ON THE MINIMAL FACTORIZATION OF THE HIGHER DIMENSIONAL QUASICONFORMAL MAPS

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ABSTRACT. With the aid of the logarithmic spiral mapping

$$s_\lambda(\rho,\theta) = (\rho,\ \theta + \lambda \log \rho),$$

we construct an n (≥ 3)-dimensional K-quasiconformal homeomorphism $f_{n,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ (to be defined in Section 1) with the following property: if $U_0 \subset \mathbb{R}^n$ is any domain and $0 \in U_0$, then the restriction K-quasiconformal map $f_{n,\lambda}|_{U_0}$ admits no "minimal" factorizations. That is, for any $0 < s < 1$, we have

$$f_{n,\lambda}|_{U_0} \neq f_2 \circ f_1,$$

where f_1 is K^s-quasiconformal and f_2 is K^{1-s}-quasiconformal.

1. INTRODUCTION

Let $n \geq 2$ be an integer. Suppose that $f : U \to V$ be a sense-preserving homeomorphism of domains in $\mathbb{R}^n$. We define

$$H_f(\zeta) \equiv \limsup_{r\to 0^+} \frac{\max_{|z-\zeta|=r} |f(z) - f(\zeta)|}{\min_{|z-\zeta|=r} |f(z) - f(\zeta)|}, \quad \zeta \in U.$$

Denote by

$$K[f] = \begin{cases} \infty, & \text{if } \sup_{\zeta\in U} H_f(\zeta) = \infty, \\ ess\sup_{\zeta\in U} H_f(\zeta), & \text{if } \sup_{\zeta\in U} H_f(\zeta) \neq \infty, \end{cases}$$

the maximal (linear) distortion of f. If

$$K[f] \leq K, \tag{1}$$

where $1 \leq K < \infty$, we call f a K-quasiconformal homeomorphism. Thus roughly speaking, a quasiconformal homeomorphism distorts the relative distance of nearby points by a bounded factor. For any quasiconformal maps f and g, we obviously have

$$K[g \circ f] \leq K[g] \cdot K[f].$$

H. Grötzsch [8] first introduced plane quasiconformal homeomorphisms in 1928. Later M. A. Lavrentieff [13], C. B. Morrey [18] generalized a classical result due to Gauss on the existence of isothermal coordinates by establishing the *Measurable Riemann Mapping Theorem* for plane quasiconformal maps. At about the same time, O. Teichmüller [22, 23] used plane quasiconformal homeomorphisms to study the deformation spaces of closed Riemann surfaces. Under the influence of the work of Teichmüller, L. Ahlfors and L. Bers [2, 3, 5] gave a systematic study of the general theory of plane quasiconformal homeomorphisms. Now plane quasiconformal maps play important roles in a variety of areas in complex analysis, including complex dynamics and kleinian groups, etc (see e.g. [5, 6, 21]).

Higher dimensional quasiconformal maps ($n \geq 3$) were first considered by M. A. Laverentieff [14], A. Markuševič [17] and M. Kreines [12] in 1938-1941. The above definition (1), free of all differentiability requirements, goes back to M. A. Laverentieff [14] in 1935. Since 1959, higher dimensional quasiconformal mappings have been studied rather extensively by C. Löwner [16], F. W. Gehring [9, 10, 11], J. Väisälä [24] and others in several countries.

But no analogous results exist for the distortion of length or angle in n (≥ 3)-dimensions because the corresponding system of partial differential equations is over determined. Geometrically, the lack of such a result reflects the lack of nontrivial conformal mappings in higher dimensional space.

If f is a plane quasiconformal homeomorphism with maximal distortion K, the Measurable Riemann Mapping Theorem implies that a "minimal" factorization $f = f_2 \circ f_1$ always exists. That is, f_1 is K^s-quasiconformal and f_2 is K^{1-s}-quasiconformal with $0 < s < 1$. In particular for any given $\epsilon > 0$, we can always write

$$f = f_m \circ f_{m-1} \circ \cdots \circ f_1, \tag{2}$$

where $K[f_i] < 1 + \epsilon$ ($1 \leq i \leq m$) and $m = m(\epsilon, K)$.

