) = e^{2\pi i\theta}\dfrac{z+\beta_n}{1+4n^2\bar{\beta}_n z}$, where $|\beta_n| < 1/(2n)$. Hence

$$|M'_n(0)| = 1 - 4n^2|\beta_n|^2, \quad |M''_n(0)| = 8n^2|\beta_n|(1 - 4n^2|\beta_n|^2). \tag{3.6}$$

Denote by $\gamma_0 = \{|z| = 3/(2n)\}$. The smallest and largest circles mutually tangent to $M_n(c_0)$ and $M_n(\gamma_0)$ have radii

$$\frac{1}{2n}\frac{1-2n|\beta_n|}{1+6n|\beta_n|} \quad \text{and} \quad \frac{1}{2n}\frac{1+2n|\beta_n|}{1-6n|\beta_n|}, \quad \text{respectively.}$$

Hence the radius r of the circle mutually tangent to $M_n(c_0)$ and $M_n(\gamma_0)$ satisfies that

$$(n+1)|1-2nr| = 8n^2|\beta_n| + O(1/n) = |M_n''(0)| + O\left(\frac{1}{n+1}\right). \tag{3.7}$$

Consider the radii r of the largest and smallest images under M_n of the six generation one circles of H_n. Then we have $s_{n+1} \leq \sup|1-2nr|$, where the supremum is taken over all choices of M_n for all n-generations subpacking H_n'. From (3.7), together with the above fact, it follows that

$$(n+1)\cdot s_{n+1} \leq (n+1)\cdot \sup\{|1-2nr|\} = \sup\{|M_n''(0)|\} + O\left(\frac{1}{n+1}\right). \tag{3.8}$$

Recall that G_n is the quasiconformal homeomorphism between the polygons P and P_n' and has the Beltrami differential μ_n. As in (2.5), we obtain a quasiconformal homeomorphism $F_n : P \to P$ with the Beltrami differential μ_n. Then we have $G_n = K_n \circ F_n : P \to P_n'$, where $K_n : P \to P_n'$ is a holomorphic mapping satisfies $K_n(0) = 0$. It immediately follows from Lemma 2.3 that, for $z_0 = \frac{1}{2n} + \frac{i}{2\sqrt{3}n}$

$$|F_n'(z_0) - 1| \leq O(1/n), \quad |F_n''(z_0)| \leq O(1/n). \tag{3.9}$$

The Riemann Mapping Theorem implies that, for $r = O(1/n)$ the region $P\backslash\{|z| \leq r\}$ has modulus

$$M(P\backslash\{|z|\leq r\}) = \frac{1}{2\pi}\log\frac{R_0}{r} + O(1/n), \tag{3.10}$$

where $R_0 = 3\sqrt[3]{4}\Gamma(\frac{2}{3})/\Gamma^2(\frac{1}{3}) = 0.89854\ldots$ is the conformal radius of P, please refer to [11,14]. Lemma 2.2 shows that the region $G_n(P\backslash\{|z| \leq 1/2n\})$ has modulus $\frac{1}{2\pi}\log 2nR_0 + O(1/n)$. Hence the region $P\backslash K_n^{-1}\{|z| \leq 1/2n\} = F_n(P\backslash\{|z| \leq 1/2n\})$ has modulus

$$\frac{1}{2\pi}\log 2nR_0 + O(1/n). \tag{3.11}$$

If $K_n(z) = a_1z + a_2z^2 + \cdots$ at the neighborhood of $z = 0$, then (3.10) implies that the region $P\backslash K_n^{-1}\{|z| \leq 1/2n\}$ has modulus $\frac{1}{2\pi}\log 2n|a_1|R_0 + O(1/n)$. Together with (3.11) we obtain that $|K_n'(0)| = |a_1| = 1 + O(1/n)$.

From the Bieberbach Theorem it follows that $|K_n''(0)| \leq 4/R_0|K_n'(0)| \leq 4/R_0 + O(1/n)$, please refer to [15]. By using (3.9) and the chain rule $G_n''(z_0) = K_n''(F_n(z_0))(F_n'(z_0))^2 + K_n'(F_n(z_0))\widetilde{F}_n''(z_0)$, we obtain

$$\begin{aligned} |M_n''(z_0)| = |G_n''(z_0)| &\leq K_n''(F_n(z_0))(F_n'(z_0))^2| + |K_n'(F_n(z_0))F_n''(z_0)| \\ &\leq (4/R_0 + O(1/n))(1 + O(1/n)) + O(1/n) \\ &= 4/R_0 + O(1/n). \end{aligned} \tag{3.12}$$

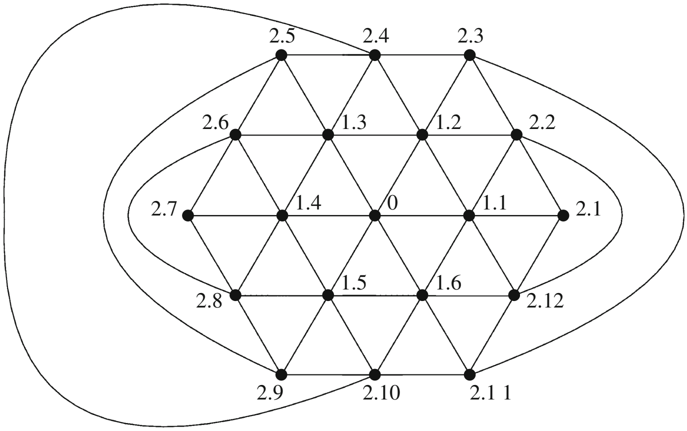

Fig. 2 The decomposition for $n = 2$

The fact (2.1) i.e. $|\beta_n| = O(1/n^2)$, implies that

$$|M_n'''(z)| = \frac{|96n^4\beta_n^2(1-4n^2\beta_n^2)|}{|1+4n^2\beta_n z|^4} \le C$$

in the δ-neighborhood of 0 (δ is independent of n). From (3.12) we see that

$$|M_n''(0)| \le |M_n''(z_0)| + O(1/n) = 4/R_0 + O(1/n). \tag{3.13}$$

Combining (3.8) with (3.13), we get the estimate

$$(n+1)\cdot s_{n+1} \le 4/R_0 + O\left(\frac{1}{n+1}\right), \quad R_0 = 3\sqrt[3]{4}\Gamma\left(\frac{2}{3}\right)/\Gamma^2\left(\frac{1}{3}\right) = 0.89854\ldots. \tag{3.14}$$

