CHARACTERIZATIONS OF CIRCLE PATTERNS AND CONVEX POLYHEDRA IN HYPERBOLIC 3-SPACE

XIAOJUN HUANG AND JINSONG LIU

ABSTRACT. In this paper we consider the characterization problem of convex polyhedrons in the three dimensional hyperbolic space $\mathbb{H}^3$. Consequently we can give a characterization of circle patterns in the Riemann sphere with *dihedral angle* $0 \le \Theta < \pi$. That is, for any circle pattern on $\hat{\mathbb{C}}$, its quasiconformal deformation space can be naturally identified with the product of the Teichmüller spaces of its interstices.

Mathematics Subject Classification. 52C26, 30F60, 30F15.

0. INTRODUCTION

Let P be a *circle pattern* in the Riemann sphere $\hat{\mathbb{C}}$. It is a collection of circles in $\hat{\mathbb{C}}$ in which no circle has its interior contained in the union of the interior of two others. Define the *contact graph* $G = G_P$ of the pattern P to be a graph whose vertices correspond to the circles in P, and an edge appears in G if and only if the corresponding circles intersect each other. Please see [5, 6, 14, 15]. Let $P(v)$ denote the circle in P corresponding to the vertex $v \in V$. For any edge $e = [v, w]$ of G, the *dihedral angle* $\Theta_P(e)$ of the pair of intersecting circles $P(v)$ and $P(w)$ is defined to be the angle in $[0, \pi)$ between the clockwise tangent of $P(v)$ and the counterclockwise tangent of $P(w)$ at a point of $P(v) \cap P(w)$. Please see Figure 1.

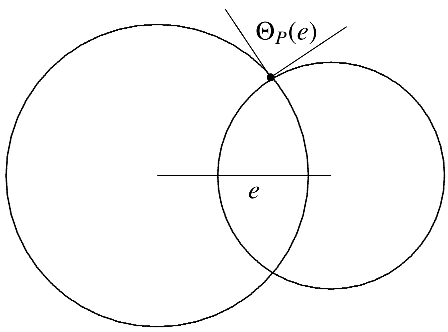

Figure 1

Let $\mathbb{H}^3$ denote the 3-dimensional hyperbolic space. If $\Pi \subset \mathbb{H}^3$ is any hyperbolic plane, then Π is the intersection of $\mathbb{H}^3$ with a sphere orthogonal to $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$. Thus

The first author was supported by the Fundamental Research Funds for the Central Universities Grant No. CQDXWL-2012-013.

The second author was supported by NSFC Grant No. 10831004.

the intersection $\partial\Pi \cap \hat{\mathbb{C}}$ is a circle. Furthermore, for any two hyperbolic planes Π, Π', if they intersect, then we have the following

Observation 0.1. *The dihedral angle of the circles $\partial\Pi \cap \hat{\mathbb{C}}$ and $\partial\Pi' \cap \hat{\mathbb{C}}$ is the dihedral angle at the edge $\Pi \cap \Pi'$.*

This simple observation will play an important role throughout this paper.

In order to state our main results, let us introduce some definitions (see [10, 11]).

A *finite convex polyhedron* $\mathcal{P} \subset \mathbb{H}^3$ is a subset with non-empty interior, which is an intersection of finite closed half-spaces. Each finite face of $\mathcal{P}$ lies in a unique hyperbolic plane. Therefore, there is a corresponding circle pattern P on the Riemann sphere.

When $\partial\mathcal{P} \cap \partial\mathbb{H}^3 \neq \emptyset$, then the intersection is the union of several closed regions. Each of these closed region I has finitely many boundary components. And each boundary component is a piece-wise smooth curve formed by finitely many circular arc or circles. Each (maximal) circular arc or circle on ∂I belongs to the boundary of some circle. Therefore, this boundary arc is marked by an element of the vertices set V. The closed region I, together with a marking of the circular arcs and /or circles on its boundary by elements of V is called an interstice of this polyhedron.

Each closed region, together with a marking of the circular arcs and/or circles on its boundary is called an *interstice*. Therefore it induces the complex structure from the Riemann sphere.

A hyperbolic plane $\Pi \subset \mathbb{H}^3$ is a support hyperplane for a polyhedron $\mathcal{P}$ if Π lies entirely on one side of $\mathcal{P}$ and Π contains at least a point of $\mathcal{P}$. In Section 1 we will give the definition of polar map $*$, which maps a hyperbolic plane Π in $\mathbb{H}^3$ to the set $\Pi^* \subset \mathbb{S}^2_1$ of outward and normal to all of its supporting hyperplanes, where $\mathbb{S}^2_1$ is the de-Sitter space. We refer to Section 1 for background on polar map and de-Sitter space. Throughout this paper, without otherwise specified, we use the same V as the vertices set of a graph on $\hat{\mathbb{C}}$. We also use V to be the index set of the finite faces of any finite convex polyhedron in $\mathbb{H}^3$.

Let (Q, g) be a metric space with boundary. (Q, g) is called finite admissible if it satisfies the following conditions.

1. The metric space Q is homeomorphic to a closed region in the Riemann sphere $\hat{\mathbb{C}}$.

2. The metric g has constant curvature 1 away from a finite collection of cone points $\{c_v\}_{v\in V}$.

3. The cone angle at each interior cone points c_v is greater than 2π.

4. The lengths of all closed local geodesics[1] of the metric space (Q, g) are greater than or equal to 2π. The set $G_{2\pi} = \{\Gamma_1, \Gamma_2, \cdots, \Gamma_N\}$ of all geodesics of length equal to 2π is finite. Furthermore, if $\Gamma \in G_{2\pi}$, then one of the connected components of $Q\backslash\Gamma$ is isometric to the standard open hemisphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z > 0\}$.

5. Each boundary component of Q consists of boundary cone points and geodesic arcs connecting pairs of adjacent boundary cone points. The length of each geodesic arc is less than or equal to π.

[1]a closed local geodesic is the image of a standard circle under a locally distance minimizing mapping. At any interior cone points it subtends an angle of at least π on either side. When the geodesic pass through a boundary cone, it subtends an angle of at least π on the side of the region Q.

For any pair of adjacent boundary cone points $\{c_1, c_2\}$ with distance π, we use $\gamma_{[c_1,c_2]}$ to denote the boundary geodesic arc joining c_1 and c_2. Note that, by our assumption, we have $L(\gamma_{[c_1,c_2]}) = \pi$.

Definition 0.2. For any local geodesic $\gamma \subset Q$ joining the cone points c_1 and c_2, if the angles between γ and $\gamma_{[c_1,c_2]}$ at c_j, $j = 1, 2$, is greater than or equal to $\pi/2$, then we call γ a semi-closed geodesic.

6. The length of any semi-closed geodesic on (Q, g) is $\geq \pi$. The number of such semi-closed geodesics is finite. Furthermore, if the length of a semi-close geodesic is precisely π, then the region between this semi-close geodesic and the geodesic arc with length π is isometric to the open region $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, y > 0, z > 0\}$.

Remark 0.3. For any semi-closed geodesic γ with length $\pi/2$, then there is at least one cone point incident to γ except for the two end points. Otherwise, there is a lung[2] abutting the region along γ. Then there are infinite many semi-closed geodesics in the lung. It would contradict to the condition 6.

Then we have the following result.

Theorem 0.4. *The polar images of finite convex polyhedra in the* 3*-dimensional hyperbolic space are precisely the finite admissible spaces.*

Fixing a polyhedron $\mathcal{P}_0$ whose polar image is (Q, g), we have

Theorem 0.5. *The set of all polyhedra whose polar image is* (Q, g) *can be characterized by the product of the Teichmüller space* $\Pi_i^k \mathcal{T}_{I_i}$*, where* $\{I_1, I_2, \cdots, I_k\}$ *are all interstices of* $\mathcal{P}_0$.