Using this fact, L. Ahlfors [1] gave a quasiconformal extension of each quasiconformal map of $\hat{\mathbb{C}}$ to the 3-dimensional upper half spaces $\mathbb{H}^3$.

Contrary to plane quasiconformal maps, little is known on factoring higher dimensional quasiconformal mappings. F. Gehring [10] conjectured that there exist higher dimensional quasiconformal maps which do not admit "minimal" factorizations. In this paper we will affirm Gehring's conjecture.

Suppose $\lambda > 0$ and let

$$s_\lambda(\rho, \theta) = (\rho,\ \theta + \lambda \log \rho) : \mathbb{R}^2 \to \mathbb{R}^2$$

be the logarithmic spiral mapping, where $(\rho,\ \theta)$ are the polar coordinates of $\mathbb{R}^2$ and log denotes natural logarithmic. Obviously s_λ has Betrami differential

$$\mu_\lambda(z) = \frac{i\lambda}{2 + i\lambda} \cdot \frac{z}{\bar{z}}.$$

Therefore s_λ has maximal dilatation

$$K = \frac{\sqrt{4 + \lambda^2} + \lambda}{\sqrt{4 + \lambda^2} - \lambda}. \tag{3}$$

We recall that a homeomorphism $s : U \to V$ of domains in $\mathbb{R}^n$ is an L-bi-lipschitz if $L \geq 1$ and

$$L^{-1}\,|z - z'| \leq |s(z) - s(z')| \leq L\,|z - z'|, \quad \forall\ z,\ z' \in U. \tag{4}$$

The smallest $L \geq 1$ for which (4) holds is called the isometric distortion of s.

By computing its Jacobian, we can easily check that s_λ is a bi-lipschitz homeomorphism with isometric distortion $\sqrt{K}$.

Furthermore, in [7] M. Freedman and the first author established that it requires at least $\lambda/\sqrt{L^2 - 1}$ factors to write s_λ into a composition of L-bi-lipschitz homeomorphisms. On the other hand, as noted in (2), the minimal factors of s_λ with conformal distortions $\leq L$ grows like $2 \log_L \lambda$ when λ is large. Thus for large λ, the number of factors with small isometric distortion needed to "unwind" the spiral map s_λ is much greater that the number of factors with the same conformal distortion.

Noting that $s_{-\lambda} = s_\lambda^{-1}$, we can handle the map $s_\lambda (\lambda < 0)$ by similar methods.

Let $f_{n,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 3)$ be a homeomorphism defined by

$$f_{n,\lambda}(z,\ t_1, \cdots,\ t_{n-2}) = (s_\lambda(z),\ t_1/\sqrt{K}, \cdots,\ t_{n-2}/\sqrt{K}). \tag{5}$$

where $z \in \mathbb{R}^2$ and $(t_1, \cdots,\ t_{n-2}) \in \mathbb{R}^{n-2}$. Obviously $f_{n,\lambda}$ is a quasiconformal homeomorphism with maximal distortion K.

Denote $\mathbb{T} = \{(z,\ t_1, \cdots,\ t_{n-2}) \in \mathbb{R}^n |\ z \equiv 0\}$. When $U_0 \subset \mathbb{R}^n$ is a domain with $U_0 \cap \mathbb{T} \neq \emptyset$, we have the following main result.

Theorem. The n-dimensional restriction quasiconformal map $f_{n,\lambda}|_{U_0} : U_0 \to V_0 = f_{n,\lambda}(U_0)$ admits no "minimal" factorizations. That is, for any given $0 < s < 1$, we have

$$f_{n,\lambda}|_{U_0} \neq f_2 \circ f_1,$$

where f_1 is a K^s-quasiconformal map and f_2 is a K^{1-s}-quasiconformal map.

