Sharp L^2 extensions and multiplier ideal sheaves

周向宇

中国科学院数学研究所
中国科学院华罗庚数学重点实验室

纪念华先生

华罗庚数学讲座，2014年11月18日

L^2 extension and multiplier ideal sheaf

natural development of

✓ Oka's solutions of Levi problem, problems of Cousin I and II

✓ Bergman's work on Bergman kernels

✓ Oka - Cartan's global theory on Stein manifolds

✓ Hörmander's L^2 method for solving $\bar{\partial}$ equation

✓ Kodaira embedding theory (Kodaira vanishing, embedding theorems)

解析延拓

“analytic continuation is a fundamental phenomena in complex analysis”

- 从区域到更大的区域（形）的延拓;
  从子流形到母流形的延拓
- 单复变:
  导致黎曼面的产生、黎曼 ζ-函数的研究、单值化定理
- 多复变：导致了多复变作为一个专门研究方向的产生

- singularity of a psh φ: $\varphi(z) = -\infty$

  e.g., for $\varphi = c\ log(|f_1|^2 + \cdots + |f_k|^2)$ is psh, where $c > 0$
- relation between a complex analytic subset and a pluripolar set:

  $f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) = \varphi^{-1}(\infty)$
- psh with analytic singularities

- singular hermitian metric on a holomorphic line bundle:

  locally $e^{-\varphi}, \varphi \in L^1_{loc}$

  curvature $\Theta = i\partial\bar{\partial}\varphi$ in the sense of currents

  pseudoeffective line bundle: $\Theta \geq 0$, i.e., φ is psh.

- singular hermitian metric on a holomorphic line bundle:
  locally $e^{-\varphi}, \varphi \in L^1_{loc}$
  curvature $\Theta = i\partial\bar{\partial}\varphi$ in the sense of currents
  pseudoeffective line bundle: $\Theta \geq 0$, i.e., φ is psh.
- psh and positive closed (1,1)-current:
  φ is psh, then $i\partial\bar{\partial}\varphi$ is a positive closed (1,1)-current
  a positive closed (1,1)-current has locally a psh potential

equivalence of singularities

- φ is more singular than ψ, denoted by $\varphi \preccurlyeq \psi$: if $\varphi \leq \psi + O(1)$
- φ and ψ have equivalent singularities: if $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$

invariants:

- Lelong number: $v(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{ln|z-x|}$
- complex singularity exponent (log canonical threshold: lct) $c_x(\varphi) = sup\{c \geq 0 : \exp^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } x\}$, and μ is the Lebesgue measure on $\mathbb{C}^n$.
- multiplier ideal sheaf: $\{f = 0| \int |f|^2 e^{-\varphi} < \infty\} = \{e^{-\varphi} \text{ not locally integrable }\} \subset \{\varphi = -\infty\}$

Oka-Cartan's global theory on Stein manifolds

Given a coherent analytic sheaf on a Stein manifold,

✓ Theorem A. the stalk of the sheaf is generated by the global section of the sheaf;

✓ Theorem B. degree ≥ 1 cohomology groups of the Stein manifold with value in the sheaf vanish;

✓ Oka-Cartan extension theorem: Given a closed complex subvariety S in the Stein manifold M, then any holomorphic section f of a holomorphic vector bundle restricting on S can be holomorphically extended to a holomorphic section F on the Stein manifold M.

a philosophy

Which kind of complex manifolds is Stein?（Levi problem）

Which kind of complex manifolds is projective algebraic? (Kodaira embedding)

Philosophy behind Levi problem, Kodaira embedding theorem, Hömander's L^2 estimate:

construction of specified holomorphic objects(functions, sections) (problems in several complex variables and complex algebraic geometry) is reduced to the construction of specified plurisubharmonic functions (real-valued, may take $-\infty$) (problems in differential geometry)

relations and a natural question

- ✓ Theorem B implies theorem A (Siu);
- ✓ Theorem B is equivalent to Oka-Cartan extension theorem
- **A natural question:**

in the above Oka-Cartan extension theorem, if the holomorphic function or section is of a special property (say, invariant w.r.t. a group action, L^p, bounded or L^2), could the holomorphic extension be still of the same special property?