Assume $G = (V, E)$ is an embedded graph $\hat{\mathbb{C}}$, where V is the set of vertices in G and E is its edges set. Suppose that $|V| > 5$. Let $\Theta : E \to [0, \pi)$ be the dihedral angle function for G. Recall that a circle pattern P is said to realize the data (G, Θ) if its contact graph is combinatorially isomorphic to a graph $\tilde{G} = (V, \tilde{E})$, and its dihedral angle function $\tilde{\Theta}$ satisfies $\Theta = \tilde{\Theta}|_E$.

If (G, Θ) satisfies the conditions $(i), (ii), (iii)$ and (iv) in Section 2, then we can construct a metric space (Q, g) (see Section 2). By using Observation 0.1 and Theorem 0.5, we have the following result. Please see Theorem 2.1 and Theorem 2.2 in Section 2.

Theorem 0.6. *If the graph* (G, Θ) *satisfies the conditions* $(i), (ii), (iii)$ *and* (iv) *in Section 2, and if the corresponding metric space* (Q, g) *is finite admissible, then there is at least one circle pattern* P_0 *realizing the data* (G, Θ).

Furthermore, the space of equivalence classes of circle patterns realizing (G, Θ) *can be natural identified with the Teichmüller space* $\Pi_i^k \mathcal{T}_{I_i}$*, where* $\{I_1, I_2, \cdots, I_k\}$ *are the interstices of* P_0.

In [5] we prove the above theorem under the condition $0 \leq \Theta \leq \pi/2$. By using Theorem 0.6, we can give a simple new proof of the following theorem. Please see Theorem 3.1 in Section 3.

[2] a lung is a piece of a sphere bounded by two geodesic arcs

Theorem 0.7. *Assume that $G = (V, E)$ is an embedded graph $\hat{\mathbb{C}}$ with $|V| > 5$. If $0 \le \Theta \le \pi/2$, then there is a circle pattern P_0 realizing (G, Θ). The space of equivalence classes of circle patterns realizing (G, Θ) can be natural identified with the product of the Teichmüller spaces of all interstices of the circle pattern P_0.*

The paper is organized as follows. In Section 1 we introduce some basic terminologies and develop various background necessary for our proofs. The object of Section 2 is to give the proof of some results on circle patterns. In Section 3 we give a new proof of the main result in [5]. The proof of Theorem 0.5 is left to the last section.

Notational Conventions. Through this paper, for any circle pattern P and vertex v, we denote by $P(v)$ the circle corresponding to v. For any circle c in the complex plane $\mathbb{C}$, we denote by $\rho(c)$ its euclidean radius.

For any finite set S, we denote by $|S|$ the number of elements in S.

Acknowledgement. We wish to thank Professor Zhengxu He for suggesting this project.

1. Preliminaries

The main purpose of this section is to define some terms we later use, and to introduce notations.

For any finite convex polyhedron $\mathcal{P} \subset \mathbb{H}^3$, we describe the Gauss map of $\mathcal{P}$ in $\mathbb{H}^3$. For any vertex $\mathcal{V}$ of $\mathcal{P}$, $\mathcal{V}$ is mapped by the Gauss map G to a spherical polygon $G(\mathcal{V})$, whose sides are the images under G of edges incident to the vertex $\mathcal{V}$, and whose angles are seen to be the angles supplementary to the planar angles of the faces incident to $\mathcal{V}$. Namely, the edges $G(e_l)$ and $G(e_z)$ meet at angle $\pi - \alpha$ whenever the edges e_l, e_2 of $\mathcal{P}$ meet at angle α.

Hodgson and Rivin [10, 11] interpret the Gauss image by using the polar map. For the sake of completeness, we give it here. Denote by $\mathbb{R}^3_1$ the linear space $\mathbb{R}^4$ equipped with the $(3, 1)$ Minkowski inner product

$$< x, y >= -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3)$. Then we can view the classical hyperbolic space

$$\mathbb{H}^3 = \{x \in R^3_1 : < x, x >= -1, x_0 > 0\},$$

which is the "sphere" of radius $\sqrt{-1}$. Let

$$\mathbb{S}^2_1 = \{x \in R^3_1 : < x, x >= 1\}.$$

be the de Sitter space. It is a semi-Riemann submanifold of the space $\mathbb{R}^3_1$. The polar map $*$ maps a hyperbolic convex polyhedron $\mathcal{P} \subset \mathbb{H}^3$ to the set $P^* \in \mathbb{S}^2_1$ of outward and normal to all of its supporting hyperbolic planes. Then P^* has the intrinsic metric induced from $\mathbb{S}^2_1$. Hodgson and Rivin prove that the intrinsic metric on the polar image $\mathcal{P}^*$ is precisely the usual metric on the Gauss image $G(\mathcal{P})$.

Moreover, Hodgson and Rivin [10, 11] prove the following results.

Theorem 1.1. *A metric space (Q, g) is the polar image of a finite volume polyhedron if and only if it satisfies the following conditions:*

$\triangleright$ *Q is homeomorphic to the Riemann sphere $\mathbb{S}^2$.*

$\triangleright$ *The metric g has constant curvature 1 away from a finite collection $\{c_v\}_{v\in V}$ of cone points.*

$\triangleright$ *The cone angle α_v at c_v greater than 2π.*

$\triangleright$ *The lengths of all closed geodesics of (Q, g) are greater than or equal to 2π.*

$\triangleright$ *The set $G_{2\pi} = \{\Gamma_1, \cdots, \Gamma_N\}$ of all closed geodesics of length equal to 2π is finite. Furthermore, if $\Gamma \in G_{2\pi}$, then one of the connected components of $Q\backslash\Gamma$ is isometric to the standard open hemisphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z > 0\}$.*

Furthermore, the hemispheres bounded by geodesics of length 2π correspond to ideal vertices.

Let $H = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1, z > 0\}$ denote the standard closed Northern hemisphere. Denote $E = \partial H = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ the equator. The North pole is the point $N = (0, 0, 1)$.

Definition 1.2. A spherical triangle $ABC \subset \mathbb{S}^2$ will be called equatorial if $A = N$ and $B, C \in E$.

Therefore, the segments AB and AC are lines of longitude with length $\pi/2$. The segment $BC \subset E$. The angles at B and C are both equal to $\pi/2$ and the angle at A is equal to the length of BC.

Proof of Theorem 0.4.

For any finite polyhedron $\mathcal{P}$, the length of a closed geodesic of $\mathcal{P}^*$ corresponds to the sum of the exterior dihedral angles of an immersed cylinder. Therefore, the polar $\mathcal{P}^*$ is a finite admissible space. Please refer to Section 3 in [11].

Conversely, suppose that (Q, g) is a finite admissible metric space. Namely, it satisfies the conditions $1-6$ in Section 0. We index the boundary components of the metric space Q by the set $\{i\}_{1\leq i\leq k}$. For any boundary component γ_i of Q, suppose that

$$\gamma_i = \bigcup_{j=1}^{N_i} \gamma_{ij},$$

where $\{\gamma_{ij}\}_{1\leq j\leq N_i}$ are geodesic arcs on γ_i connecting two adjacent boundary cone points on γ_i. Their lengths satisfy $0 < L(\gamma_{ij}) \leq \pi$.