Note that each L-bi-lipschitz homeomorphism is an L^2-quasiconformal homeomorphism. The above Theorem immediately implies that that $f_{n,\lambda}|_{U_0} \neq s_2 \circ s_1$, for any $K^{\frac{s}{2}}$-bi-Lipschitz map s_1 and $K^{\frac{1-s}{2}}$-bi-Lipschitz map s_2.

We note that the following problem is open even for $U = \mathbb{R}^n$.

Open Problem. (Gehring [10, 11]). Supposing that $1 < \widetilde{K} < K$, is there an n-dimensional $(n \geq 3)$ quasiconformal map $f : U \to V$ with maximal dilatation K and

$$f \neq f_1 \circ f_2 \circ \cdots \circ f_m,$$

for any $\widetilde{K}$-quasiconformal maps $f_i,\ 1 \leq i \leq m$?

There are examples show that the above open problem is almost certainly not true without further restrictions on the region U. See e.g. [7].

Notational conventions.

Through the paper, for any matrix A we denote by A^T the transpose of A.

The symbol $SO(n)$ will denote the n-dimensional special orthogonal group, i.e., $Q \in SO(n)$ if and only if $Q \cdot Q^T = I_n$ (the n-dimensional identity matrix) and with determinant $\det(Q) = 1$.

For $\lambda_i \in \mathbb{R}$ $(1 \leq i \leq n)$, we denote diag $(\lambda_1,\ \lambda_2, \cdots, \lambda_n)$ to be the diagonal matrix

$$\text{diag}\,(\lambda_1,\ \lambda_2, \cdots, \lambda_n) = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

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2. Basic Materials

A quasiconformal homeomorphism $f : U \to V$ possesses the following properties, see e.g. [24].

(1). f is A. C. L (Absolutely Continuous on Lines). Also it is differentiable with Jacobian $J_f(\zeta) > 0$ almost everywhere;

(2). For any measurable set $E \subset U$, the measure $m(E) = 0$ implies $m(f(E)) = 0$.

Suppose f is differentiable at $\zeta \in U$ with Jacobian $J_f(\zeta) > 0$. Take the normalized frame $\{e_1, e_2, \cdots, e_n\}$ in the tangent space $T_\zeta U \cong \mathbb{R}^n$, where

$$e_1 = (1, 0, \cdots, 0), \quad e_2 = (0, 1, \cdots, 0), \quad \cdots, \quad e_n = (0, 0, \cdots, 1).$$

Then we have

$$\begin{aligned} \left.\frac{\partial f}{\partial e_1}\right|_\zeta &= a_{11}\, e_1 + a_{21}\, e_2 + \cdots + a_{n1}\, e_n, \\ \left.\frac{\partial f}{\partial e_2}\right|_\zeta &= a_{12}\, e_1 + a_{22}\, e_2 + \cdots + a_{n2}\, e_n, \\ &\cdots \\ \left.\frac{\partial f}{\partial e_n}\right|_\zeta &= a_{1n}\, e_1 + a_{2n}\, e_2 + \cdots + a_{nn}\, e_n. \end{aligned}$$

Hence the Jacobian matrix of f at ζ is

$$Df(\zeta) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{with} \quad J_f(\zeta) = \det(Df(\zeta)) > 0.$$

The following result is well known. Its proof, included here for completeness, is elementary in linear algebra.

