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Koebe Problems and Teichmüller Theory

Jin Song LIU
Institute of Mathematics, Academy of Mathematics & Systems Science, Chinese Academy of Sciences, Beijing 100080*, P. R. China*
E-mail: liujsong@math.ac.cn

Abstract In this paper we study the deformation space of certain Kleinian groups. As a result, we give a new proof of the finite Koebe theorem on Riemann surfaces from a viewpoint of Teichmüller theory.

Keywords Koebe problem, Teichmüller space, Circle
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1 Introduction

A Riemann surface $\tilde{S}$ is topologically finite if its fundamental group is finitely generated. The topologically finite Riemann surface $\tilde{S}$ is of type (g, n, m), if there is a closed Riemann surface S of genus g and a holomorphic embedding $i : \tilde{S} \to S$, so that the set $S \backslash i(\tilde{S})$ includes m closed disks and n points. When $m = 0$, we called $\tilde{S}$ conformally finite.

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane, which is holomorphically equivalent to the standard Riemann sphere. In the spherical metric, a domain on the Riemann sphere is said to be a round domain if each of its boundary components is either a circle or a point. In [1] Koebe proposed a conjecture: Any domain $\Omega \subset \hat{\mathbb{C}}$ can be holomorphically realized as a round domain on $\hat{\mathbb{C}}$. Later he [2] gave an affirmative answer to this problem when Ω is of finite type.

On the Riemann sphere, after giving an appropriate orientation to any circle, one can talk about its interior and exterior (the interior lies to the left of the circle). For a collection of circles $\{C_i\}$ on $\hat{\mathbb{C}}$, if they could be given an appropriate orientation so that they have disjoint closures pairwise, we call $\{C_i\}$ allowable.

Fixing allowable circles $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ on $\hat{\mathbb{C}}$, we denote by Ω_0 the resulting Riemann surface by deleting $\mathscr{C}$ and its interiors from $\hat{\mathbb{C}}$. Suppose $\mathscr{D}_m$ is the space of equivalent classes of allowable m-circles on $\hat{\mathbb{C}}$ (defined in §2). We have the following:

Theorem 1 *For the region Ω_0 on the Riemann sphere $\hat{\mathbb{C}}$, the space $\mathscr{D}_m$ is homeomorphic to the Teichmüller space $T(\Omega_0)$ of Ω_0.*

Also this result will be generalized to closed Riemann surfaces. For simplicity, here we assume its genus $\mathrm{g} > 1$.

Let S be a closed Riemann surface of genus g. Then it can be represented as a quotient of its universal covering space $\mathbb{D}$ by the covering transformations group Γ. Then S inherits the hyperbolic metric from that of $\mathbb{D}$. Similarly one can define the allowable m-circles on S.

Let $\mathscr{D}_m$ be the space consisting of equivalent classes of allowable m-circles on Riemann surfaces of genus g (defined in §2). Denoting S_* the Riemann surface by cutting away the allowable circles $\mathscr{C}$ and its interiors from S, we obtain

Theorem 2 *When the genus $\mathrm{g} > 1$, the space $\mathscr{D}_m$ is homeomorphic to the Teichmüller space $T(S_*)$ of S_*.*

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As an immediate consequence of these two results, in this paper we will provide a new approach to the finite Koebe Theorem: Any Riemann surface of finite type can be realized as a round domain on some compact Riemann surface of the same genus, and this realization is unique up to conformal mappings between Riemann surfaces.

The main object of this paper is to give a systematic method for solving these problems. Furthermore the proofs imply that we can solve the Koebe problem by deformation method.

2 Proofs and Main Results

Recall that $\tilde{S}$ is a Riemann surface of type (g, n, m). For simplicity here we assume that $\tilde{S}$ is not of the following type: $(\mathrm{g}, n, m) \neq (0,0,0), (0,1,0), (1,0,0), (0,2,0), (0,0,1), (0,1,1), (0,0,2)$.

Definition *The (reduced) Teichmüller space $T(\tilde{S})$ is defined to be the space of Teichmüller deformations of the complex structure $\tilde{S}$.*

The Teichmüller metric $d_T(\cdot,\cdot)$ on $T(\tilde{S})$ is defined as $d_T(\tilde{S}_1, \tilde{S}_2) = \log K$, where K is the maximal dilatation of the Teichmüller deformation map between $\tilde{S}_1$ and $\tilde{S}_2$. In the metric topology the space $T(\tilde{S})$ is homeomorphic to the Euclidean space $\mathbb{R}^{6\mathrm{g}+2n+3m-6}$, see [3].