For any $1 \leq i \leq k,\ 1 \leq j \leq N_i$, we can construct a singular spherical triangle H_{ij} whose sides have lengths $\{L(\gamma_{ij}), \pi, \pi\}$. More precisely, we construct three equatorial triangles with lengths

$$\{L(\gamma_{ij}), \pi/2, \pi/2\},\ \{\pi, \pi/2, \pi/2\},\ \{\pi, \pi/2, \pi/2\}.$$

For each of the above three equatorial triangles, two sides have the same length $\pi/2$. Both of the angles opposite the sides are equal to $\pi/2$. Then the singular spherical triangle H_{ij} is glued out of these three equatorial triangles along their sides with length $\pi/2$. The triangle H_{ij} is like a hemisphere. The equator of H_{ij} consists of three geodesic arc with lengths $\{L(\gamma_{ij}), \pi, \pi\}$. The equator has lengths $2\pi + L(\gamma_{ij})$, while correspondingly there is a cone point $N(H_{ij})$ with angle $2\pi + L(\gamma_{ij})$ in the interior of H_{ij}, where $1 \leq i \leq k, 1 \leq j \leq N_i$.

Now we can view each singular spherical triangle H_{ij} as an isoceles spherical triangle whose sides have equal length π. Each H_{ij} has the intrinsic orientation induced from the equatorial triangles. By using their orientation, and using $\gamma_i = \bigcup_{j=1}^{N_i} \gamma_{ij}$ as a skeleton, we can glue $\{H_{i1}, H_{i2}, \cdots, H_{iN_i}\}$ into a metric space Q_{i0}. The boundary $\tilde{\gamma}_i = \partial Q_{i0}$ consists of N_i geodesic arcs $\{\tilde{\gamma}_{ij}\}_{1\le j\le N_i}$ with lengths

$$L(\tilde{\gamma}_{ij}) = L(\gamma_{ij}), \ \ 1 \le j \le N_i.$$

Recall that $\gamma_i = \bigcup_{j=1}^{N_i} \gamma_{ij}$ are the boundary component of Q. By using their natural orientation, we can glue Q_{i0} with Q by identifying γ_{ij} with the corresponding side $\tilde{\gamma}_{ij}$. Using this way, we obtain a metric space

$$Q_0 = Q \cup \bigcup_{i=1}^{k} Q_{i0}.$$

The space Q_0 is homeomorphic to the standard sphere $\hat{\mathbb{C}}$. Furthermore, since the metric space Q satisfies the conditions $1-6$ in Section 0, we can easily check that the metric space Q_0 satisfies the conditions of Theorem 1.1. It follows from Theorem 1.1 that there is a finite volume hyperbolic polyhedron $\tilde{\mathcal{P}}_0$ whose polar image is precisely Q_0.

Recall that $\{c_v\}_{v\in V}$ are the cone points of Q. Let F_v be the face of $\tilde{\mathcal{P}}_0$ corresponding to the cone point c_v. Let the plane of the face F_v be Π_v, $v \in V$. Denote by $\mathcal{P}_0$ the finite polyhedron bounded by the hyperbolic planes Π_v, $v \in V$. In general, it is very hard to determine the combinatorial of the convex polyhedron in the de Sitter space. But in the case of ideal vertices, it is easy to determine the combinatorics structure of $\mathcal{P}_0$ around the ideal vertices from the metric space $\tilde{Q}_0$. Please refer to Theorem 1.1.

Therefore we can deduce that $\mathcal{P}_0$ is a finite convex polyhedron with polar image Q, which implies Theorem 0.4. *q.e.d.*

The polyhedron $\mathcal{P}_0$ has k ideal faces $\{I_1, I_2, \cdots, I_k\}$, which are interstices in the Riemann sphere $\hat{\mathbb{C}}$. From the construction of the polar images Q_{0i}, $1 \le i \le k$, it follows that all the circular arcs on ∂I_{0i} are tangent to a common center circle. But we will not use this fact in our paper.

Let $U, W \subset \mathbb{C}$ be any two domains.

Definition 1.3. An orientation preserving map $f : U \to W$ is called quasiconformal if and only if, for some $K \ge 1$,

$$\limsup_{r\to 0^+} \frac{\max_{|z-\zeta|=r} |f(z) - f(\zeta)|}{\min_{|z-\zeta|=r} |f(z) - f(\zeta)|} \le K, \qquad \zeta \in U.$$

We refer to [2, 7, 8] for general background on quasiconformal mappings.

For each ideal face $I \in \{I_1, I_2, \cdots, I_k\}$, it has the natural complex structure induced from $\hat{\mathbb{C}}$. Two quasiconformal mappings $h_1, h_2 : I \to \hat{\mathbb{C}}$ are called equivalent if $h_2 \circ (h_1)^{-1} : h_1(I) \to h_2(I)$ is isotopic to a conformal homeomorphism g such that for each circular arc or circle $\gamma \subset \partial I$, the homeomorphism g maps $h_1(\gamma)$ onto $h_2(\gamma)$.

Definition 1.4. The *Teichmüller space* of I, denoted by $\mathcal{T}_I$, is the space of all equivalence classes of quasiconformal mappings $h : I \to \hat{\mathbb{C}}$.

If the interstice I is a k-sided polygon, it follows from the classical Teichmüller theory that the space $\mathcal{T}_I$ is diffeomorphic to the euclidean space $\mathbb{R}^{k-3}$. See e.g. [2, 7, 8].

2. Circle patterns

For any $0 \leq \theta < \pi$, for notational simplicity, in this section and the next section we will always denote by $\theta_\pi = \pi - \theta$. Then $0 < \theta_\pi \leq \pi$.

Recall that $G = (V, E)$ is an embedded graph on $\hat{\mathbb{C}}$. We can regard G as being of a cell decomposition of $\hat{\mathbb{C}}$. For any triangle of G with vertices $\{u, v, w\}$ and dihedral angles $\{\theta([u,v]), \theta([v,w]), \theta([w,u])\}$, we assume that $\{\theta([u,v]), \theta([v,w]), \theta([w,u])\}$ satisfy one of the following conditions:

(i). $\theta([u,v]) + \theta([v,w]) + \theta([w,u]) > \pi$ and

$$\theta([u,v]) + \theta([v,w]) - \pi \leq \theta([w,u]) \leq \pi - |\theta([u,v]) - \theta([v,w])|.$$

(ii). $\theta([u,v]) + \theta([v,w]) + \theta([w,u]) = \pi$ and $0 < \theta([u,v]), \theta([v,w]), \theta([w,u]) < \pi$.

(iii). $\theta([u,v]) + \theta([v,w]) + \theta([w,u]) = \pi$, and one of them is 0, and the others are in $(0, \pi)$.

(iv). $\theta([u,v]) + \theta([v,w]) + \theta([w,u]) < \pi$.

Equivalently,

(i)$'$. $\theta([u,v])_\pi + \theta([v,w])_\pi + \theta([w,u])_\pi < 2\pi$ and

$$|\theta([u,v])_\pi - \theta([v,w])_\pi| \leq \theta([w,u])_\pi \leq \theta([u,v])_\pi + \theta([v,w])_\pi.$$

(ii)$'$. $\theta([u,v])_\pi + \theta([v,w])_\pi + \theta([w,u])_\pi = 2\pi$, and $0 < \theta([u,v])_\pi, \theta([v,w])_\pi, \theta([w,u])_\pi < \pi$.

(iii)$'$. $\theta([u,v])_\pi + \theta([v,w])_\pi + \theta([w,u])_\pi = 2\pi$, and one of them is π, and the others are in $(0, \pi)$.

(iv)$'$. $\theta([u,v])_\pi + \theta([v,w])_\pi + \theta([w,u])_\pi > 2\pi$.