Lemma 2.1. If A is a $n \times n$ real matrix with determinant $\det(A) > 0$, then there exist $P,\ Q \in SO(n)$ such that

$$P \cdot A \cdot Q = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

Proof: Since the determinant $\det(A) > 0$, the symmetric matrix AA^T is positive definite. Therefore there exists $P \in SO(n)$ such that

$$P \cdot AA^T \cdot P^T = \text{diag}\,(\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Denote

$$Q = A^T \cdot P^T \cdot \text{diag}\,(\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_n^{-1}).$$

Then $Q^T \cdot Q = I_n$ (the $n \times n$ identity matrix). Consequently $P,\ Q \in SO(n)$ and

$$P \cdot A \cdot Q = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

as desired. $q.e.d.$

With respect to any n-dimensional quasiconformal homeomorphism $f : U \to V$, Lemma 2.1 implies that there exist $P(\zeta),\ Q(\zeta) \in SO(n)$ such that

$$Df(\zeta) = P(\zeta) \cdot \text{diag}\,\big(\lambda_1(\zeta), \lambda_2(\zeta), \cdots, \lambda_n(\zeta)\big) \cdot Q(\zeta), \quad a.e.\ \ \zeta \in U,$$

where $\lambda_1(\zeta) \geq \lambda_2(\zeta) \geq \cdots \geq \lambda_n(\zeta) > 0$. From the definition (1), it follows that f is K-quasiconformal if and only if

$$\frac{\lambda_1(\zeta)}{\lambda_n(\zeta)} \leq K, \quad a.e.\ \ \zeta \in U. \tag{6}$$

See e.g. Thm 34.6 in [24].

Hence an n-dimensional K-quasiconformal homeomorphism maps an infinitesimal $(n-1)$-sphere to an $(n-1)$-ellipsoid almost everywhere, with the ratio of the lengths of longest semi-axis and shortest semi-axis bounded from above by K.

We recall that 1-quasiconformal maps of plane domains are holomorphic homeomorphisms. For $n\ (\geq 3)$-dimensional 1-quasiconformal homeomorphisms we have the following generalized Liouville Theorem due to F. W. Gehring [9] and Yu. G. Reshetnyak [19]. This result involves no priori differentiability hypotheses.

Liouville Theorem. An $n\ (\geq 3)$-dimensional quasiconformal homeomorphism $f : U \to V$ is 1-quasiconformal if and only if f is the restriction to U of a Möbius transformation, i.e. the composition of even reflections in $(n-1)$-spheres or planes.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2 \backslash 0$ be a conformal covering map defined by

$$\pi(\tau,\ \theta) = (e^\tau \cos\theta,\ e^\tau \sin\theta)\,, \quad \forall\ \ \tau + i\,\theta \in \mathbb{R}^2. \tag{7}$$

Consider the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{l_\lambda} & \mathbb{R}^2 \\ \pi\downarrow & & \pi\downarrow \\ \mathbb{R}^2\backslash 0 & \xrightarrow{s_\lambda} & \mathbb{R}^2\backslash 0, \end{array}$$

where the homeomorphism $l_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $l_\lambda(\tau,\ \theta) = (\tau,\ \theta + \lambda\,\tau)$. Then we have

$$Ds_\lambda(z) = D\pi(l_\lambda(\pi^{-1}(z))) \cdot Dl_\lambda(\pi^{-1}(z)) \cdot D\pi^{-1}(z), \quad \forall\, z \in \mathbb{R}^2\backslash 0.$$

Therefore, if $(\rho,\ \theta)$ is the polar coordinate of $z \in \mathbb{R}^2\backslash 0$ and $\theta' = \theta + \lambda \log \rho$, we obtain that

$$\begin{aligned} Ds_\lambda(z) &= \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= A(\lambda,\theta') \cdot \text{diag}\,(\sqrt{K},\ 1/\sqrt{K}) \cdot B(\lambda,\theta), \end{aligned} \tag{8}$$

where $K = \dfrac{\sqrt{4+\lambda^2}+\lambda}{\sqrt{4+\lambda^2}-\lambda}$, as given in (3), and

$$A(\lambda,\theta') = \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} & -\sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} \\ \sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} & \sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} \end{pmatrix} \in SO(2),$$

and

$$B(\lambda,\theta) = \begin{pmatrix} \sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} & \sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} \\ -\sqrt{\frac{2}{\lambda^2+4+\lambda\sqrt{\lambda^2+4}}} & \sqrt{\frac{2}{\lambda^2+4-\lambda\sqrt{\lambda^2+4}}} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2).$$

3. Proof of the Theorem

Now we can begin the proof of the Theorem. We will prove this theorem for $n = 3$ and the general case is similar.