Theorem. **(Zhou, Proc. of ICM 2002)**

Let X be a Stein manifold, $G_{\mathbb{C}}$ be a connected complex reductive Lie group holomorphically acting on X, D be a $G_{\mathbb{R}}$-invariant Stein domain which is orbit connected and π-saturated, where $\pi : X \to X//G_{\mathbb{C}}$ is GIT (categorically) quotient. Then $G_{\mathbb{C}} \cdot D$ is Stein if and only if the invariant version of Oka-Cartan extension theorem holds.

✓ Let A be a $G_{\mathbb{R}}$-invariant analytic subset in D, f be a $G_{\mathbb{R}}$-invariant holomorphic function, then there exists a $G_{\mathbb{R}}$-invariant holomorphic function F on D which is a holomorphic extension of f.

L^2 extension problem

- **L^2 extension problem:** for a suitable pair (M, S), in the above Oka-Cartan extension theorem, if the function or section f is further L^2 on S, find an L^2 extension F on M together with a good or even optimal estimate?

  • see Demailly: Analytic methods in algebraic geometry, Higher Education Press, Beijing, 2010.

some known progress

✓ L^2 extension theorem: L^2 analytic continuation of L^2 holomorphic functions on analytic subvarieties to the Stein manifolds (Ohsawa-Takegoshi, Ohsawa, Siu, Berndtsson, Manivel, Demailly, McNeal-Varolin,...)

✓ many important applications in several complex variables and algebraic geometry

- Y.-T. Siu, Some recent transcendental techniques in algebraic and complex geometry, Proc. of ICM. 2002.
- J.P. Demailly, Kähler manifolds and transcendental techniques in algebraic geometry. Proc. of ICM. 2006.

L^2 extension problem with optimal estimate

- left a problem – L^2 extension problem with optimal estimate
- other proofs of Ohsawa-Takegoshi's theorem by Berndtsson, Demailly, Ohsawa, Y.-T. Siu etc, also obtained explicit good estimate
- our previous work (Guan-Zhou-Zhu, Comptes Rendus 2011, and J. Math. Pures Appl. 2012): introduced a different method from previous ones to solve the above problem (method of undetermined function via solving ODE to determine the function)

- **method of undetermined functions:**

✓take undetermined function for twist factor or weight,

✓do calculation according to twisted Bochner-Kodaira Identity or the method using complete Kähler metric,

✓get naturally extremal relation of the function, get suitably divided two sum terms,

✓naturally asking one term to be zero, obtain ODE and determine the functions by solving the ODE

- one key difference between our method and previous methods (Ohsawa-Takegoshi, Ohsawa, Siu, Berndtsson, Demailly,...): undetermined functions and ODE (explicit functions and calculus)

Lemma.(see [Demailly's book])

- (X, ω) is a Kähler manifold of dimension n with a Kähler metric ω,
- $\eta, g > 0$ are smooth functions on X,
- $\alpha \in \mathcal{D}(X, \Lambda^{n,q} T_X^* \otimes E)$ (space of smooth differential forms with values in E with compact support)
- $\bar{\partial}^*$ means the Hilbert adjoint operator of $\bar{\partial}$

Then we have

$$\|(\eta + g^{-1})^{\frac{1}{2}} \bar{\partial}^* \alpha\|^2_{\tilde{h}} + \|\eta^{\frac{1}{2}} \bar{\partial} \alpha\|^2_{\tilde{h}}$$
$$\geq \ll [\eta \sqrt{-1} \Theta_E - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega] \alpha, \alpha \gg_{\tilde{h}} .$$

optimal estimate of Ohsawa's paper III

Continuing our previous work, in the framework of Ohsawa series paper III on L^2 extension theorem, Guan and Zhou obtained the L^2 extension theorem with optimal estimate

- T. Ohsawa, On the extension of L^2 holomorphic functions. III. Negligible weights. Math. Z. 219 (1995), no. 2, 215–225.
- Qi'an Guan, Xiangyu Zhou: Optimal constant problem in the L^2 extension theorem, C. R. Acad. Sci. Paris, Ser. I 350 (2012), 753–756

Theorem 1. (Guan, Zhou)

- M Stein manifold of dimension n.
- $\varphi+\psi$, ψ: plurisubharmonic functions on M.
- w holomorphic function on M such that

$\sup_M(\psi+2\log|w|)\leq 0$ and dw not vanish identically on any branch of $w^{-1}(0)$.