Let Γ be a finitely generated discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. The set of accumulation points of orbit $\Gamma z = \{\gamma(z) : \gamma \in \Gamma\}$ is called the limit set $\Lambda(\Gamma)$. Its complement $\Omega(\Gamma) = \hat{\mathbb{C}} - \Lambda(\Gamma)$ is the region of discontinuity. The quotient space $\Omega(\Gamma)/\Gamma$ is a union of Riemann surfaces. The Ahlfors Finite Theorem shows that $\Omega(\Gamma)/\Gamma$ includes finite components and each component is a conformally finite Riemann surface.

Given a finitely generated discrete group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$, if there exists a quasi-conformal homeomorphism $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that $\Gamma' = \Phi\Gamma\Phi^{-1}$ is also a discrete group of $\mathrm{PSL}(2,\mathbb{C})$, then we call Γ' a qc-deformation of Γ. The following is a typical way for constructing such a qc-deformation of the discrete group Γ.

Suppose μ is a Beltrami differential on the Riemann surfaces $\Omega(\Gamma)/\Gamma$ with $\|\mu\|_\infty \leq k < 1$. By lifting μ to the region of discontinuity $\Omega(\Gamma)$, and extending it to be 0 on the limiting set $\Lambda(\Gamma)$, we obtain a new Beltrami differential $\tilde{\mu}$. The differential $\tilde{\mu}$ is invariant under the action of group Γ. The classical Ahlfors–Bers Theorem states that there exists a quasi-conformal mapping $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that $\Phi_{\bar{z}} = \tilde{\mu}\Phi_z$. Therefore, the group $\Gamma' = \Phi\Gamma\Phi^{-1} \subset \mathrm{PSL}(2,\mathbb{C})$ is a quasi-conformal deformation of the group Γ.

The Theorem of Sullivan [4] implies the above typical constructions include all *qc*-deformations of Γ. This Theorem plays a crucial role throughout this paper.

Theorem (Sullivan) *For any finitely generated discrete group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$, the space of qc-deformations of Γ is homeomorphic to the Teichmüller space of the Riemann surfaces $\Omega(\Gamma)/\Gamma$, where $\Omega(\Gamma)$ is the region of discontinuity of Γ acting on $\hat{\mathbb{C}}$.*

Recall that $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ is m allowable circles on the Riemann sphere $\hat{\mathbb{C}}$. Each circle in the set $\mathscr{C}$ corresponds to three real parameters-one complex parameter (two real numbers) corresponds to the center of the circle, and one real parameter corresponds to for the radius of the circle. We denote by z_i and $r_i (r_i > 0)$ the center coordinate and radius of the circle C_i, respectively.

Denote by $\mathscr{D}_m$ the set consisting of all allowable m-circles, up to Möbius transformations (orientation preserving). For $m \geq 3$ and any $\mathscr{C} = \{C_1, C_2, \ldots, C_m\} \in \mathscr{D}_m$, without lose of the generality, we assume the circles C_1, C_2 and C_3 have centers 0, 1 and ∞, respectively. Then each $\mathscr{C} \in \mathscr{D}_m$ is uniquely determined by $3m-6$ numbers r_i $(1 \leq i \leq 3)$ and $(z_4, r_4, \ldots, z_m, r_m)$. We can view the set $\mathscr{D}_m$ as a subset of $\mathbb{R}^{3m-6}$. Thus set $\mathscr{D}_m$ is a topological space.

After denoting by D_i the closure of the interior of C_i $(i = 1, 2, \ldots, m)$, we set $\Omega_0 = \hat{\mathbb{C}} \setminus \bigcup_{i=1}^m D_i$. It is an open Riemann surface of type $(0, 0, m)$.

With the above notation, we are now ready to give

Theorem 1 *When $m \geq 3$, the space $\mathscr{D}_m$ is homeomorphic to the Teichmüller space $T(\Omega_0)$ of Ω_0.*

Proof Let γ_i be the element generated by reflection of the circle C_i and let $\Gamma_{\mathscr{C}} = \langle\gamma_1, \gamma_2, \ldots, \gamma_m\rangle$ be the group of Möbius transformations (some of them reverse the orientations). It follows from the Poincaré Polyhedron Theorem that the action of $\Gamma_{\mathscr{C}}$ on $\hat{\mathbb{C}}$ is discrete. The surface $\Omega_0 = \hat{\mathbb{C}} \setminus \bigcup_{i=1}^m D_i$ is a fundamental domain for $\Gamma_{\mathscr{C}}$ acting on $\hat{\mathbb{C}}$.