In the case $(i)'$ one can construct a spherical triangle (or degenerating spherical triangle) with lengths $\{\theta([u,v])_\pi, \theta([v,w])_\pi, \theta([w,u])_\pi\}$. In the case $(ii)'$ it is a hemisphere with all its edges lying in the equator. Its angles are $\{\pi, \pi, \pi\}$. In the case $(iii)'$, without loss of generality we assume that $\theta([u,v])_\pi = \pi$ and

$$0 < \theta([v,w])_\pi, \theta([w,u])_\pi < \pi, \quad \theta([v,w])_\pi + \theta([w,u])_\pi = \pi.$$

We can construct a lung which is a one-quarter sphere. One of its edge is a semi great circle and the other semi great circle consists of two circular arcs with lengths $\theta([v,w]), \theta([w,u])$. We can regard this lung as being of a triangle with lengths

$$\Big\{\theta([u,v])_\pi = \pi, \;\; \theta([v,w]), \;\; \theta([w,u])\Big\}.$$

Its angles are $\{\pi, \pi/2, \pi/2\}$. In the cases $(ii)'$ and $(iii)'$, we also call them spherical triangles.

Giving the data (G, Θ), we assume that all the triangles of G satisfy one of the conditions (i), (ii), (iii) and (iv). For each triangle of G which satisfies one of the conditions (i), (ii), (iii), we can construct the corresponding spherical triangle. By using the dual graph G^* of G as the skeleton, we can combine all there spherical triangles into a space $Q = Q(G, \Theta)$. It is a metric space with boundary. If the metric space Q satisfies the conditions $1 - 6$ in Section 0, by using Theorem 0.5, there is a polyhedron $\mathcal{P}_0 = \mathcal{P}(G, \Theta)$ whose polar is Q. Consequently, the corresponding circle pattern P_0 realizes the embedded graph G. Note that each face of G corresponds to an interstice of $\mathcal{P}_0$ (or P_0) if it satisfies either of the conditions:

(a). It has 3 edges and the dihedral angles satisfy the condition (iv).

(b). It has more than 3 edges.

Then Theorem 0.4 and Theorem 0.5 imply the following two results.

Theorem 2.1. *Let $P = \{P(v), v \in V\}$ and $Q = \{Q(v), v \in V\}$ be circle patterns in $\hat{\mathbb{C}}$ whose embedded contact graphs are both equivalent to G. Suppose that $\Theta_P = \Theta_Q : E \to [0, \pi)$, and that each interstice of P is conformally equivalent to the corresponding interstice of Q. Then P and Q are Mobiüs equivalent.*

Theorem 2.2. *The set of equivalent classes of circle patterns whose embedded graph is isotopic to G, and whose dihedral angle function is equal to Θ, can be naturally identified with the product of the Teichmüller space of the interstices of the circle pattern P_0.*

3. A SIMPLE NEW PROOF OF THEOREM 0.6 WHEN $0 \leq \Theta \leq \pi/2$

In this section we will give a simple proof of the main result in [5]. Let G be an embedded graph in $\hat{\mathbb{C}}$ with vertex V and edge set E. Let $\Theta : E \to [0, \pi/2]$ be the dihedral function. Furthermore, suppose that (G, Θ) satisfies the following condition.

(C). If the number of elements $|V| > 5$, and if a simple loop formed by the edges $e_1, e_2, \cdots, e_k$ separates the vertices of the embedded graph G, then

$$\sum_{j=1}^{k} \Theta(e_j)_\pi > 2\pi.$$

Under the hypotheses above, we have

Theorem 3.1. *There is circle pattern P_0 realizing the contact graph (G, Θ) when $0 \leq \Theta \leq \pi/2$. Moreover, the set of equivalent classes of circle patterns whose embedded graph is isotopic to G, and whose dihedral angle function is equal to Θ, can be naturally identified with the product of the Teichmüller space of the interstices of P_0.*

In order to prove the above theorem 3.1, the following two lemmas are needed. Let ABC be any spherical triangle whose edge lengths are greater than or equal to $\pi/2$.

Lemma 3.2. *Each angle of ABC is greater than or equal to the length of its opposite edge. In particular, all its angles are greater than or equal to $\pi/2$.*

Proof. Denote by $\{a, b, c\}$ the lengths of ABC. Then $\pi/2 \leq a, b, c \leq \pi$. By elementary spherical geometry, we have

$$\cos a = \cos c \cos b + \sin c \sin b \cos A. \tag{1}$$

Therefore $\cos A \leq 0$, which implies that $A \geq \pi/2$. Moreover from (1) it follows that $|\cos A| \geq |\cos a|$, which implies that $A \geq a$. *q.e.d.*

Lemma 3.3. *Let D be any point in the edge BC. Then the length of the line AD is greater than or equal to $\pi/2$.*

Proof. Let $A = (0, 0, 1)$ be the North pole. Then B and C are in the South hemisphere. Obviously $AD \geq \pi/2$. *q.e.d.*

Proof of Theorem 3.1. Since the arguments are rather technical, the proof will be sketchy.

If $0 \leq \Theta \leq \pi/2$, then (G, Θ) obviously satisfies the conditions (i), (ii), (iii) and (iv) in Section 2. Let $Q = Q(G, \Theta)$ be the corresponding metric space. To prove Theorem 3.1, by using Theorem 0.6, it is only need to verify that (Q, g) satisfies the conditions 1 – 6 in Section 0. Obviously it satisfies the conditions 1, 2, 5 in Section 0.

Suppose that $c \in Q$ is any interior cone point. If there are at least 5 triangles adjacent to c, from Lemma 3.2 it follows that its cone angle is greater than 2π. If there are only 3 or 4 triangles adjacent to c, Lemma 3.2 and the above condition (C) prove that the cone angle at c is greater than 2π, which implies the condition 3 in Section 0.

For any cone points $c \in Q$, we denote by $star(c)$ the union of all spherical triangles adjacent to c. Let $int(star(c))$ denote the interior of $star(c)$.

Let $\gamma \subset Q$ be a closed geodesic. If $\gamma \cap int(sta(c)) = \emptyset$ for any cone points $\{c\} \subset Q$, then γ consists of several edges of the spherical triangles. From $0 \leq \Theta \leq \pi/2$ and the condition (C), it follows that its length $L(\gamma) > 2\pi$.

Now suppose that $\gamma \cap int(sta(c_v)) \neq \emptyset$ for some cone point c_v. If $c_v \in \gamma$, then Lemma 3.3 implies that the length of $\gamma \cap int(sta(c_v)) \geq \pi$. If $c_v \notin \gamma$, by using elementary spherical geometry we also prove that the length of $\gamma \cap int(sta(c_v)) \geq \pi$. When $\gamma \cap int(sta(c_v)) \neq \emptyset$ and $\gamma \cap int(sta(c_w)) \neq \emptyset$ for some cone points c_v, c_w with $int(sta(c_v)) \cap int(sta(c_w)) = \emptyset$, we deduce that the length $L(\gamma) \geq 2\pi$. Otherwise, we have $\gamma \cap int(sta(c_v)) \neq \emptyset$ and $\gamma \subset star(c_v)$ for some cone point c_v. When $c_v \in \gamma$, we have $L(\gamma) = 2\pi$. If c_v is in the interior of γ, and $\gamma \subset int(star(c_v))$, then by using elementary spherical geometry we have $L(\gamma) > 2\pi$. The only remaining case is

$$\gamma = (\cup_k e_k) \cup (\cup_j \gamma'_j),$$

where $\{e_k\}$ are the edges on $\partial star(c_v)$ and $\{\gamma'_j\}$ are the components $\gamma \cap int(star(c_v))$. Note that the length of each edge e_i is $\geq \pi/2$ and $L(\gamma') \geq \pi$. We can easily deduce $L(\gamma) \geq 2\pi$ when $i > 1$ or $j > 1$. When $i = 1$ and $j = 1$, we have $\gamma = e \cup \gamma'$. Recall that the angle of c is greater than 2π. Note that the edge length of a spherical triangle with fixed sides is a strictly increasing function of the opposite angle. Thus

$$L(\gamma') > 2\pi - L(e).$$

Therefore $L(\gamma) = L(e) + L(\gamma') > 2\pi$, which verifies the condition 4 of Section 0.