4 Estimate of the lower bound of s_n

This section will begin the estimate of the upper bound of s_n. As in [6], we construct the Koebe Packing $\mathcal{KH}_n$. Let

$$\Lambda_n = \{(a/n) + (b/n)e^{i\pi/3} : a, b \in \mathbb{Z}\}$$

consist of the points of the hexagonal lattice. Then Λ_n determines a triangulation of P by equilateral triangles. We modify this triangulation to get a decomposition of the entire 2-sphere $\hat{\mathbb{C}}$ as follows. For each vertex $v \neq \pm 1$ on the boundary of P, add an edge joining v to its complex conjugation $\bar{v}$. This complex yields a decomposition of the 2-sphere into triangles and quadrilaterals. Moreover, by adding a vertex to the interior of each quadrilateral and connect it to all four vertices of this quadrilateral, we obtain a triangulation of the 2-sphere $\hat{\mathbb{C}}$, see Fig. 2. The Andreev–Koebe–Thurston Theorem in [19] shows that there is circle packing on $\hat{\mathbb{C}}$ realizing this triangulation (see, e.g., Fig. 3). Also it is unique up to Möbius transformations.

We select a particular realization as follows. We require that the disk bounded by circle that corresponds to the vertex of $1 \in \Lambda_n$ should be a right half plane, and the circle corresponding to the vertex of $-1 \in \Lambda_n$ should have its center on the real axis and with left-hand

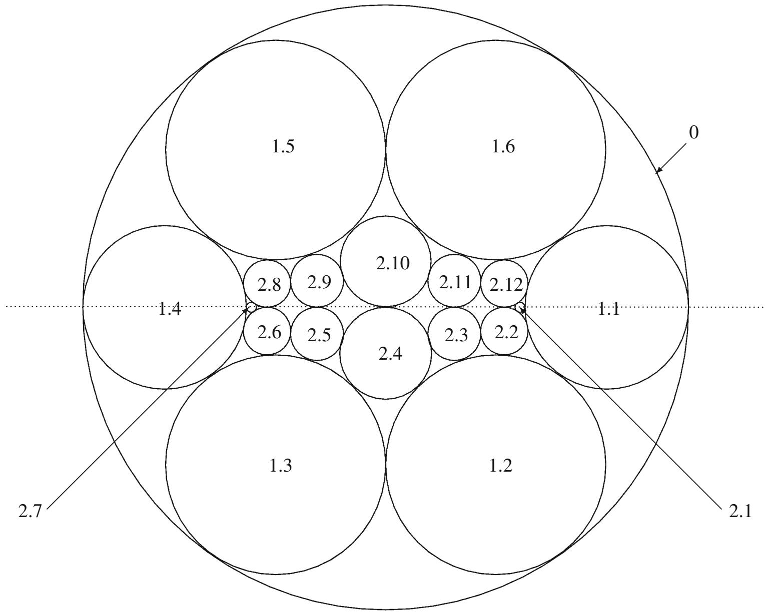

Fig. 3 Allowable realization for $n = 2$

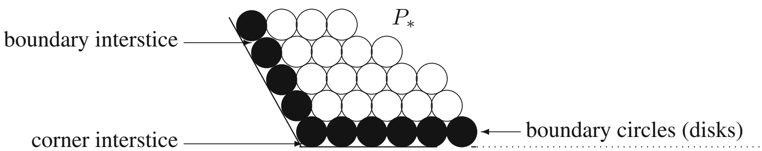

Fig. 4 Boundary disks, boundary interstice and corner interstice

endpoint at $-R_0/4$. Also the circle corresponding to the vertex $0 \in \Lambda_n$ is centered at the origin 0.

This particular allowable circle packing is called the Koebe packing and is denoted by $\mathcal{KH}_n$. Note that $\mathcal{KH}_n$ may not be a normalized n-generation circle packing since the radius of its center circle may not be $1/2n$.

Let $\mathcal{KP}_n$ be the polygon of $\mathcal{KH}_n$. Then as in (2.4), we obtain a quasiconformal homeomorphism $\mathcal{G}_n : P \to \mathcal{KP}_n$. Let P_* be the minimum regular hexagon containing H_n (see Fig. 4), which has side length $1 + \frac{1}{\sqrt{3}n}$.

In order to obtain the lower bound in our estimate, we need the following result. Its proof is also postponed to Sect. 5.

Lemma 4.1 *We can extend the quasiconformal mapping $\mathcal{G}_n : P \to \mathcal{KP}_n$ to a quasiconformal homeomorphism $P_* \to W \equiv \{-\infty < z \le -R_0/4\}$ (where $R_0 = 3\sqrt[3]{4}\Gamma(\frac{2}{3})/\Gamma^2(\frac{1}{3})$), still denoted by $\mathcal{G}_n$. Furthermore, if μ_n is the Beltrami differential of $\mathcal{G}_n : P_* \to W$, then $\mu_n(z) = O(1/n^2)$, a.e. $|z| \le 3/4$. Also we have*

$$\iint_{P_*} |\mu_n(z)|\, \mathrm{d}x\mathrm{d}y = O(1/n), \quad \textit{where } z = x + iy.$$

Let us write $\mathcal{G}_n = \mathcal{K}_n \circ \mathcal{F}_n : P_* \to W$, where $\mathcal{F}_n : P_* \to P$ is the quasiconformal mapping with Beltrami differential μ_n. Also $\mathcal{F}_n$ satisfies that $\mathcal{F}_n^{-1}(0) = \mathcal{G}_n^{-1}(0)$ and $\mathcal{F}_n(1/2) > 0$. And $\mathcal{K}_n : P \to W$ is the conformal mapping with $\mathcal{K}_n(0) = 0,\ \mathcal{K}_n(\mathcal{F}_n(1/2)) > 0$. Thus $\mathcal{K}_n'(0) = 1,\ \mathcal{K}_n''(0) = 4/R_0$.