At first we assume that $(0,0,0) \in U_0$.

Suppose that $D_r \times (-l,l) \subset U_0$, where $D_r = \{|z| < r\}$ $(r > 0)$ and $l > 0$. Without loss of generality we assume that

$$U_0 = D_r \times (-l,l).$$

Let $f_{3,\lambda}$ be the quasiconformal map given in (5) (where $n = 3$). That is,

$$f_{3,\lambda}(z,\ t) = \left(s_\lambda(z),\ t/\sqrt{K}\right).$$

We assume, by contradiction, that

$$f_{3,\lambda}|_{U_0} = f_2 \circ f_1, \tag{9}$$

for some K^s-quasiconformal map f_1 and K^{1-s}-quasiconformal map f_2, where $0 < s < 1$.

Now we have the following result. Its proof will be postponed to Section 4.

Lemma 3.1. For almost all $\zeta = (z,\ t) \in U_0$, there exist $P(\zeta) \in SO(3)$ and $a_\zeta > 0$ such that

$$Df_1(\zeta) = a_\zeta \cdot P(\zeta) \cdot \begin{pmatrix} K^s & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} B(\lambda,\ \theta)_{2\times 2} & \\ & 1 \end{pmatrix},$$

where $B(\lambda,\ \theta) \in SO(2)$ is defined by (8).

Let us proceed with the proof of the Theorem.

Recall that the logarithmic spiral mapping s_λ has Betrami differential

$$\mu_\lambda(z) = \frac{i\lambda}{2+i\lambda} \cdot \frac{z}{\bar{z}}.$$

We set

$$0 < c \equiv \frac{K^s-1}{K^s+1} \cdot \frac{\sqrt{4+\lambda^2}}{\lambda} < 1. \tag{10}$$

Then we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{l_{c,\lambda}} & \mathbb{R}^2 \\ \pi\Big\downarrow & & \pi\Big\downarrow \\ \mathbb{R}^2\backslash 0 & \xrightarrow{s_{c,\lambda}} & \mathbb{R}^2\backslash 0, \end{array}$$

where $l_{c,\lambda} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$l_{c,\lambda}(\tau,\ \theta) = \left(\frac{1+\frac{(1-c^2)\lambda^2}{4}}{1+\frac{(1-c)^2\lambda^2}{4}} \cdot \tau, \quad \theta + \frac{c\lambda}{1+\frac{(1-c)^2\lambda^2}{4}} \cdot \tau \right), \quad \forall\ \tau + i\,\theta \in \mathbb{R}^2.$$

The map $s_{c,\lambda} : \mathbb{R}^2\backslash 0 \to \mathbb{R}^2\backslash 0$ is a quasiconformal homeomorphism with Beltrami differential $c \cdot \mu_\lambda$. Therefore $s_\lambda = \left(s_\lambda \circ s_{c,\lambda}^{-1}\right) \circ s_{c,\lambda}$ is a "minimal" factorization. That is, $s_{c,\lambda}$ is K^s-quasiconformal and $s_\lambda \circ$