Given any Beltrami differentials $[\mu] \in T(\Omega_0)$ with $||\mu||_\infty \leq k < 1$, one can construct a new Beltrami differential $\tilde{\mu}$ on $\hat{\mathbb{C}}$: For any $z \in \gamma^{-1}(\Omega_0)(\gamma \in \Gamma)$, we set

$$\tilde{\mu} = \tilde{\mu}(z)\frac{d\bar{z}}{dz} \equiv \begin{cases} \mu(\gamma z)\frac{\overline{d\gamma(z)}}{d\gamma(z)}, & \gamma \text{ preserves the orientation;} \\ \overline{\mu(\overline{\gamma z})\frac{d\gamma(z)}{\overline{d\gamma(z)}}}, & \gamma \text{ reverses the orientation.} \end{cases}$$

And place $\tilde{\mu} = 0$ on other regions of $\hat{\mathbb{C}}$.

From the Ahlfors–Bers Theorem, there is a quasi-conformal homeomorphism $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, whose dilation is just $\tilde{\mu}$. The group $\Gamma'_{\mathscr{C}} = \Phi\Gamma_{\mathscr{C}}\Phi^{-1} = \langle \gamma'_1, \gamma'_2, \ldots, \gamma'_m \rangle$ is a finitely generated discrete group of Möbius transformations. In general it is not true that the image of a circle under a quasi-conformal mapping is still a circle. But the sets $C'_i = \Phi(C_i)$ are fixing points of the Möbius transformation $\gamma'_i = \Phi\gamma_i\Phi^{-1}$. The fixing points set of an orientation-reversing Möbius transformation with order 2 is either an empty set (for example $z \to -1/\bar{z}$) or a circle (for example $z \to 1/\bar{z}$). Therefore $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_m\}$ are also allowable m-circles.

Thus we have a mapping $H : T(\Omega_0) \to \mathscr{D}_m$, defined as $H([\mu]) = \mathscr{C}' \in \mathscr{D}_m$.

Let $\overline{\Gamma}_{\mathscr{C}} \subset \Gamma_{\mathscr{C}}$ be the subgroup consisting of orientation-preserving elements. Subgroup $\overline{\Gamma}_{\mathscr{C}}$ and the element γ_1 together generate $\Gamma_{\mathscr{C}}$. The quotient Riemann surface associated with the discrete group $\overline{\Gamma}_{\mathscr{C}}$ is $\Omega(\overline{\Gamma}_{\mathscr{C}})/\overline{\Gamma}_{\mathscr{C}} = \Omega_0 \cup_{\gamma_1} \gamma_1(\Omega_0)$. It is a closed Riemann surface of genus $m-1$.

For any allowable m-circles $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_m\}$, we denote by $\Gamma_{\mathscr{C}'}$ the group generated by the reflection in the circles $\mathscr{C}'$. Since $\Gamma_{\mathscr{C}'}$ and $\Gamma_{\mathscr{C}}$ are geometrically finite groups, they are quasiconformally conjugate ([5]). Then $\Gamma_{\mathscr{C}'} = \Phi\Gamma_{\mathscr{C}}\Phi^{-1}$, for some quasi-conformal mapping Φ. The Theorem of Sullivan implies that the Beltrami differential μ_Φ of Φ can be obtained by the extension of a Beltrami differential on $\Omega(\overline{\Gamma}_{\mathscr{C}})/\overline{\Gamma}_{\mathscr{C}}$ through the action of group $\overline{\Gamma}_{\mathscr{C}}$. The fact that $\Phi\gamma_1\Phi^{-1}$ is a Möbius transformation implies $\gamma_1(\mu_\Phi) = \overline{\mu}_\Phi$ on $\Omega(\overline{\Gamma}_{\mathscr{C}})/\overline{\Gamma}_{\mathscr{C}}$. Then $\mu_\Phi \in T(\Omega_0)$, which implies that the deformation space of $\Gamma_{\mathscr{C}}$ is uniquely determined by the complex structures on Ω_0. That is, the mapping H is surjective.

On the other hand, Sullivan Theorem implies that $H : T(\Omega_0) \to \mathscr{D}_m$ is continuous. By Brouwer's theorem on the invariance of domain, we conclude that H is homeomorphic.

Theorem 1 immediately leads to the finite Koebe theorem on the Riemann sphere $\hat{\mathbb{C}}$.

Corollary 1 *Any domain $\tilde{\Omega} \subset \hat{\mathbb{C}}$ of finite type is holomorphically homeomorphic to a round domain on the Riemann sphere. Furthermore it is unique up to orientation-preserving Möbius transformations.*

Proof First we assume that the open Riemann surface $\tilde{\Omega}$ is of type $(0, 0, m)$.

The case $m = 1$ is just the Riemann Mapping Theorem.

When $m = 2$, $\tilde{\Omega} \subset \hat{\mathbb{C}}$ is a 2-connected domain. Let $r(r > 0)$ be the conformal modulus of $\tilde{\Omega}$. Then $\tilde{\Omega}$ is conformally equivalent to the round domain $\{z : 1 < z < e^r\} \subset \hat{\mathbb{C}}$.