- $S=w^{-1}(0)$ and $S_0=\{x\in H : dw(x)\neq 0\}$.

- Then $\exists$ a uniform constant $\mathbf{C} = 1$ s.t.,
- $\forall$ holomorphic $(n-1)$-form f on S_0 satisfying

$$c_{n-1} \int_{S_0} e^{-\varphi-\psi} f \wedge \bar{f} < \infty,$$

where $c_k = (-1)^{\frac{k(k-1)}{2}} i^k$ for $k \in \mathbb{Z}$,

- $\exists$ a holomorphic n-form F on M satisfying $F = dw \wedge \tilde{f}$ on S_0 with $\theta^* \tilde{f} = f$ and
-

$$c_n \int_X e^{-\varphi} F \wedge \bar{F} \le 2\mathbf{C}\pi c_{n-1} \int_{S_0} e^{-\varphi-\psi} f \wedge \bar{f},$$

where $\theta : S_0 \longrightarrow M$ is the inclusion map.

motivation of optimal estimate of Ohsawa's paper III

✓ Hörmander's solution of Bergman's conjecture:

boundary behavior of the Bergman kernel on bounded strictly pseudoconvex domain

- method of inscribed ball, L^2 existence theorem
- method of hyperplane section, L^2 extension theorem with negligible weight

✓ optimal estimate of L^2 extension theorem with negligible weight recovers Hörmander's solution of Bergman's conjecture

optimal estimate of Ohsawa's paper V

- in the framework of Ohsawa series paper V on L^2 extension theorem, Guan and Zhou obtained the L^2 extension theorem with optimal estimate
- as a corollary, give a proof of a conjecture of Ohsawa which is an extension of the Suita conjecture to high dimensional manifolds and high codimensional subvarieties

• T. Ohsawa, On the extension of L^2 holomorphic functions. V. Effects of generalization. Nagoya Math. J. 161 (2001), 1–21

• Qi'an Guan, Xiangyu Zhou: Generalized L^2 extension theorem and a conjecture of Ohsawa, C. R. Acad. Sci. Paris, Ser. I

Theorem 2. (Guan, Zhou)

Let (M, S) satisfy condition (ab), S be of pure codim k, h be a smooth metric on a holomorphic vector bundle E on M with rank r. Then, for any negative polar function Ψ on M such that $\Psi \in C^\infty(M \setminus S)$ and $\Theta_{he^{-\Psi}} \geq 0$ and $\Theta_{he^{-(1+\delta)\Psi}} \geq 0$ on $M \setminus S$ in the sense of Nakano,

$\exists$ a uniform constant $\mathbf{C} = 1$ s.t., $\forall$ holomorphic section f of $K_M \otimes E|_S$ on S satisfying

$$\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi] < \infty,$$

$\exists$ a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M |F|_h^2 dV_M \leq \mathbf{C}(1+\delta^{-1})\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi].$$

Especially, if Ψ is furthermore psh, $\exists$ a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M |F|_h^2 dV_M \leq \mathbf{C}\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi].$$

a pair (M, S) satisfying condition (ab)

includes

- M is a Stein manifold, and S is any closed complex subvariety of M;
- M is a complex projective algebraic manifold, and S is any closed complex subvariety of M;
- M is a projective family, and S is any closed complex subvariety of M.

polar function

a class of upper-semi-continuous function Ψ from M to the interval $[-\infty, A)$ where $A \in (-\infty, +\infty]$, such that

- $\Psi^{-1}(-\infty) \supset S$, and $\Psi^{-1}(-\infty)$ is a closed subset of M;
- If S is $l-$dimensional around a point $x \in S_{reg}$, there exists a local coordinate $(z_1, \cdots, z_n)$ on a neighborhood U of x such that $z_{l+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} \left|\Psi(z) - (n-l)\log\sum_{l+1}^{n}|z_j|^2\right| < \infty.$$

associated measure on subvariety

For each polar function Ψ, Ohsawa introduced a positive measure $dV_M[\Psi]$ on S as the minimum element of the partial ordered set of positive measures $d\mu$ satisfying

$$\int_{S_l} f d\mu \geq \limsup_{t\to\infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_M$$

for any nonnegative continuous function f with compact support on M.