Lemma 4.1 implies that, for $z_0 = \frac{1}{2n} + \frac{i}{2\sqrt{3}n}$,

$$|\mathcal{F}_n'(z_0) - 1| \le O(1/n), \quad |\mathcal{F}_n''(z_0)| \le O(1/n). \tag{4.15}$$

By using (4.15) and the chain rule, it turns out that

$$\begin{aligned} |\mathcal{G}_n''(z_0)| &\ge |\mathcal{K}_n''(\mathcal{F}_n(z_0))(\mathcal{F}_n'(z_0))^2| - |\mathcal{K}_n'(\mathcal{F}_n(z_0))\mathcal{F}_n''(z_0)| \\ &\ge (4/R_0 + O(1/n))(1 + O(1/n)) + O(1/n) \\ &= 4/R_0 + O(1/n). \end{aligned} \tag{4.16}$$

As in (3.10), for $r = O(1/n)$ the region $W \backslash \{|z| \le r\}$ have modulus

$$M(W \backslash \{|z| \le r\}) = \frac{1}{2\pi} \log \frac{R_0}{r} + O(1/n). \tag{4.1}$$

Let D_n be the disk bounded by the center circle of the Koebe packing $\mathcal{KH}_n$. By applying Lemma 2.2 and 4.1, we obtain

$$M\left(P_* \backslash \{|z| \le 1/2n\}\right) = M\left(W \backslash D_n\right) + O(1/n). \tag{4.2}$$

If the radius of D_n is r_n, (3.10), (4.1), and (4.2) immediately give

$$n\left|r_n - \frac{1}{2n}\right| = O(1/n). \tag{4.3}$$

By scaling $\mathcal{KH}_n$ we obtain a normalized circle packing $\mathcal{H}_n$. Let $\mathcal{P}_n$ be the polygon of $\mathcal{H}_n$. The quasiconformal mapping between P and $\mathcal{P}_n$ is

$$\mathcal{G}_n/(2nr_n) : P \to \mathcal{P}_n.$$

We denote by $\mathcal{M}_n$ the Möbius transformation which maps I_1 to the interstice bounded by the corresponding circles of $\mathcal{H}_n$. From (4.3) and (4.16) we deduce that

$$\begin{aligned} |\mathcal{M}_n''(z_0)| &= \frac{1}{2nr_n}|\mathcal{G}_n''(z_0)| \ge \frac{1}{2nr_n}(4/R_0 + O(1/n)) \\ &= (1 + O(1/n))(4/R_0 + O(1/n)) \\ &= 4/R_0 + O(1/n). \end{aligned} \tag{4.4}$$

Moreover, the fact $|\mathcal{M}_n(0)| = O(1/n^2)$ implies that $|\mathcal{M}_n'''(z)| \le C$ in the δ-neighborhood of 0 (δ is independent of n). Hence $|\mathcal{M}_n''(0)| \ge |\widetilde{\mathcal{M}}_n''(z_0)| + O(1/n)$.

By the similar argument as in (3.7), we deduce that the radii r_0 of circles mutually tangent to $\mathcal{M}_n(c_0)$ and $\mathcal{M}_n(\gamma_0)$ satisfy $n|1 - 2nr_0| = |\mathcal{M}_n''(0)| + O(1/n)$. It implies that

$$ns_n \ge n|1 - 2nr_0| = |\mathcal{M}_n''(0)| + O(1/n),$$

which together with (4.4) implies

$$ns_n \ge |\mathcal{M}_n''(0)| + O(1/n) \ge 4/R_0 + O(1/n). \tag{4.5}$$

Summing up (3.14) and (4.5), we obtain Theorem 1.

5 Proof of the lemmas

Now we can begin the proofs of the lemmas. For later use we will need the following elementary lemma.

Lemma 5.1 *Consider the rectangles* $R = [0, m] \times [0, 1]$ *and* $R' = [0, m'] \times [0, 1]$, *which have conformal moduli* m *and* m', *respectively. Also we assume* $1/C < m,\ m' < C$ *for some* $C > 1$. *Let* $f : R \to R'$ *be a* K*-quasiconformal mapping with maximal dilatation* K, *which maps the corners of* R *to the corresponding corners of* R'.

If there exists an integer $n \geq 1$ *such that the Beltrami differential* $\mu = \mu_f$ *satisfies*

$$\iint_R |\mu(z)|\,\mathrm{d}x\mathrm{d}y \leq O(1/n),$$

then $|m - m'| \leq O(1/n)$.

Proof Let J_f be the Jacobian of f. For any $y \in [0, 1]$, we have

$$m' = \int_0^m \frac{\partial f(x, y)}{\partial x}\,\mathrm{d}x \leq \int_0^m \left|\frac{\partial f(x, y)}{\partial x}\right|\,\mathrm{d}x \leq \int_0^m K_f^{1/2} J_f^{1/2}\,\mathrm{d}x. \tag{5.1}$$

Squaring both sides of (5.1) and by applying the Schwartz inequality gives

$$\begin{aligned}(m')^2 &\leq \left(\int_0^1\int_0^m K_f^{1/2} J_f^{1/2}\,\mathrm{d}x\mathrm{d}y\right)^2\\ &\leq \int_0^1\int_0^m K_f\,\mathrm{d}x\mathrm{d}y \cdot \int_0^1\int_0^m J_f\,\mathrm{d}x\mathrm{d}y\\ &= m'\int_0^1\int_0^m K_f\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Since $K_f(z) - 1 = \dfrac{2|\mu(z)|}{1 - |\mu(z)|} \leq C_1|\mu(z)|$, we deduce that

$$m' \leq \int_0^1\int_0^m [1 + (K_f - 1)]\,\mathrm{d}x\mathrm{d}y \leq m + O(1/n). \tag{5.2}$$

Similarly, by considering the rectangles $[0, 1] \times [0, \frac{1}{m}]$ and $[0, 1] \times [0, \frac{1}{m'}]$ we conclude that

$$1/m' \leq 1/m + O(1/n). \tag{5.3}$$

It follows by (5.2) and (5.3) that $|m - m'| \leq O(1/n)$, as desired. □

Proof of Lemma 2.1 For any circle $c \in H_m \backslash H_{m-1}$ $(1 \leq m \leq n - 1)$, it is the center circle of some $(n - m)$-generations regular hexagonal circle packing. That means the configuration of the $(n - m)$-generation circles in H_n around c is combinatorially equivalent to H_{n-m}.