By applying the similar reasoning, we can also verify the condition 6 in Section 0. *q.e.d.*

4. Proof of Theorem 0.5

This section is devoted to Theorem 0.5.

Let $I \in \{I_1, I_2, \cdots, I_k\}$ be any interstices of the finite convex polyhedron $\mathcal{P}_0$. Denote $\partial I = \bigcup_{j=1}^{N} \delta_j$. Recall that V is the index set of all finite faces of $\mathcal{P}_0$. Then we have the corresponding circle pattern P_0. Without loss of generality, we assume that $\{P_0(v_1), P_0(v_2), \cdots, P_0(v_N)\}$ are circles which share circular arcs $\{\delta_1, \delta_2, \cdots, \delta_N\}$ with ∂I, where $v_1, v_2, \cdots, v_N \in V$. Note that $\mathcal{P}_0$ is a finite convex polyhedron with polar image Q. It follows that the length of the geodesic arc between the boundary cone points $c_{v_j}, c_{v_{j+1}} \in Q$ is precisely

$$\pi - \theta([v_j, v_{j+1}]), \ \ 1 \leqq j \leqq N,$$

where $\theta([v_j, v_{j+1}])$ is the dihedral angle of the circles $P_0(v_j), P_0(v_{j+1})$.

Let $\{\tau_1, \cdots, \tau_k\} \in \Pi_{i=1}^{k} \mathcal{T}_{I_i}$ be any point in the product of the Teichmüller spaces. For the given complex structure $[\tau : I \to \hat{\mathbb{C}}]$, there are N vertices on $\tau(I)$, denoted by $\{a_1, a_2, \cdots, a_k\}$. By post-composition with a Mobiüs transformation, we may assume that the region $\tau(I)$ is a bounded domain $\mathbb{C}$.

Lay down a regular hexagonal packing of circles in $\mathbb{C}$, say each of radius $1/n$. By a small translation we can move the circle packing so that each of $\{a_1, a_2, \cdots, a_k\}$ is inside a circle. By using the boundary component $\partial\tau(I)$ like a cookie-cutter, we obtain a circle packing P_n which consists of all the circles intersecting the closed region $\tau(I)$. Denote K_n the contact graph of P_n, where $n = 1, 2 \cdots$. By using K_n, we can construct a new graph $\mathcal{K}_n$ as follows. For $j = 1, 2, \cdots, N$, we add the vertex v_j and edges $[v_j, v_{j+1}]$ to the graph K_n. We also add edges $[v_j, v]$, where $v \in \{v : P_n(v) \cap \tau(\gamma_j) \neq \emptyset\}$. Obviously the resulting graph $\mathcal{K}_n$ is isomorphic to the one-skeleton of a triangulation of a closed topological disk D. Therefore it induces a cell decomposition $(\mathcal{K}_n; \mathcal{E}_n, \mathcal{F}_n)$ of the closed topological disk D, where $\mathcal{E}_n$ is the set of edges and $\mathcal{F}_n$ is the set of faces. The region $\tau(I)$ induces a natural orientation on the cell decomposition.

Now we can define a (singular) metric $g = g(I, n)$ on the closed region D. Define the metric g on each edge of $\mathcal{E}_n$ by setting $L(e) = \pi - \theta(e)$ if $e = [v_j, v_{j+1}]$, $1 \le j \le N$, where $\theta([v_j, v_{j+1}])$ is the dihedral angle of the circles $P_0(v_j), P_0(v_{j+1})$. Otherwise, set $L(e) = \pi$. For each face $F \in \mathcal{F}_n$ with three edges, say $\{e_1, e_2, e_3\}$, we can define the metric g on F so that it becomes a singular triangle with edge lengths $L(e_1), L(e_2), L(e_3)$. Please see the proof of Theorem 0.4 in Section 1. In this metric each face F looks like a "hemisphere" with three edges on its equator. The North pole of F is a cone point with angle

$$L(e_1) + L(e_2) + L(e_3) > 2\pi.$$

Therefore we obtain a metric space (D, g).

For any $1 \le i \le k$ and $n \ge 1$, by using the above construction, let Q_{in} denote the resulting metric space (D, g_{in}), where $g_{in} = g(I_i, n)$. From the above construction the boundary ∂Q_{in} consists of N_i geodesic arcs with lengths

$$\{L(\gamma_{i1}), L(\gamma_{i2}), \cdots, L(\gamma_{iN_i})\}.$$

Recall that $\gamma_i = \bigcup_{j=1}^{N_i} \gamma_{ij}$, $1 \le i \le k$, are the boundary components of Q. By using their natural orientation, we can glue Q_{in} to Q by identifying γ_{ij} with the corresponding side on ∂Q_{in}. Using this way, we obtain a metric space

$$Q_n = Q \cup \bigcup_{i=1}^{k} Q_{in}, \quad n = 1, 2, \cdots.$$

The space Q_n is homeomorphic to the standard sphere $\hat{\mathbb{C}}$. Since Q satisfies the conditions $1-6$, the metric spaces $\{Q_n\}$ satisfy the conditions in Theorem 1.1. Hence there is a finite volume hyperbolic polyhedron $\tilde{\mathcal{P}}_n$ whose polar is Q_n.

Recall that $c_v, v \in V$, are cone points of Q. We call c_v, $v \in V$, ordinary cone points. Other cone points of $Q_n \backslash Q$ are called special cone points. Let $\{F_{vn}\}$ be the face of $\tilde{\mathcal{P}}_n$ corresponding to the ordinary cone point $\{c_v\}$. Let the hyperbolic plane of the face F_{vn} be Π_{vn}, $v \in V$. $\{\Pi_{vn}\}_{v \in V}$ are called ordinary planes. Other hyperbolic planes corresponding to special cone points of $\tilde{\mathcal{P}}_n$ are called special planes.

We can move $\tilde{\mathcal{P}}_n$ by an isometry if it is necessary. By using a similar argument as in [10] (Section 5-8), we have

Lemma 4.1. *There is an $N_0 > 0$ and $R > 0$ such that all of the ordinary planes of $\tilde{\mathcal{P}}_n$ intersect the ball $B_R(0)$ for $n > N_0$.*

In general, it is hard to determine the combinatorial structure of the polyhedron $\tilde{\mathcal{P}}_n$. Observe that each special cone point of Q_n is incident to a closed geodesic of $\tilde{Q}_n$ with length 2π. From Theorem 1.1, it follows that the intersections of all special planes with $\hat{\mathbb{C}}$ are k hexagonal circle packing and the dual circles of triples of mutually intersecting circles in these k hexagonal packing. Applying Length-Area Lemma ([12]), we have

Lemma 4.2. *For any $R > 0$, there is an $N_0 = N_0(R)$ such that for all special planes Π of $\tilde{\mathcal{P}}_n$, we have $\Pi \cap B_R(0) = \emptyset$ for any $n > N_0$.*

From Lemma 4.1 and 4.2, it follows that $\tilde{\mathcal{P}}_n \to \mathcal{P}_\tau$ in the Hausdorff topology as $n \to \infty$. It is easily to verify that $\mathcal{P}_\tau$ is a finite convex polyhedron with polar Q. From the following Lemma 4.3, it follows that the interstices of $\mathcal{P}_\tau$ have the complex structures τ, which proves the existence part of Theorem 0.5. *q.e.d.*

Lemma 4.3. *The interstices of $\mathcal{P}_\tau$ do not degenerate.*

Proof. If at least interstice of $\mathcal{P}_\tau$ degenerates, say I_1, then it is an ideal vertex of $\mathcal{P}_\tau$. Recall that $\gamma_1 = \cup_{j=1}^{N_1} \gamma_{1j}$ is a boundary component of Q. Therefore, in the polar of $\mathcal{P}_\tau$ the boundary component γ_1 will be the boundary of a hemisphere.