a solution of a conjecture of Ohsawa

- Let $G(\cdot, S)$ be the nontrivial generalized pluricomplex Green function. Let dV_M be a continuous volume form on M and let $\{\sigma_j\}_{j=1}^{\infty}$ (resp. $\{\tau_j\}_{j=1}^{\infty}$) be a complete orthogonal system of $A^2(M, K_M, dV_M^{-1}, dV_M)$ and $A^2(S, K_M|_S, dV_M^{-1}, dV_M[G(\cdot, S)])$
- denote by $\kappa_M = \sum_{j=1}^{\infty} \sigma_j \otimes \bar{\sigma}_j \in C^{\omega}(M, K_M \otimes \bar{K}_M)$ (resp. $\kappa_{M/S} = \sum_{j=1}^{\infty} \tau_j \otimes \bar{\tau}_j \in C^{\omega}(S, K_M \otimes \bar{K}_M)$.)
- **A conjecture of Ohsawa**: *$\kappa_{M/S}(x) \leq (\pi^k/k!)\kappa_M(x)$ for any $x \in S_{n-k}$.*
- **Corollary 1. (Guan, Zhou)**

  The above conjecture of Ohsawa holds.

a general setting

• Qi'an Guan, Xiangyu Zhou: A solution of an L^2 extension problem with optimal estimate and applications. published online, Ann. of Math., 2014.

▶ Let h be a smooth metric on a holomorphic vector bundle E on M with rank r which is semi-positive in the sense of Nakano.

▶ an hermitian metric h on E is said to be semi-positive in the sense of Nakano if the Chern curvature tensor Θ_h associated to the Chern connection is semi-positive definite as an hermitian form on $T_X \otimes E$, i.e. if for every $u \in T_X \otimes E$, $u \neq 0$, we have $\Theta_h(u,u) \geq 0$

a general sharp L^2 extension theorem

Theorem 3. (Guan, Zhou: Ann. of Math.)

There exists a uniform constant $\mathbf{C} = 1$ such that, for any holomorphic section f of $K_M \otimes E|_S$ on S of pure codimension k satisfying

$$\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi] < \infty,$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M c_A(-\Psi)|F|_h^2 dV_M \leq \mathbf{C}(\int_{-A}^{\infty} c_A(t)e^{-t}dt)(\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi]).$$

Let $c_A(t)$ be a positive smooth function on $(-A, +\infty)$ $(A \in (-\infty, +\infty])$ satisfying

- $\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$, and
- $$(\int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1)^2 > c_A(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1dt_2,$$

for any $t \in (-A, +\infty)$.

- It's easy to see that when $c_A(t)e^{-t}$ is decreasing with respect to t and A is finite, the above inequality holds.

- Inequality part of Suita's conjecture in 1972: on any open Riemann surface Ω as above, $(c_\beta(z))^2|dz|^2 \leq \pi\kappa_\Omega(z)$ (Blocki; Guan-Zhou).
- logarithmic capacity $c_\beta(z)$: locally defined by

$$c_\beta(z) := \exp \lim_{\xi \to z} (G_\Omega(\xi, z) - \log|\xi - z|)$$

a complete solution of a conjecture of Suita

In the same paper, Suita also posed the following:

- **A conjecture of Suita, 1972**: $(c_\beta(z))^2|dz|^2 = \pi\kappa_\Omega(z)$ *for* $z \in \Omega$ *if and only if* Ω *is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.*

- **Corollary 2. (Guan, Zhou: Ann. of Math.)**

  The above conjecture of Suita holds.

a relation to Berndtsson's theorem

- M a pseudoconvex domain in $\mathbb{C}^{n+m}$ with coordinate (z,t), Y is a domain in $\mathbb{C}^m$ with coordinate t,

$$p(z,t)=t;$$

- M is a projective manifold, and Y is a complex manifold, and p is a fibration.
- Let e be the local frame of L. Let κ_{M_t} be the Bergman kernel of $K_{M_t}\otimes L$ on M_t, and $\kappa_{M_t}:=B_t(z)dz\otimes e\otimes d\bar{z}\otimes\bar{e}$ locally

▶ We found in our above Annals paper that our L^2 extension theorem 3 with optimal estimate implies

Theorem (Berndtsson). $\log B_t(z)$ is a plurisubharmonic function with respect to (z, t).