Denote by D the disk bounded by c. Let $I_{D,j}$, $1 \leq j \leq 6$, be the 6 interstices adjacent to the disk D. Let $M_{D,j}$ be the Möbius transformation which satisfies that

$M_{D,j}|_{I_{D,j}} = \Phi_{n+1}|_{I_{D,j}}$. Without loss of generality we assume $I_D \equiv I_{D,1}$ and $M_D \equiv M_{D,1}$. Then the mapping $F_D|_c$ has the form

$$M_D^{-1} \circ M_{D,j}(z) = e^{2\pi i\theta} \frac{z + \beta_j}{1 + 4n^2 \bar{\beta}_j z}, \quad z \in c,$$

for some $1 \leq j \leq 6$. By using (2.3) and the maximal principle we obtain $|\beta_j| = O\left(\dfrac{1}{n(n-m)^2}\right)$.

Let $\mathbb{D} = \{|w| < 1\}$. We consider the conjugation transformation $\mathbb{G}(w) = 2nF_D(w/2n) : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. We see that $\mathbb{G}(w) = e^{2\pi i\theta} \dfrac{w + \alpha_j}{1 + \bar{\alpha}_j w}$, $w \in \partial\mathbb{D}$, where $\alpha_j = 2n\beta_j$. Therefore $\mathbb{G} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ has the form

$$\mathbb{G}(w) = 2nF_D(w/2n) = e^{2\pi i\theta} |w| \frac{\frac{w}{|w|} + \alpha_j}{1 + \bar{\alpha}_j \frac{w}{|w|}} = e^{2\pi i\theta} w \frac{|w| + \alpha_j \bar{w}}{|w| + \bar{\alpha}_j w}, \quad \forall\, w \in \mathbb{D}\backslash\{0\}. \tag{5.4}$$

By straightforward computation we obtain at once

$$\mathbb{G}_w(w) = e^{2\pi i\theta} \frac{2|w|^2 + \alpha_j \bar{w}|w| + \bar{\alpha}_j w|w|}{2(|w| + \bar{\alpha}_j w)^2} = 1 + O(1/(n-m)^2),$$

and

$$\mathbb{G}_{\overline{w}}(w) = e^{2\pi i\theta} \frac{\alpha_j \bar{w}|w| + \bar{\alpha}_j |w| w + 2|\alpha_j|^2 |w|^2}{2(\bar{w} + \bar{\alpha}_j |w|)^2} = O(1/(n-m)^2). \tag{5.5}$$

Then $\mu_{\mathbb{G}}(w) = O(1/(n-m)^2)$, $\ w \in \mathbb{D}\backslash\{0\}$, and hence

$$|\mu_n(z)| = |\mu_{F_D}(z)| = O(1/(n-m)^2), \quad z \in D. \tag{5.6}$$

In particular, we obtain

$$\mu_n(z) = O(1/n^2), \qquad \text{a.e. } |z| \leq 3/4. \tag{5.7}$$

Moreover, by using (5.6) and (5.7) we get the desired result

$$\iint_P |\mu_n(z)|\, \mathrm{d}x\mathrm{d}y = \frac{C_1}{n^2} + \sum_{m=[n/2]}^{n-1} \frac{6mC_2}{(n-m)^2} \pi r^2$$
$$\leq \frac{C_1}{n^2} + 6\pi C_2 n r^2 \sum_{m=1}^{[n/2]} \frac{1}{m^2} \leq O\left(\frac{1}{n}\right), \quad \text{where } r = 1/2n. \tag{5.8}$$

□

Proof of Lemma 2.2 Consider the diagram

$$\begin{array}{ccc} P\backslash\{|z| \leq 1/2n\} & \xrightarrow{F_n} & F_n(P\backslash\{|z| \leq 1/2n\}) \\ \pi \big\downarrow & & \overline{\pi} \big\downarrow \\ R_n = \{w : r_n < |w| < 1\} & \xrightarrow{\overline{F}_n} & \overline{R}_n = \{\zeta : \overline{r}_n < |\zeta| < 1\}, \end{array}$$

where π and $\overline{\pi}$ are the conformal homeomorphisms. The quasiconformal homeomorphism $\overline{F}_n : R_n \to \overline{R}_n$ induces a Beltrami differential μ_n' on R_n.

The circle $\{|w| = 1/10\} \subset R_n$ divides R_n into two ring domains $R_n' = \{r_n < |w| < 1/10\}$ and $R_n'' = \{1/10 < |w| < 1\}$. We have $|\mu_n'(w)| = O(1/n^2)$, $w \in R_n'$, which implies that

$$M(\overline{F}_n(R_n')) = M(R_n') + O(1/n^2) \cdot M(R_n') = M(R_n') + O(1/n). \tag{5.9}$$