From Lemma 4.1, it follows that all ordinary faces of $\tilde{\mathcal{P}}_n$ will not degenerate. Hence, in the limiting case, all boundary cone points $\{c_{v_j}\}_{v_j \in V} \subset \gamma_i$ will become interior cone points of the polar of $\mathcal{P}_\tau$. Hence their cone angles are greater than 2π. It follows that in Q the angle subtend by γ_1 at c_{v_j} are greater than π. From Condition 6, we can deduce that $\gamma_1 \subset \partial Q$ is a closed geodesic. Hence its length is greater than 2π, which contradicts to the fact that γ_1 is the boundary of a hemisphere. *q.e.d.*

In order to prove the uniqueness part of Theorem 0.5, the following two results are needed. The proofs, included here for completeness, are elementary. Please see [4, 9, 16].

Let $C_1, C_2, C_3 \subset \mathbb{C}$ be a tripe of mutually intersecting circles such that their dihedral angles $\{\theta_{12}, \theta_{23}, \theta_{31}\}$ satisfy one of the conditions (i), (ii),(iii) and (iv) in Section 2. We also suppose that no circle has its interior contained in the union of the interior of two others.

Let L_{ij} be the line through $C_i \cap C_j$, and tangent to C_i and C_j at $C_i \cap C_j$ in case $\theta_{ij} = 0$. Then the lines L_{12}, L_{23} and L_{31} meet in a point O, which is called the dual center of this tripe of circles. Obviously, if there is a circle orthogonal to these 3 circles, then O is the center of this circle.

Lemma 4.4. *Suppose the circles C_1, C_2, C_3 meet pairwise in the fixed dihedral angles $0 \le \theta_{12}, \theta_{23}, \theta_{31} < \pi$. If C_1 and C_2 are held constant but C_3 is varied in such a way that the dihedral angles $\theta_{12}, \theta_{23}, \theta_{31}$ are fixed. Let A_1, A_2, A_3 denote the centers of circles C_1, C_2, C_3.*

For $j = 1, 2, 3$, denote $\rho_j = \rho(C_j)$. Let $\Theta_j = \Theta_j(\rho_1, \rho_2, \rho_3, \theta_{12}, \theta_{23}, \theta_{31})$ denote the angle of the triangle $A_1A_2A_3$ at the vertices A_j. Then

$$\frac{\partial \Theta_3}{\partial \log \rho_1} = \frac{h_{13}}{|A_1 - A_3|},$$

where h_{13} is the oriented distance from O to the edge A_1A_3.

In particular, $\dfrac{\partial \Theta_3}{\partial \log \rho_1}$ does not change if we switch 1 and 3.

Proof. Fix A_3 as the origin. We also assume that A_2 lies on the positive real axis. Let z denote the complex coordinate of A_1. Hence $z = l_2 \cdot e^{i\Theta_3}$, where $l_2 = |A_1 A_3|$. By applying the argument of Lemma 3 in [9], we have

$$\frac{\partial z}{\partial \log \rho_1} = \overrightarrow{OA_1}. \tag{2}$$

Moreover,

$$\frac{\partial z}{\partial \log \rho_1} = \frac{\partial (l_2 \cdot e^{i\Theta_3})}{\partial \log \rho_1} = e^{i\Theta_3}\left(\frac{\partial l_2}{\partial \log \rho_1} + i l_2 \cdot \frac{\partial \Theta_3}{\partial \log \rho_1}\right). \tag{3}$$

By combining (2) with (3), we conclude that

$$\frac{\partial \Theta_3}{\partial \log \rho_1} = \frac{h_{13}}{|A_1 - A_3|}.$$

q.e.d.

Suppose that $C_1, C_2, C_3, C_4 \subset \mathbb{C}$ are 4 circles such that C_1, C_2, C_3 are mutually intersecting and C_1, C_3, C_4 are mutually intersecting and the tripes of dihedral angles $\{\theta_{12}, \theta_{23}, \theta_{31}\}$ and $\{\theta_{13}, \theta_{34}, \theta_{41}\}$ satisfy one of the conditions (i), (ii),(iii) and (iv) in Section 2. We also assume that no circle has its interior contained in the union of the interior of two others.

Recall that O is the dual center of $\{C_1, C_2, C_3\}$. Let O' denote the dual center of $\{C_1, C_3, C_4\}$. By using Lemma 4.4, we have

Corollary 4.5. *Let A_j denote the center of C_j, $1 \le j \le 4$. If we fix the dihedral angles $0 \le \theta_{12}, \theta_{23}, \theta_{31}, \theta_{14}, \theta_{34} < \pi$, then*

$$\frac{\partial \angle A_2 A_3 A_4}{\partial \log \rho_1} = \frac{|OO'|}{|A_1 A_3|}.$$

In particular, $\dfrac{\partial \angle A_2 A_3 A_4}{\partial \log \rho_1} \ge 0.$

Now we are ready to prove the uniqueness part of Theorem 0.5.

If this is not the case, we assume by contrary that there are two polyhedron $\mathcal{P}$ and $\mathcal{P}'$ with the same polar (Q, g), and their corresponding interstices have the same complex structure τ.

Then there are holomorphic homeomorphisms $\phi_i : I_i \to \tilde{I}_i$, $1 \le i \le p$, between the pairs of corresponding interstices. ϕ_i maps circular arcs or circles on ∂I_i to the corresponding circular arcs or circles on $\partial I_i'$.

Recall that $*$ is the polar map. Then, as metric spaces, both of $\mathcal{P}^*, \mathcal{P}'^*$ are isometric to (Q, g). Note that $\mathcal{P}^*$ and $\mathcal{P}'^*$ may not have the same combinatorial structure.

If $\mathcal{P}^*$ and $\mathcal{P}'^*$ are combinatorially equivalent, then there is nothing to do.

Now we assume that the polar images $\mathcal{P}^*$, $\mathcal{P}'^* \subset \mathbb{S}^2_1$ are not combinatorially equivalent. Hence $\mathcal{P}$, $\mathcal{P}' \subset \mathbb{H}^3$ are not combinatorially equivalent. $\mathcal{P}^*$, $\mathcal{P}'^*$ induce two different cell decompositions $\mathcal{G}, \mathcal{G}'$ of Q with the same vertex set V. By using these cell decompositions $\mathcal{G}, \mathcal{G}'$, we can give a new cell decomposition $G = G(\mathcal{G}, \mathcal{G}')$ of Q by "superimposing" the original cell decompositions $\mathcal{G}, \mathcal{G}'$. For a more detailed version of this argument, please see [1, 11].