▶ one dimensional case due to Maitani-Yamaguchi

- B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, Annals of Math, 169 (2009), 531–560.
- B. Berndtsson and M. Păun, Bergman kernels and the pseudoeffectivity of relative canonical bundles, Duke Math. J. 145 (2008), 341–378

other applications:

- obtain the L^p $(0 < p \leq 2)$ extension theorem with optimal estimates for line bundle twisted pluricanonical extension
- interpolating submanifolds in Bargmann-Fock spaces
- answer a question of Ohsawa

• T. Ohsawa, On the extension of L^2 holomorphic functions. VI. a limiting case, Contemporary Mathematics, Vol 332, (2003), 235–239.

By taking different $c_A(t)$, obtain optimal estimate versions of various well known L^2 extension theorems due to Ohsawa, Siu, Manivel, Demailly, Berndtsson, McNeal -Varolin, Demailly-Hacon-Păun,...

• Ohsawa, On the extension of L^2 holomorphic functions. II. Publ. Res. Inst. Math. Sci. 24 (1988), no. 2, 265–275.

• Demailly-Hacon-Păun L^2 extension theorem, in Extension theorems, non-vanishing and the existence of good minimal models, Acta Mathematica 2013.

Theorem (Zhu, Zhou, 2013)

for weakly pseudoconvex Kähler manifolds, in the setting of Manivel-Demailly,

optimal estimate for L^2 extension holds

multiplier idea sheaf

- Definition (Nadel 1990 Ann. of Math.): to a plurisubharmonic function φ, is associated an ideal subsheaf $\mathcal{I}(\varphi)$ of $\mathcal{O}$: germs of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 e^{-2\varphi}$ is locally integrable.
    - origin goes back to Hörmander, Bombieri, Skoda
- Nadel theorem: $\mathcal{I}(\varphi)$ is coherent
- Nadel vanishing theorem

- unified treatment:

▲ algebraic geometry: Kodaira vanishing theorem, Kodaira embedding theorem, Kawamata-Vieweg vanishing theorem

▲ several complex variables: Oka-Cartan extension theorem, Grauert's solution of Levi problem for complex manifolds, Cousin problems I and II

Strong openness conjecture

Denote by

$$\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi).$$

Strong openness conjecture of Demailly: For any plurisubharmonic function φ on X, one has $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$.

- Under the assumption that $e^{-2\varphi}$ is locally integrable, proved by Berndtsson arXiv:1305.5781
- *dimX* < 3, proved by Favre, Jonsson

C. Favre and M. Jonsson, Invent. Math. 2005;

C. Favre and M. Jonsson, J. Amer. Math. Soc. 2005

a solution of the strong openness conjecture

Theorem 4. (Guan, Zhou) The strong openness conjecture of Demailly holds.

- Q.A. Guan, X.Y. Zhou, Strong openness conjecture for plurisubharmonic functions, arXiv:1311.3781.
- Q.A. Guan and X.Y. Zhou, Strong openness conjecture and related problems for plurisubharmonic functions, arXiv:1401.7158.
- Q.A. Guan and X.Y. Zhou, Effectiveness of Demailly's strong openness conjecture and related problems, arXiv:1403.7247.

a remark on the strong openness conjecture

Let $\{\psi_j\}_{j=1,2,\cdots}$ be a sequence of plurisubharmonic functions on Δ^n, which is increasingly convergent to φ on Δ^n, when $j \to \infty$.

The same proof produces the following:

Corollary. **(Guan, Zhou)** One has $\cup_{j=1}^{\infty}\mathcal{I}(\psi_j) = \mathcal{I}(\varphi)$.

In particular, let $\psi_j = \varphi + \frac{1}{j}\varphi_0$, then we solve Kim's modified version of the strong openness conjecture in

• D. Kim, The exactness of a general Skoda complex, arXiv:1007.0551.

Corollary. **(Guan, Zhou)** Let φ be a negative plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and $\varphi_0 \not\equiv -\infty$ be a negative plurisubharmonic function on Δ^n. Then

$\cup_{\varepsilon>0}\mathcal{I}(\varphi + \varepsilon\varphi_0) = \mathcal{I}(\varphi)$.

main idea of the proof of the SOC

✓ a use movably of Ohsawa-Takegoshi's L^2 extension theorem.

✓ induction method

- Since $\mathcal{I}_+(\varphi) \subset \mathcal{I}(\varphi)$, sufficient to prove $\mathcal{I}(\varphi) \subset \mathcal{I}_+(\varphi)$.