Since the holomorphic map π behaves like $z \to \zeta = z^{3/2}$ near each of the six corners of P, we get

$$\iint_{R_n''} |\mu_n'| \, \mathrm{d}A_w = O(1/n),$$

where $\mathrm{d}A_w$ denotes the area element on R_n. It follows from Lemma 5.1 that $M(\overline{F}_n(R_n'')) = M(R_n'') + O(1/n)$, which together with (5.9) shows that

$$M(\overline{R}_n) \geq M(R_n) + C_3/n. \tag{5.10}$$

Similarly, the circle $\{|\varpi| = 1/10\} \subset \overline{R}_n$ divides $\overline{R}_n$ into two regions $\overline{R}_n' = \{\overline{r}_n < |\varpi| < 1/10\}$ and $\overline{R}_n'' = \{1/10 < |\varpi| < 1\}$. Since $\mu_n'(w) = O(1/n^2)$, $w \in \overline{F}_n^{-1}(\overline{R}_n')$, we obtain that

$$M(\overline{F}_n^{-1}(\overline{R}_n')) = M(\overline{R}_n') + O(1/n). \tag{5.11}$$

Let $\widetilde{\pi} : \{\widetilde{r}_n < |z| < 1\} \to \overline{F}_n^{-1}(\overline{R}_n'')$ be the conformal map. It is obvious that $0 < c < \widetilde{r}_n < C < 1$ for some positive constants c and C. The quasiconformal map $F_n \circ \widetilde{\pi} : \{\widetilde{r}_n < |z| < 1\} \to \overline{R}_n''$ induces a Beltrami differential μ_n'' on $\{\widetilde{r}_n < |z| < 1\}$. Let $\mathrm{d}\tilde{A}$ denotes the area element on $\{\widetilde{r}_n < |z| < 1\}$. By applying Lemma 2.1 and 5.1, we get

$$\begin{aligned}\iint_{\{\widetilde{r}_n < |z| < 1\}} |\mu_n''(z)| \, \mathrm{d}\tilde{A} &= \iint_{\{\widetilde{r}_n < |z| < C\}} |\mu_n''(z)| \, \mathrm{d}\tilde{A} + \iint_{\{C < |z| < 1\}} |\mu_n''(z)| \, \mathrm{d}\tilde{A} \\ &= O(1/n^2) + O(1/n) = O(1/n).\end{aligned} \tag{5.12}$$

Hence $M(F_n^{-1}(\overline{R}_n'')) = M(\overline{R}_n'') + O(1/n)$, which together with (5.11) implies that

$$M(\overline{R}_n) \leq M(R_n) + C_4/n. \tag{5.13}$$

Using (5.10) and (5.13) we have the desired result

$$M(F_n(P \backslash \{|z| \leq 1/2n\})) = M(P \backslash \{|z| \leq 1/2n\}) + O(1/n). \qquad \square$$

Proof of Lemma 2.3 Let $H : P \to \mathbb{D}$ be the Riemann mapping with $H(0) = 0$ and $H'(0) > 0$.

The quasiconformal homeomorphism $F_H \equiv H \circ F_n \circ H^{-1} : \mathbb{D} \to \mathbb{D}$ induces a Beltrami differential μ_H on $\mathbb{D}$. Also $F_H^{-1}(0) = H \circ G_n^{-1}(0) = O(1/n^2)$ and $F_H(w_0) > 0$, where $w_0 = H(1/2) > 0$. Since the map H behaves like $z \mapsto z^{3/2}$ near each of the six corners of P, we deduce $|H'(z)| \leq C$, $\forall z \in P$. Therefore it follows from Lemma 2.1 that, on the unit disk $\mathbb{D}$,

$$\begin{aligned}\iint_{\mathbb{D}} |\mu_H(w)| \, \mathrm{d}A &= \iint_P |\mu_H(w(z))| |H'(z)|^2 \, \mathrm{d}x\mathrm{d}y \\ &\leq C^2 \iint_P |\mu_n(z)| \, \mathrm{d}x\mathrm{d}y = O(1/n).\end{aligned} \tag{5.14}$$

Using inversion on $\partial\mathbb{D}$, we may extend F_H to a quasiconformal mapping $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, still denoted by F_H. In other words, $F_H(w) = 1/\overline{F_H(\frac{1}{\overline{w}})}$. From the definition and (2.1) it follows that, in the spherical metric $\mathrm{d}_\rho(\cdot,\cdot)$ on $\hat{\mathbb{C}} \cong \mathbb{S}^2$,

$$\mathrm{d}_\rho\left(F_H^{-1}(0),\ 0\right) = O(1/n^2),\ \ \mathrm{d}_\rho\left(F_H^{-1}(\infty),\ \infty\right) = O(1/n^2). \tag{5.15}$$

Now the quasiconformal mapping $F_H : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ induces a Beltrami differential

$$\begin{cases} \mu_H(w), & \text{if } |w| \le 1, \\ \mu_H\,(1/\overline{w})\cdot w^2/\overline{w}^2, & \text{if } |w| > 1. \end{cases}$$

still denoted by μ_H. Obviously in the spherical metric ρ on $\hat{\mathbb{C}}$ we have

$$||\mu_H||_{\hat{\mathbb{C}}} \equiv \iint\limits_{\hat{\mathbb{C}}} |\mu_H(w)|\,\mathrm{d}A_\rho = O(1/n),$$

where $\mathrm{d}A_\rho$ denotes the area element on $\hat{\mathbb{C}} \cong \mathbb{S}^2$.

The quasiconformal map F_H maps the four-punctured sphere $\hat{\mathbb{C}}\backslash\{F_H^{-1}(0), w_0, 1/w_0, F_H^{-1}(\infty)\}$ onto the four-punctured sphere $\hat{\mathbb{C}}\backslash\{0, F_H(w_0), 1/F_H(w_0), \infty\}$, where $w_0 = F_H(1/2) > 0$. These punctured spheres are doubly covered by the four-punctured tori T_1 and T_2 (via some elliptic functions π_1 and π_2). Thus F_H can be lifted to a quasiconformal mapping $\mathcal{F} : T_1 \to T_2$, which has Beltrami differential $\mu_{\mathcal{F}}$.