$s_{c,\lambda}^{-1}$ is K^{1-s}-quasiconformal (please see Thm 4.7 in [15]). Moreover, by using the polar coordinate of $z \in \mathbb{R}^2 \backslash 0$, we have

$$Ds_{c,\lambda}(z) = k \cdot \rho^{\sigma} \cdot A_c(z) \cdot \begin{pmatrix} K^s & \\ & 1 \end{pmatrix} \cdot B(\lambda, \theta), \quad z \in \mathbb{R}^2 \backslash 0, \tag{11}$$

where $B(\lambda, \theta) \in SO(2)$ is defined in (8), $A_c(z) \in SO(2)$ and

$$\sigma = \frac{\frac{c(1-c)\lambda^2}{2}}{1 + \frac{(1-c)^2\lambda^2}{4}} > 0, \quad k = \frac{1 + \frac{(1-c^2)\lambda^2}{4}}{1 + \frac{(1-c)^2\lambda^2}{4}} \cdot \frac{\sqrt{4+\lambda^2} - c \cdot \lambda}{\sqrt{4+\lambda^2} + c \cdot \lambda}. \tag{12}$$

Please refer to Lemma 3.1.

Recall that $U_0 = D_r \times (-l, l)$. We set $D_r' \equiv D_r \backslash \{-r < x \le 0\}$ and denote

$$U_0' \equiv D_r' \times (-l, l) \subset \mathbb{R}^3.$$

Then the domain U_0' is simply connected. Namely, any closed curve in U_0' can be contracted to a point in U_0'.

With respect to $\sigma > 0$ defined in (12), we can always select a single-valued branch of the analytic function $\zeta^{\frac{1}{1+\sigma}}$ on the simply connected domain $s_{c,\lambda}(U_0')$. Therefore we have a map

$$F_1 \equiv \left((s_{c,\lambda})^{\frac{1}{1+\sigma}}, \ c_0 \cdot t \right) \Big|_{U_0'} : U_0' \to F_1(U_0'), \tag{13}$$

where

$$c_0 = \left(1 + \frac{(1-c)^2\lambda^2}{4} \right) \cdot \left(\frac{\sqrt{4+\lambda^2} - c\lambda}{\sqrt{4+\lambda^2} + c\lambda} \right) \cdot \left(1 + \frac{(1-c^2)\lambda^2}{4} \right)^{-1}.$$

It is obvious that $F_1 : U_0' \to F_1(U_0')$ is quasiconformal. Moreover, by (11) it follows that

(14)

$$DF_1(\zeta) = c_0 \cdot B_c(\zeta) \cdot \begin{pmatrix} K^s & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} B(\lambda, \ \theta)_{2\times 2} & \\ & 1 \end{pmatrix}, \ \ a.e. \ \ \zeta \in U_0',$$

where $B_c(\zeta) \in SO(3)$.

From Lemma 3.1 and (14) we deduce that

$$D(f_1|_{U_0'} \circ F_1^{-1})(\eta) = \lambda_\eta \cdot Q_\eta, \qquad a.e. \ \ \eta \in F_1(U_0'),$$

where $\lambda_\eta > 0$ and $Q_\eta \in SO(3)$. Therefore Liouville Theorem implies $f_1|_{U_0'} = \gamma \circ F_1$ for some Möbius transformation γ.

Choose the closed curve $C_r \equiv \{|z| = r/2\} \times \{0\} \subset U_0$. Then the open curve $C'_r \equiv C_r \backslash \{z = -r/2\} \subset U'_0$. Therefore

$$\gamma^{-1} \circ f_1|_{U'_0}(C'_r) = F_1(C'_r). \tag{15}$$

The left hand of (15) is an open curve by cutting off a point from the closed curve $\gamma^{-1} \circ f_1(C_r)$. By the definition of F_1, the right hand of (15) is an open circle arc spanning an angle $\dfrac{2\pi}{1+\sigma}$. This is a contradiction, which implies the quasiconformal map $f_{3,\lambda}|_{U_0} : U_0 \to V_0 = f_{n,\lambda}(U_0)$ admits no "minimal" factorizations.

When $(0,0,0) \notin U_0$ but $U_0 \cap \mathbb{T} \neq \emptyset$, we assume that $(0,0,t_0) \in U_0$. Let A be the conformal affine map defined by $A(z,\ t) = (z,\ t - t_0)$. By applying the same argument to the quasiconformal map

$$f_{3,\lambda} \circ A^{-1}|_{A(U_0)} : A(U_0) \to V_0,$$

we deduce that $f_{3,\lambda}|_{U_0}$ also admits no "minimal" factorizations.