The vertex set of G is the union of V with the set V' of the intersections of edges of $\mathcal{P}^*$ with those of $\mathcal{P}'^*$. The edges of G will be segments of the edges of $\mathcal{P}^*$ and $\mathcal{P}'^*$. In this way some of edges of $\mathcal{P}^*, \mathcal{P}'^* \subset \mathbb{S}^2_1$ may have an associated dihedral angle π. We will call these degenerate edges. Under the polar map $* : \mathbb{S}^2_1 \to \mathbb{H}^3$, the degenerate edges in $\mathbb{S}^2_1$ are sent to edges of length 0 in $\mathbb{H}^3$. The polar images $\mathcal{P}^*, \mathcal{P}'^* \subset \mathbb{S}^2_1$ will have two kinds of vertices and edges degeneracy:

(1) At vertices of the cell decomposition G, some degenerate edges may enter the vertex, but the vertex itself is non-degenerate;

(2) At each vertex of the cell decomposition G, a non-degenerate edge intersects a degenerate one, and the vertex itself is flat. That is, the vertex is degenerate.

In any case, we can regard $\mathcal{P}^*, \mathcal{P}'^*$ as being of the same combinatorial type.

For convenience, to prove the uniqueness we consider the corresponding circle patterns P, P' on $\hat{\mathbb{C}}$. Recall that $\{I_1, I_2, \cdots, I_p\}$ (resp. $\{I'_1, I'_2, \cdots, I'_p\}$) are interstices of P (resp. P'). By using a Mobiüs transformation, we may assume $\infty \in I_p$ is a fixed point of $\phi_p : I_p \to \tilde{I}_p$. Then ϕ_p has the form

$$\phi_p(z) = z + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \tag{4}$$

near ∞. If there is at least some $z \in \{I_i\}$ such that $|\phi'_i(z)| \neq 1$, without loss of generality, we assume that $|\phi'_i(z)| < 1$. Denote by M_0 the following

$$\min\left\{\log|\phi'_1|, \log|\phi'_2|, \cdots, \log|\phi'_k|, \log\frac{\rho(P'(v_0))}{\rho(P(v_0))}, \log\frac{\rho(P'(v_1))}{\rho(P(v_1))}, \cdots, \log\frac{\rho(P'(v_{|V|-1}))}{\rho(P(v_{|V|-1}))}\right\}, \tag{5}$$

where $\rho(P(v))$ (resp. $\rho(P'(v))$) are the radii of the circles $P(v)$ (resp. $P'(v)$) in the patterns, where $v \in V$. Then $M_0 < 0$. Then we have the following maximum principle. Its proof is left to the last part of this section

Lemma 4.6. *Let $\mathcal{P}, \mathcal{P}'$ be two finite hyperbolic polyhedra with the same polar. Then the maximum (or minimum) of*

$$\frac{\rho(P'(v))}{\rho(P(v))}$$

is attained at a boundary vertex $v \in V$, where P, P' are the corresponding circle patterns.

Let us proceed with the proof of the uniqueness part of Theorem 0.5.

By using Lemma 4.6, it follows that there is some ϕ_i, say ϕ_1, such that

$$\log|\phi'_1(z_0)| = M_0 \tag{6}$$

or some boundary circles, say $P(v_0), P'(v_0)$, such that

$$\log\frac{\rho(P'(v_0))}{\rho(P(v_0))} = M_0. \tag{7}$$

Assume the first case holds. If z_0 is the intersecting point of two adjacent circles $P(v_j), P(v_{j+1})$ with dihedral angle $\theta_j = \Theta([v_j, v_{j+1}]) > 0$, or if z_0 is not the intersecting point of two adjacent circles, then the strong maximal principle[3] immediately implies

$$0 > \frac{\partial}{\partial n}\log|\phi'_1(z_0)| = \frac{1}{\rho(P(v_{j_0}))} - \frac{|\phi'_1(z_0)|}{\rho(P'(v_{j_0}))}.$$

[3]The strong maximal principle states that: if a non-constant bounded harmonic function attains its minimal (resp. maximal) at $z_0 \in \partial\Omega$, and the boundary $\partial\Omega$ satisfies an interior sphere condition at z_0, then the outer normal derivative of u at z_0, if it exists, satisfies the strict inequality $\frac{\partial u}{\partial n}(z_0) > 0$ (resp. < 0). See e.g. Lemma 3.4 of [3].

Here $P(v_0)$ denote some boundary circle of I_1 such that $z_0 \in \partial I_1 \cap P(v_0)$. See [13, 5]. It is immediately that $\rho(P'(v_{j_0}))/\rho(P(v_{j_0})) < |\phi_1'(z_0)| < 1$, which is a contradiction to (6). If z_0 is the tangent point of two adjacent circles $P(v_j), P(v_{j+1})$, a simple calculation shows that

$$|\phi_1'(z_0)| = \left(\frac{1}{\rho(P(v_j))} + \frac{1}{\rho(P(v_{j+1}))}\right) \cdot \left(\frac{1}{\rho(P'(v_j))} + \frac{1}{\rho(P'(v_{j+1}))}\right)^{-1}. \tag{8}$$

We can reduce this case to the second case.

Now we assume that the second case holds. Then there is a boundary circle $P(v_0)$ such that $\log \dfrac{\rho(P'(v_0))}{\rho(P(v_0))}$ attains the minimal value M_0. Recall that G is the contact graph. Without loss of generality, let $\{v_1, v_2, \cdots, v_p\}$ denote the vertices of G which is adjacent to the vertex v_0. For $0 \le j \le p$, let A_j(resp. A_j') denote the center of the circle $P(v_j)$ (resp. $P'(v_j)$). Let $A, \tilde{A}$ (resp. $A', \tilde{A}'$) denote the end points of the circular arcs of the interstices adjacent to the circle $P(v_0)$ (resp. $P'(v_0)$). Then by using Corollary 4.5, we have, for $2 \le j \le p-1$,

$$\angle A_{j-1}'A_j'A_{j+1}' \le \angle A_{j-1}A_jA_{j+1}, \tag{9}$$

and

$$\angle A'A_1'A_2' \le \angle AA_1A_2, \quad \angle A_{p-1}'A_p'\tilde{A}' \le \angle A_{p-1}A_p\tilde{A}. \tag{10}$$

Please see the following Figure 2. Note that we use Corollary 4.5 with 2 degenerating circles in (10) .

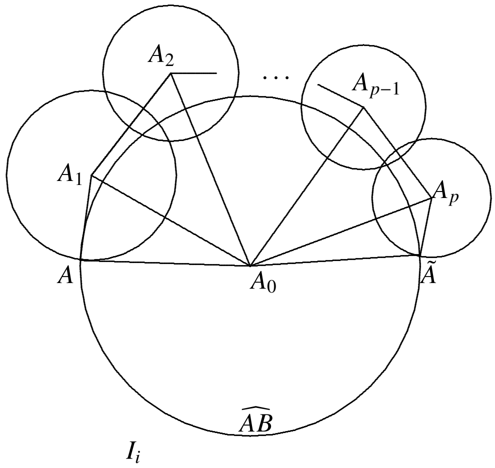

Figure 2

On the other hand, we have

$$\angle A_0'A'A_1' = \angle A_0AA_1, \quad \angle A_0'\tilde{A}'A_p' = \angle A_0\tilde{A}A_p, \tag{11}$$

which implies that

$$\angle A'A_0'\tilde{A}' \ge \angle AA_0\tilde{A},$$

where $\angle AA_0\tilde{A}$ (resp. $\angle A'A_0'\tilde{A}'$) is the angle opposite to the p circles $\{P(v_j)\}_{1\le j\le p}$ (resp. $\{P'(v_j)\}_{1\le j\le p}$). It follows from (6) and (7) that on the arc $\widehat{AB}$, we have

$$M_0 = \frac{\rho(P'(v_0))}{\rho(P(v_0))} < |\phi_i'(z)|, \quad z \in \widehat{AB} \subset \partial I_i. \tag{12}$$

Integrating (12) over the circular arc $\widehat{AB}$ gives

$$2\pi - \angle A'A_0'\tilde{A}' > 2\pi - \angle AA_0\tilde{A}. \tag{13}$$

Combining (11) and (13), we reach a contradiction. Therefore, $|\phi_i'(z)| \equiv 1, z \in \{I_i\}$, from which we deduce that

$$\phi(z) \equiv z, \ \ z \in \{I_i\}.$$

Hence there is an isometry $M : \mathbb{H}^3 \to \mathbb{H}^3$ such that $M(\mathcal{P}) = \mathcal{P}'$. Namely, the polyhedron $\mathcal{P}$ is congruent to $\mathcal{P}'$, which completes the proof of Theorem 0.5. *q.e.d.*

Proof of Lemma 4.6.