If $w \in \hat{\mathbb{C}}$ is near one of the punctures $\{F_H^{-1}(0), w_0, 1/w_0, F_H^{-1}(\infty)\}$, it follows from (5.7) that $|\mu_H(w)| = O(1/n^2)$. In addition, the branched covering $\pi_1 : T_1 \to \hat{\mathbb{C}}$ is smooth at the pre-image (by π_1) of the region bounded away from these punctures $\{F_H^{-1}(0), w_0, 1/w_0, F_H^{-1}(\infty)\}$. So, if T_1 is endowed with the flat metric of total volume uniformly bounded from above, then we have the estimate

$$\iint\limits_{T_1} |\mu_{\mathcal{F}}(z)|\,\mathrm{d}A_1 = O(1/n), \tag{5.16}$$

where $\mathrm{d}A_1$ denotes the flat area element in T_1.

The following process is analogous to [7]. For the sake of completeness we give it here.

Let ζ_1 (resp. ζ_2) be the conformal modulus of T_1 (resp. T_2). Then $T_j = \mathbb{C}/\{m+n\ \zeta_j,\ m,n \in \mathbb{Z}\},\ \ j = 1,2$. If $\widetilde{\mathcal{F}} : \mathbb{C} \to \mathbb{C}$ is the lift of $\mathcal{F} : T_1 \to T_2$, then it should satisfy that

$$\widetilde{\mathcal{F}}(z+1) = \widetilde{\mathcal{F}}(z) + 1,\ \ \widetilde{\mathcal{F}}(z+\zeta_1) = \widetilde{\mathcal{F}}(z) + \zeta_2.$$

Therefore,

$$\begin{aligned} 1 = \widetilde{\mathcal{F}}(iy+1) - \widetilde{\mathcal{F}}(iy) &= \int_0^1 \frac{\partial\widetilde{\mathcal{F}}}{\partial x}(x+iy)\,\mathrm{d}x \\ &\le \int_0^1 \left|\frac{\partial\widetilde{\mathcal{F}}}{\partial x}(x+iy)\right|\,\mathrm{d}x \le \int_0^1 K^{1/2}J^{1/2}\,\mathrm{d}x, \end{aligned}$$

where $K(x+iy)$ is the maximal dilatation of $\widetilde{\mathcal{F}}$, and $J(x+iy)$ is the Jacobi of $\widetilde{\mathcal{F}}$. Integrating the above inequality with respect to $y \in [0, y_1]$ $(y_1 = \Im(\zeta_1))$ gives

$$y_1 \leq \int_0^{y_1}\int_0^1 K^{1/2}J^{1/2}\,\mathrm{d}x\mathrm{d}y = \iint_{T_1} K^{1/2}J^{1/2}\,\mathrm{d}A_1.$$

By an obvious application of the Schwarz inequality we conclude that

$$y_1^2 \leq \iint_{T_1} K\,\mathrm{d}A_1 \cdot \iint_{T_1} J\,\mathrm{d}A_1 = \iint_{T_1} K\,\mathrm{d}A_1 \cdot \mathrm{Area}(T_2)$$
$$= \iint_{T_1} K\,\mathrm{d}A_1 \cdot (\Im(\zeta_2)).$$

Now Ring Lemma [17] implies $K-1 = \dfrac{2|\mu_{\widetilde{\mathcal{F}}}|}{1-|\mu_{\widetilde{\mathcal{F}}}|} \leq C_5|\mu_{\widetilde{\mathcal{F}}}|$. Together with (5.16) we have

$$\iint_{T_1} K\,\mathrm{d}A_1 = \mathrm{Area}(T_1) + \iint_{T_1}(K-1)\,\mathrm{d}A_1$$
$$\leq \Im(\zeta_1) + (C_6-1)\iint_{T_1}|\mu_{\mathcal{F}}|\,\mathrm{d}A_1 = \Im(\zeta_1) + O(1/n).$$

Hence

$$\Im(\zeta_1)^2 = y_1^2 \leq [\Im(\zeta_1) + O(1/n)]\cdot\Im(\zeta_2).$$

Since $w_0 = H(1/2)$ is bounded away from $\{0, \infty\}$, and ζ_1 lies on a compact subset of the upper half plane, we immediately obtain

$$\Im(\zeta_1) \leq \Im(\zeta_2) + O(1/n). \tag{5.17}$$

Similarly, letting α_1 and α_2 be integers, then

$$|\alpha_1 + \alpha_2\zeta_2| \leq \int_0^1 \left|\frac{\partial\widetilde{\mathcal{F}}(x + t(\alpha_1+\alpha_2\zeta_1))}{\partial t}\right|\mathrm{d}t$$
$$\leq |\alpha_1+\alpha_2\zeta_1|\cdot\int_0^1 K(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\cdot J(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\,\mathrm{d}t.$$

Integrating this inequality over $x \in [0, 1]$ yields

$$|\alpha_1+\alpha_2\zeta_2| \leq |\alpha_1+\alpha_2\zeta_1|\cdot\int_0^1\int_0^1 K(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\cdot J(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\,\mathrm{d}t\mathrm{d}x.$$

From the Schwarz inequality we get

$$
\begin{aligned}
&|\alpha_1+\alpha_2\zeta_2|^2\\
&\le |\alpha_1+\alpha_2\zeta_1|^2\cdot\int_0^1\int_0^1 K(x+t(\alpha_1+\alpha_2\zeta_1))\,\mathrm{d}t\mathrm{d}x\cdot\int_0^1\int_0^1 J(x+t(\alpha_1+\alpha_2\zeta_1))\,\mathrm{d}t\mathrm{d}x\\
&= |\alpha_1+\alpha_2\zeta_1|^2\cdot\frac{\Im(\zeta_2)}{\Im(\zeta_1)}\cdot\int_0^1\int_0^1 K(x+t(\alpha_1+\alpha_2\zeta_1))\,\mathrm{d}t\mathrm{d}x\\
&\le |\alpha_1+\alpha_2\zeta_1|^2\cdot\frac{\Im(\zeta_2)}{\Im(\zeta_1)}\cdot(1+O(1/n)).
\end{aligned}
$$