Summing up the above cases, we conclude that the quasiconformal homeomorphism $f_{3,\lambda}|_{U_0}$ admits no "minimal" factorizations when $U_0 \cap \mathbb{T} \neq \emptyset$. *q.e.d.*

Note that a region $U_1 \subset \mathbb{R}^n$ is said to be convex if and only if $\zeta_1,\ \zeta_2 \in U_1$ implies that

$$\kappa \cdot \zeta_1 + (1-\kappa) \cdot \zeta_2 \in U_1, \quad 1 \leq \kappa \leq 1.$$

For any convex domain $U_1 \subset \mathbb{R}^n$ with $U_1 \cap \mathbb{T} = \emptyset$, by applying the similar argument as in (13), we obtain

Corollary. The n-dimensional K-quasiconformal map $f_{n,\lambda}|_{U_1}$ admits "minimal" factorizations. More precisely, for each $0 < s < 1$, there exist a K^s-quasiconformal map g_1 and a K^{1-s}-quasiconformal map g_2 such that

$$f_{3,\lambda}|_{U_1} = g_2 \circ g_1.$$

4. Proof of Lemma 3.1

Now we begin to prove Lemma 3.1

Lemma 2.1 shows that, for almost all $\zeta \in U_0$ and $\eta \in f_1(U_0)$, there exist

$$P_1(\zeta),\ Q_1(\zeta),\ P_2(\eta),\ Q_2(\eta) \in SO(3)$$

such that

$$\begin{aligned} Df_1(\zeta) &= P_1(\zeta)\cdot \mathrm{diag}\big(\mu_1(\zeta),\ \mu_2(\zeta),\ \mu_3(\zeta)\big)\cdot Q_1(\zeta),\\ Df_2(\eta) &= P_2(\eta)\cdot \mathrm{diag}\big(\gamma_1(\eta),\ \gamma_2(\eta),\ \gamma_3(\eta)\big)\cdot Q_2(\eta), \end{aligned}$$

with $\mu_1(\zeta)\ge\mu_2(\zeta)\ge\mu_3(\zeta)>0$ and $\gamma_1(\eta)\ge\gamma_2(\eta)\ge\gamma_3(\eta)>0$. By (6) it follows that

$$\frac{\mu_1(\zeta)}{\mu_3(\zeta)}\le K^s,\quad \frac{\gamma_1(\eta)}{\gamma_3(\eta)}\le K^{1-s},\quad a.e.$$

From $Df_{3,\lambda}(\zeta)=Df_2(f_1(\zeta))\cdot Df_1(\zeta)$, we deduce that

$$\begin{aligned} &\begin{pmatrix} \mathrm{A}(\lambda,\ \theta')_{2\times2} & \\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{K} & & \\ & 1/\sqrt{K} & \\ & & 1/\sqrt{K}\end{pmatrix}\begin{pmatrix} B(\lambda,\ \theta)_{2\times2} & \\ & 1\end{pmatrix}\\ =\ & P_2(\eta)\cdot\begin{pmatrix}\gamma_1(\eta) & & \\ & \gamma_2(\eta) & \\ & & \gamma_3(\eta)\end{pmatrix}\cdot Q_2(\eta)\cdot P_1(\zeta)\cdot\begin{pmatrix}\mu_1(\zeta) & & \\ & \mu_2(\zeta) & \\ & & \mu_3(\zeta)\end{pmatrix}\cdot Q_1(\zeta), \end{aligned}$$