Now if $m > 2$, we use the notation as in the proof of Theorem 1. The Riemann surfaces $\tilde{\Omega}$ and Ω_0 are quasi-conformally equivalent. We choose a qc-mapping $\Omega_0 \to \tilde{\Omega}$ with Beltami differential μ. Theorem 1 implies $\tilde{\Omega}$ can be realized as a round domain cutting away $H([\mu])$ and its interiors from the Riemann sphere.

When $\tilde{\Omega}$ is of type $(0, n, m)$, there exist an open Riemann surface Ω' of type $(0, 0, m)$ and a holomorphic embedding $i : \tilde{\Omega} \to \Omega'$. The above assertion implies that Ω' can be realized as a round domain in $\hat{\mathbb{C}}$. Thus $\tilde{\Omega}$ can also be realized as a round domain on $\hat{\mathbb{C}}$. It is unique up to orientation-preserving Möbius transformations.

Now we generalize these results to compact Riemann surfaces of genus g > 1.

Let S_0 be a smooth closed surface of genus g. We denote by $\tilde{\mathscr{D}}_m$ the set consisting of all pairs $\{(S, \mathscr{C})\}$, where S is a conformal structure on S_0 and $\mathscr{C} = \{\mathscr{C}, \mathscr{C}, \ldots, \mathscr{C}\}$ are allowable m-circles on S. The equivalent relation " $\sim$ " in $\tilde{\mathscr{D}}_m$ is defined as:

$(S,\mathscr{C}) \sim (\mathscr{S}',\mathscr{C}')$ if and only if there exists a conformal homeomorphism $h: S \to S'$, which is homotopic to the identity mapping, so that $h(C_i) = C_i'$ for $1 \le i \le m$.

The set $\mathscr{D}_m = \tilde{\mathscr{D}}_m/\sim$ consists of all equivalent classes of allowable m-circles on closed Riemann surfaces of genus g. As in the case of $g = 0$, $\mathscr{D}_m$ is a topological space.

Suppose $\mathscr{C} = \{C_1, C_2, \dots, C_m\}$ are allowable m-circles on the Riemann surface S. Let D_i be the closed disk including C_i and its interior, where $i = 1, 2, \dots, m$. The open Riemann surface $S_* = S \setminus \bigcup_{i=1}^m D_i$ is of type $(\mathrm{g}, 0, m)$. We proceed to show

Theorem 2 *When the genus* $\mathrm{g} > 1$*, the space* $\mathscr{D}_m$ *is homeomorphic to the Teichmüller space* $T(S_*)$*.*

Remark Since the real dimension of the Teichmüller space $T(S_*)$ is $6\mathrm{g} + 3m - 6$, Theorem 2 implies that the real dimension of $\mathscr{D}_m$ is also $6\mathrm{g} + 3m - 6$.

Proof Assume $\pi : \mathbb{D} \to S$ is the universal covering mapping of S and Γ is the transformation group of π. Then we have $S = \mathbb{D}/\Gamma$.

Denote by $\{\bar{C}_i\}$ the pre-images $\{\pi^{-1}(C_i)\}$ lying in a fundamental domain Ω_Γ of the group Γ. Let $\bar{\gamma}_i$ be the element generated by reflecting of the circle $\bar{C}_i$, where $i = 1, 2, \dots, m$. We define the group $\Gamma_{\mathscr{C}}$ to be $\Gamma_{\mathscr{C}} = \langle \bar{\gamma}_1, \bar{\gamma}_2, \dots, \bar{\gamma}_m \rangle * \langle \Gamma \rangle * \langle \gamma_0 \rangle$, where γ_0 is the element generated by reflection of the unit circle in $\hat{\mathbb{C}}$. From the Poincaré Polyhedron Theorem, it follows that $\Gamma_{\mathscr{C}}$ is a finitely generated discrete Möbius transformations group.

Let $\bar{\Gamma}_{\mathscr{C}} \subset \Gamma_{\mathscr{C}}$ be the group consisting of all orientation-preserving elements. Geometrically it is clear that $\Omega(\bar{\Gamma}_{\mathscr{C}})/\bar{\Gamma}_{\mathscr{C}} = S_* \cup_{\bar{\gamma}_1} \bar{\gamma}_1(S_*)$. The rest of this proof is similar to that of Theorem 1.

We conclude the paper with the following finite Koebe Theorem on closed Riemann surfaces:

Corollary 2 *Any Riemann surface* $\tilde{S}$ *of type* (g, n, m) *can be realized as a round domain* $S^\star$ *on some closed Riemann surface* S*. Furthermore the pair* $(S^\star, S)$ *is unique up to conformal mappings.*

Since the proof of Corollary 2 is similar to that of Corollary 1, we omit it here.

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