This means

$$
\frac{|\alpha_1+\alpha_2\zeta_2|^2}{\Im(\zeta_2)}\le\frac{|\alpha_1+\alpha_2\zeta_1|^2}{\Im(\zeta_1)}\cdot(1+O(1/n)). \tag{5.18}
$$

The claim (5.18) holds for any rational numbers $\alpha_1,\ \alpha_2$ and hence for any $\alpha_1,\alpha_2\in\mathbb{R}$. By taking $\alpha_1=-\Re(\zeta_1)$ and $\alpha_2=1$, the claim $\Im(\zeta_2)\le\Im(\zeta_1)(1+O(1/n))$ follows. With this estimate, together with (5.17), we have at once

$$
|\Im(\zeta_2)-\Im(\zeta_1)|\le O(1/n). \tag{5.19}
$$

It follows easily by (5.18) and (5.19) that $|\alpha_1+\alpha_2\zeta_2|^2\le|\alpha_1+\alpha_2\zeta_1|^2(1+O(1/n))$, which together with (5.19) yields the desired estimate $|\zeta_2-\zeta_1|\le O(1/n)$.

Recall that $w_0=H(1/2)$. Since w_0 and $F_H(w_0)$ depend smoothly on the moduli ζ_1 and ζ_2 respectively, we obtain

$$
|F_H(w_0)-w_0|\le O(1/n). \tag{5.20}
$$

Hence $|F_H(1/w_0)-(1/w_0)|\le O(1/n)$.

Suppose $w\in\mathbb{D}$ stays away from $\partial\mathbb{D}$, say $\ge 1/8$ (in the spherical metric). Now, if w stays away from the points $\{0,\ w_0\}$, by considering the four-punctured spheres

$$
\hat{\mathbb{C}}\backslash\{F_H^{-1}(0),w_0,w,F_H^{-1}(\infty)\}\ \text{ and }\ \hat{\mathbb{C}}\backslash\{0,F_H(w_0),F_H(w),\infty\},
$$

then we obtain that the cross ratio of $(F_H^{-1}(0),w_0,w,F_H^{-1}(\infty))$ is $O(1/n)$ close to the cross ratio of $(0,F_H(w_0),F_H(w),\infty)$. By combining (5.15) with (5.20) we have $|F_H(w)-w|\le O(1/n)$.

Otherwise, if w is close to one of the points $\{0,w_0\}$, say w_0, by considering the four-punctured spheres $\hat{\mathbb{C}}\backslash\{F_H^{-1}(0),1/w_0,w,F_H^{-1}(\infty)\}$ and $\{0,F_H(1/w_0),F_H(w),\infty\}$, we also obtain that $|F_H(w)-w|\le O(1/n)$.

In both cases we deduce that $|F_H(w)-w|\le O(1/n),\quad\forall\,|z|\le 7/8$. It implies that

$$
|F_n(z)-z|\le O(1/n),\quad z\in\{|z|\le 3/4\}\subset P. \tag{5.21}
$$

Let $\Gamma=\{|z|=1/2\}$ and let Ω be the disk bounded by Γ. The following Pompeiu formula

$$
F_n(z_0)=\frac{1}{2\pi i}\oint_\Gamma\frac{F_n(z)}{z-z_0}\,\mathrm{d}z-\frac{1}{\pi}\iint_\Omega\frac{\partial_{\bar z}F_n(z)}{z-z_0}\,\mathrm{d}x\mathrm{d}y.
$$

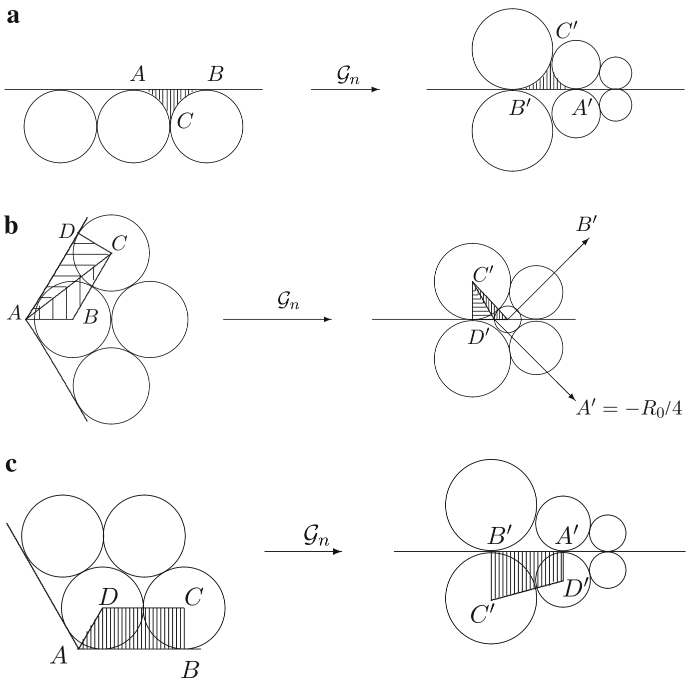

Fig. 5 The extension of G to the regions $P_*\backslash P$

holds for $z_0 = \frac{1}{2n} + \frac{i}{2\sqrt{3}n}$. Since F_n is holomorphic in the interstice containing z_0, we have

$$F_n'(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{F_n(z)}{(z-z_0)^2}\,\mathrm{d}z - \frac{1}{\pi}\iint_\Omega \frac{\partial_{\bar z}F_n(z)}{(z-z_0)^2}\,\mathrm{d}x\mathrm{d}y. \tag{5.22}$$

Using the C^1 convergence of circle packings to the Riemann map (see, e.g., [5,6]) and (5.5), we obtain $|\partial_{\bar z}F_n(z)| = O(1/n^2),\ z\in\Omega$. From (5.21) and (5.22), it follows that

$$|F_n'(z_0) - 1| \le \frac{1}{2\pi}\oint_\Gamma \left|\frac{F_n(z)-z}{(z-z_0)^2}\,\mathrm{d}z\right| + \frac{1}{\pi}\iint_\Omega \left|\frac{\partial_{\bar z}F_n(z)}{(z-z_0)^2}\right|\mathrm{d}x\mathrm{d}y.$$

That is $|F_n'(z_0) - 1| \le O(1/n)$. Similarly, with the aid of (5.21) and the following equality

$$F_n''(z_0) = \frac{1}{\pi i}\oint_\Gamma \frac{F_n(z)}{(z-z_0)^3}\,\mathrm{d}z - \frac{2}{\pi}\iint_\Omega \frac{\partial_{\bar z}F_n(z)}{(z-z_0)^3}\,\mathrm{d}x\mathrm{d}y,$$

we conclude that $|F_n''(z_0)| \le O(1/n)$, as desired. □

Proof of Lemma 4.1 The extension of the quasiconformal mapping $\mathcal{G}_n$ to regions $P_*\backslash P$ is broken up into several steps.