• Let $F \in \mathcal{I}(\varphi)$. Take a movable hyperplane close to the origin. Using Ohsawa-Takegoshi L^2 extension theorem, one can extend the restricted function of F on the movable hyperplane, together with an upper bound which is variable for the L^2 norm of the extended holomorphic function.

• If the conjecture were not true, the extended function would have been divided by F (at least along an analytic curve).

• By the induction method, we observe that the norm of such extended function has a lower bound which is also variable. And the ratio of the lower bound and upper bound would have gone to infinity, this is a contradiction.

consequences on the strong openness conjecture

Corollary A: $\{\mathcal{I}(\varphi)\} = \{\mathcal{I}(\varphi_A)\}$, where φ is a p.s.h. function and φ_A is a p.s.h. function with analytic singularities.

together with Demailly's equi-singular approximation theorem

Corollary B: Vanishing theorem of Kawamata - Vieweg - Nadel type holds on compact Kähler manifolds.

Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n, and $nd(L, \varphi)$ be the numerical dimension of (L, φ).

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $q \geq n - nd(L, \varphi) + 1$.

together with Demailly-Peternell's and Cao's results

Corollary C: relation between invariants for psh singularities

- $\mathcal{I}(c\varphi) = \mathcal{I}(c\psi)$, for any $c > 0$
- Lelong numbers up to proper modifications are the same: for all proper modifications $\pi : X \to \mathbb{C}^n$ above 0 and all points $p \in \pi^{-1}(0)$, we have $v(\varphi \circ \pi, p) = v(\psi \circ \pi, p)$ together with Boucksom-Favre-Jonsson's result

Demailly and Kollár posed the following conjecture:

Conjecture D-K: Let φ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and K be compact subset of Δ^n. If $c_K(\varphi) < +\infty$, then

$$\frac{1}{r^{2c_K(\varphi)}}\mu(\{\varphi < \log r\})$$

has a uniform positive lower bound independent of $r \in (0,1)$, where $c_K(\varphi) = sup\{c \geq 0 : \exp^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } K\}$, and μ is the Lebesgue measure on $\mathbb{C}^n$.

The above conjecture implies the openness conjecture.

Theorem 4. **(Guan, Zhou)** Let φ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. Let F be a holomorphic function on Δ^n. Assume that $|F|^2 e^{-\varphi}$ is not locally integrable near o, Then

$$\int_{\Delta^n} \mathbb{I}_{\{-(R+1)<\varphi<-R\}} |F|^2 e^{-\varphi} d\lambda_n$$

has a uniform positive lower bound independent of $R >> 0$. Especially, if $F = 1$, then

$$e^R \mu(\{-(R+1) < \varphi < -R\})$$

has a uniform positive lower bound independent of $R >> 0$.

Corollary. The above conjecture of Demailly-Kollár is true

Let I be an ideal of $\mathcal{O}_{\Delta^n,o}$, which is generated by $\{f_j\}_{j=1,\cdots,l}$.

Jonsson and Mustată posed the following conjecture for level sets of plurisubharmonic functions:

• M. Jonsson and M. Mustată, An algebraic approach to the openness conjecture of Demailly and Kollár, J. Inst. Math. Jussieu (2013), 1–26.

Conjecture J-M: Let ψ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. If $c_o^I(\psi) < +\infty$, then

$$\frac{1}{r^{2c_o^I(\psi)}} \mu(\{c_o^I(\psi)\psi - \log|I| < \log r\})$$

has a uniform positive lower bound independent of $r \in (0,1)$, where

$$\log|I| := \log \max_{1 \le j \le l} |f_j|,$$

$c_o^I(\psi) = sup\{c \ge 0 : |I|^2 e^{-2c\psi}$ is L^1 on a neighborhood of $o\}$ is the jumping number in the above paper, and μ is the Lebesgue measure on $\mathbb{C}^n$.

For $n \leq 2$, the above conjecture was proved by Jonsson and Mustată.

• M. Jonsson and M. Mustată, Valuations and asymptotic invariants for sequences of ideals, Annales de l'Institut Fourier A. 2012, vol. 62, no.6, pp. 2145–2209.

The above conjecture implies the strong openness conjecture.

Theorem 5. *(Guan, Zhou)* The above conjecture of Jonsson-Mustată holds.

Thank You!