Observe that the polyhedra $\mathcal{P}$, $\mathcal{P}'$ have the same polar (Q, g), which is regarded as an abstract metric space. We can regard $\mathcal{P}^*, \mathcal{P}'^*$ as being of the same combinatorial type.

We assume the maximum (or minimum) of $\frac{\rho(P'(v))}{\rho(P(v))}$ is attained at an interior boundary $v_0 \in V$. Let $v_1, v_2, \cdots, v_p, v_{p+1} \equiv v_1$ be the chain of neighboring vertices. For $0 \le j \le p$, let A_j denote the center of the circle $P(v_j)$. And set $\rho_j = \rho(P(v_j))$. Let O_j denote the dual center of the triple $\{P(v_{j-1}), P(v_j), P(v_{j+1})\}$. Then it follows from Corollary 4.5 that, for all $1 \le j \le p$,

$$\frac{\partial \angle A_{j-1}A_jA_{j+1}}{\partial \log \rho_0} = \frac{|O_{j-1}O_j|}{|A_0A_j|} \ge 0. \tag{14}$$

Since v_0 is an interior point, if v_0 is not a degenerate vertex, the $'\ \ge'$ in (14) can not be $''\ =\ ''$ for all $1 \le j \le p$. Otherwise, all the dual centers of the triples $\{P(v_0), P(v_j), P(v_{j+1})\}_{1 \le j \le p}$ are the same, we can deduce that c_{v_0} is flat in the space (Q, g). This is a contradiction since c_{v_0} is a cone point.

Now we assume that v_0 is an interior point of G and it is a degenerate vertex of G. Then, v_0 is the intersection of a non-degenerate edge with a degenerate edge in $\mathbb{S}^2_1$. Without loss of generality, we suppose $p = 4$. Then $v_0 = [v_1, v_3] \cap [v_2, v_4]$, where $[v_1, v_3]$ is non-degenerate edge in $\mathcal{P}^*$ and $[v_2, v_4]$ is degenerate edge in $\mathcal{P}^*$. Please see Figure 3.

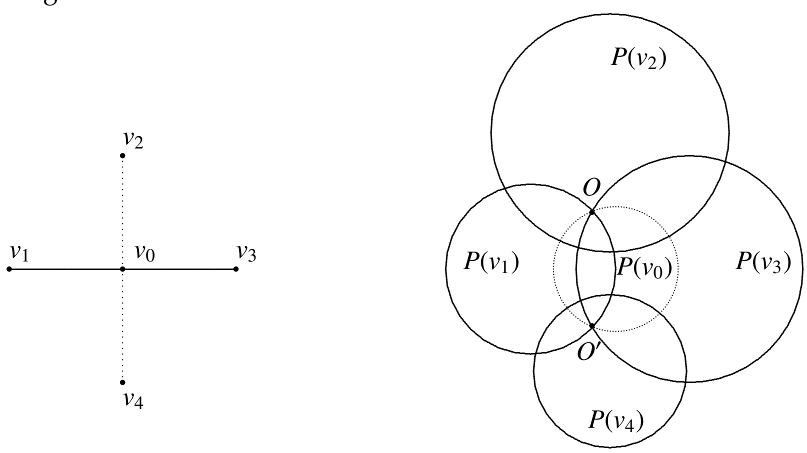

Figure 3

If all the $'\ \ge'$ in (14) are $'\ ='$, all the dual centers of triples $\{P(v_0), P(v_j), P(v_{j+1})\}_{1 \le j \le 4}$ are the same. Hence the center is O or O', say O. In the metric space (Q, g), the cone pints $c_{v_j}, 1 \leq j \leq 4$, are incident to a closed geodesic with length 2π. Theorem 1.1 implies this closed geodesic corresponds to ideal vertices of the polyhedra $\mathcal{P}$, $\mathcal{P}'$. This contradicts to our assumption that $v_1, v_2, \cdots, v_p, v_{p+1} \equiv v_1$ is the chain of neighboring vertices of v_0.

In any way, the sum $\sum_{j=1}^{p} \angle A_{j-1}A_jA_{j+1}$ is strictly increasing in ρ_0.

On the other hand, since v_0 is an interior vertex, we have $\sum_{j=1}^{p} \angle A_{j-1}A_jA_{j+1} = (p-2)\cdot\pi$. Therefore the maximum principle follows. *q.e.d.*

References

[1] Aleksandrov, A.D.: Convex Polyhedra (in Russian) Moscow: GITTL 1950; (German translation: Berlin: Akademie Verlag 1958).

[2] L. Ahlfors, Lectures on quasiconformal mappings, (D. Van. Nostrand Math. Studies 1966).

[3] D. Gilbarg & N. S. Trudinger, Elliptic differential equations of second order , (Springer-Verlag, Berlin, 1998).

[4] Z. X. He, 'Rigidity of infinite disk pattern', Ann. Math., 149 (1999) 1-33.

[5] Z. X. He. & J. S. Liu, 'On the Teichmller theory of circle patterns', Trans. Amer. Math. Soc., 365 (2013) 6517-6541.

[6] Z. X. He. & O. Schramm, 'On the convergence of circle packing', Invent. Math., 125 (1996) 285-305.

[7] Y. Imayoshi & M. Taniguchi, An introduction to Teichmüller spaces, (Springer-Verlag, Tokyo, 1992).

[8] O. Lehto. & K. I. Virtanen., Quasiconformal Mappings in the plane, (Springer-Verlag, New York, 1973).

[9] A. Marden & B. Rodin, 'On Thurston's formulation and proof of Andreev's theorem', In Computational Methods and Function Theory, Lecture Notes in Mathematics 1435. Springer, (1989) 103-115.

[10] I. A. Rivin., 'A characterization of ideal polyhedra in hyperbolic 3-space', Ann. Math., 143 (1996) 51-70.

[11] I. A. Rivin. & C. Hodgson, 'A characterization of compact convex polyhedra in hyperbolic 3-space', Invent. Math., 111 (1993) 77-111.

[12] B. Rodin. & D. Sullivan, 'The convergence of circle packings to the Riemann mapping', J. Diff. Geom., 26 (1987) 349-360.

[13] M. Schiffer. & N. S. Hawley., 'Connections and conformal mappings', Acta. Math., 107 (1962) 175-274.

[14] O. Schramm, 'Rigidity of infinite (circle) packings', J. Amer. Math. Soc., 4 (1991) 127-149.

[15] K. Stephenson, 'Circle packings in the approximation of conformal mappings', Bull. A.M.S., 23(1990) 407-415.

[16] W. Thurston, The geometry and topology of three manifolds, (Princeton University Press, 1977).

College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China

Mathematical Sciences Research Institute in Chongqing, Chongqing 401331, China
E-mail address: `hxj@cqu.edu.cn`

HUA Loo-Keng Key Laboratory of Mathematics, Chinese Academic of Sciences, Beijing 100190, China

Institute of Mathematics, Academic of Mathematics & System Sciences, Chinese Academic of Sciences, Beijing 100190, China
E-mail address: `liujsong@math.ac.cn`