Step 1. Let ABC be any *boundary interstices*[3] in P^*, which is not adjacent to *corner interstices*[4]. There is a conformal map from the boundary interstice ABC to the corresponding

[3] the bounded region bounded by an edge of P^* and two mutually tangent boundary circles of H_n, as in Fig. 4.

[4] the bounded region bounded by two intersecting edges of P^* and a boundary circle of H_n, as in Fig. 4.

interstice $A'B'C'$, see Fig. 5a. Then we can extend the mapping $\mathcal{G}_n$ radially on each boundary interstice adjacent to ABC.

Step 2. For any quadrangle $ABCD$ near the corner point $-1-\frac{1}{\sqrt{3}n} \in \partial P_*$, we define $\mathcal{G}_n$ to be the piecewise linear map from the interior of ABC (resp. ACD) to the interior of the triangle $A'B'C'$ (resp. $A'C'D'$), see Fig. 5b. By symmetrically extended along AB and $A'B'$, the map $\mathcal{G}_n$ can be extended to the neighborhood of $-1-\frac{1}{\sqrt{3}n}$.

Similarly, we can define $\mathcal{G}_n$ in the neighborhood of the point $1+\frac{1}{\sqrt{3}n} \in \partial P_*$.

Step 3. On the quadrilateral $ABCD$ which is near corner point $A \neq \pm\,(1+\frac{1}{\sqrt{3}n})$, we define $\mathcal{G}_n$ to be the linear map from $ABCD$ to the corresponding quadrilateral $A'B'C'D'$, as show in Fig. 5c. By the similar construction, we can extend $\mathcal{G}_n$ to the neighborhoods of the corner points A.

Therefore we obtain a quasiconformal homeomorphism $\mathcal{G}_n : P^* \to \mathbb{C}\backslash\{-\infty < z \leq -R_0/4\}$. Let μ_n be the Beltrami differential of $\mathcal{G}_n$. Obviously $|\mu_n(z)| = O(1/n^2), \quad \forall\, |z| \leq 3/4$. In addition,

$$\iint\limits_{P_*} |\mu_n(z)|\,\mathrm{d}x\mathrm{d}y = O(1/n),$$

as desired. □

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References

1. Aharonov, D.: The hexagonal packing lemma and Rodin-Sullivan conjecture. Trans. Am. Math. Soc. **343**(1), 157–167 (1994)
2. Andreev, E.M.: On convex polyhedra in lobačhevskii space, Mat. Sb. (N.S) **81**(123) 445–478 (1970); English transl. in Math. USSR Sb. **12**, 413–440 (1970)
3. Andreev, E.M.: Convex polyhedra of finite volume in lobačhevskii space, Mat. Sb. (N.S) **83**(125) 256–260 (1970); English transl. in Math. USSR Sb. **12**, 255–259 (1970)
4. Beardon, A.F., Stephenson, K.: The uniformization theorem for circle packings. Indiana Univ. Math. J. **39**, 1383–1425 (1990)
5. Doyle, P., He, Z.X., Rodin, B.: Second derivatives of circle packings and conformal mappings. Discret. Comput. Geom. **11**, 35–49 (1994)
6. Doyle, P., He, Z.X., Rodin, B.: The asymptotic value of the circle packing rigidity constants s_n. Discret. Comput. Geom. **12**, 105–116 (1994)
7. He, Z.X.: An estimate for hexagonal circle packings. J. Diff. Geom. **33**, 395–412 (1991)
8. He, Z.X., Rodin, B.: Convergence of circle packings of finite valence to Riemann mappings. Commun. Anal. Geom. **1**, 1–10 (1993)
9. He, Z.X., Schramm, O.: The C^∞-convergence of hexagonal disk packings to the Riemann map. Acta. Math. **180**(2), 219–245 (1998)
10. He, Z.X., Schramm, O.: On the convergence of circle packings to the Riemann map. Invent. Math. **125**(2), 285–305 (1996)
11. Koebe, P.: Kontaktprobleme der konformen abbildung. Math-Phys. Klasse. **88**, 141–164 (1936)
12. Lehto, O., Virtanen, K.I.: Quasi-conformal Mappins in the Plane. Springer, Berlin (1973)
13. Marden, A., Rodin, B.: On Thurston's formulation and proof of Andreev's theorem. In: computational Methods and Function Theory, LMN 1435, pp. 103–115. Springer, New York (1989)
14. Nehari, Z.: Conformal Mappings. McGraw-Hill, New York (1952)
15. Pommerenke, C.: Univalent Functions. Vandenhoeck and Ruprecht, Göttingen (1975)

16. Rodin, B.: Schwarz's lemma for circle packings, (II). J. Diff. Geom. **30**, 539–554 (1989)
17. Rodin, B., Sullivan, D.: The convergence of circle packings to the Riemann mapping. J. Diff. Geom. **26**, 349–360 (1987)
18. Stephenson, K.: Circle packings in the approximation of conformal mappings. Bull. Am. Math. Soc. **23**, 407–415 (1990)
19. Thurston, W.: The Geometry and Topology of Three Manifolds. Princeton University Notes, Princeton NJ (1979)
20. Thurston, W.: The Finite Riemann Mapping Theorem, Invited Address, International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University, 1980