where $\eta=f_1(\zeta)$. That is, for a.e. $\zeta\in U_0$,

$$\begin{aligned} &\begin{pmatrix}\sqrt{K} & & \\ & 1/\sqrt{K} & \\ & & 1/\sqrt{K}\end{pmatrix}\\ =\ & T_1(\zeta)\cdot\begin{pmatrix}\gamma_1(\eta) & & \\ & \gamma_2(\eta) & \\ & & \gamma_3(\eta)\end{pmatrix}\cdot T_2(\zeta)\cdot\begin{pmatrix}\mu_1(\zeta) & & \\ & \mu_2(\zeta) & \\ & & \mu_3(\zeta)\end{pmatrix}\cdot T_3(\zeta), \end{aligned} \tag{16}$$

where

$$T_1(\zeta)=\begin{pmatrix} \mathrm{A}(\lambda,\ \theta')_{2\times2} & \\ & 1\end{pmatrix}^{-1}\cdot P_2(f_1(\zeta))\in SO(3),$$

$T_2(\zeta)=Q_2(f_1(\zeta))\cdot P_1(\zeta)\in SO(3)$ and

$$T_3(\zeta)=Q_1(\zeta)\cdot\begin{pmatrix} B(\lambda,\ \theta)_{2\times2} & \\ & 1\end{pmatrix}^{-1}\in SO(3). \tag{17}$$

Now consider the actions of the left (right) matrix of (16) on the column vector $(1,0,0)^T$. By computing the Euclidean lengthes of the resulting column vectors, we immediately obtain that

$$\sqrt{K}\le\gamma_1(f_1(\zeta))\cdot\mu_1(\zeta),\quad a.e.\ \zeta\in U_0. \tag{18}$$

Similarly, by considering the actions on the column vector $(0,0,1)^T$, we have

$$1/\sqrt{K} \geq \gamma_3(f_1(\zeta)) \cdot \mu_3(\zeta), \quad a.e.\ \zeta \in U_0. \tag{19}$$

From (18) and (19) it follows that

$$K \leq \frac{\gamma_1(f_1(\zeta)) \cdot \mu_1(\zeta)}{\gamma_3(f_1(\zeta)) \cdot \mu_3(\zeta)} \leq K^s \cdot K^{1-s}, \quad a.e. \tag{20}$$

Hence all " $\leq$ " or " $\geq$ " in (18), (19) and (20) must be " $=$ ". Together with these facts, and by computing the determinants of the matrices in (16), we deduce that

$$\gamma_1(f_1(\zeta))/K^{1-s} = \gamma_2(f_1(\zeta)) = \gamma_3(f_1(\zeta)), \;\; \mu_1(\zeta)/K^s = \mu_2(\zeta) = \mu_3(\zeta). \quad \text{a.e.}$$

In addition we obtain that

(21)

$$T_j(\zeta) = \begin{pmatrix} 1 & \\ & R_j(\zeta)_{2\times 2} \end{pmatrix}, \quad \text{where } \; R_j(\zeta) \in SO(2), \;\; j = 1,2,3,$$

with $R_1(\zeta) \cdot R_2(\zeta) \cdot R_3(\zeta) = I_2$ (the 2×2 identity matrix). If setting

$$a_\zeta \equiv \mu_2(\zeta), \quad P(\zeta) \equiv P_1(\zeta) \cdot \begin{pmatrix} 1 & \\ & R_3(\zeta)_{2\times 2} \end{pmatrix},$$

and using (17) and (21), we conclude that, for almost all $\zeta \in U_0$,

$$\begin{aligned} Df_1(\zeta) &= P_1(\zeta) \cdot \begin{pmatrix} K^s \cdot \mu_2(\zeta) & & \\ & \mu_2(\zeta) & \\ & & \mu_2(\zeta) \end{pmatrix} \cdot Q_1(\zeta), \\ &= a_\zeta \cdot P(\zeta) \cdot \begin{pmatrix} K^s & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} B(\lambda,\ \theta)_{2\times 2} & \\ & 1 \end{pmatrix}, \end{aligned}$$

as desired. *q.e.d.*

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