Gromov-Witten invariants and Algebraic Geometry (II)

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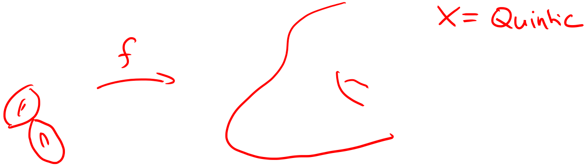
GW invariants of quintic Calabi-Yau threefolds

Quintic Calabi-Yau threefolds:

$$X = \{\mathbf{w}_5 = x_1^5 + \cdots + x_5^5 = 0\} \subset \mathbf{P}^4$$

For d, $g \in \mathbb{Z}$, form the moduli of stable maps

$$\overline{M}_g(X, d) = \{[f, C] \mid f : C \to X, \text{such that } ....\}$$



Form virtual cycle

$$[\overline{M}_g(X,d)]^{virt} \in A_0\overline{M}_g(X,d)$$

The GW invariant

$$N_g(d) = \int_{[\overline{M}_g(X,d)]^{virt}} 1 \in \mathbb{Q}.$$

The generating function

$$f_g(q) = \sum N_g(d) q^d$$

- Determining it is a challenge to mathematicians

Recent progress toward

an effective algorithm for all genus invariants

using Mixed-Spin-P (MSP) fields.

A joint work with Huailiang Chang, Weiping Li, and Mellisa Liu.

This work is inspired by Witten's vision that

GW invariants of quintics

and

Witten's spin class invariants

are equivalent via a wall crossing.

Witten's vision

- $\mathbb{C}^*$ acts on $\mathbb{C}^5 \times \mathbb{C}$ of weight $(1, 1, 1, 1, 1, -5)$;
- $(x_1^5 + \cdots + x_5^5) \cdot p : \mathbb{C}^5 \times \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}^*$ equivariant;
- the quotient $\mathbb{C}^5 \times \mathbb{C}/\mathbb{C}^*$ is pretty bad;

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- the quotient $\mathbb{C}^5 \times \mathbb{C}/\mathbb{C}^*$ is pretty bad;

$\exists \; w \in \mathbb{C}^5 \times \mathbb{C}$

s.t. $\mathrm{stab}_{\mathbb{C}^*}(w)$ infinite.

Witten's vision

$[\mathbb{C}^6/\mathbb{C}^*]$ has two GIT quotients:

- $(\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^* = K_{\mathbb{P}^4}$;
- $\mathbb{C}^5 \times (\mathbb{C} - 0)/\mathbb{C}^* = \mathbb{C}^5/\mathbb{Z}_5$;
- we call $(\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*$ and $\mathbb{C}^5 \times (\mathbb{C} - 0)/\mathbb{C}^*$ related by a simple wall crossing.

$(\mathbb{C}^5 \setminus 0)/\mathbb{C}^* = \mathbb{P}^4$, $\quad (\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*$

$\downarrow$ a line bundle

$(\mathbb{C}^5 - 0)/\mathbb{C}^* = \mathbb{P}^4$

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$\mathbb{C}^* \circlearrowright \mathbb{C} \setminus 0$ has weight -5

stab $= \mathbb{Z}_5$

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Wall crossing $\approx$ differ by a low dimensional subsets.

$(\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*$ $\qquad$ $\mathbb{C}^5 \times (\mathbb{C} \setminus 0)/\mathbb{C}^*$

$[\mathbb{C}^5 \times \mathbb{C}/\mathbb{C}^*]$ Artin stack

Witten's vision

Witten:

- a field theory valued in $K_{\mathbb{P}^4}$ is the GW of quintics;
- a field theory valued in $\mathbb{C}^5/\mathbb{Z}_5$ is the Witten's spin class (FJRW invariants);
- these two theories are equivalent via a wall crossing.

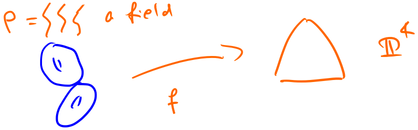
- developed a (MSP) field theory realizing this wall crossing,
- an algorithm, conjecturally determine all genus invariants.

One side of this wall crossing: LG theory of $K_{\mathbf{P}^4}$

(with HL Chang) We constructed the GW invariants of stable maps with p-fields:

- $\overline{M}_g(\mathbf{P}^4, d)^p = \{[f, C, \rho] \mid [f, C] \in \overline{M}_g(\mathbf{P}^4, d),$
  $\rho \in H^0(C, f^*\mathcal{O}(5) \otimes \omega_C)\}$
- form its virtual cycle $[\overline{M}_g(\mathbf{P}^4, d)^p]^{virt}_{loc}$
- define $N_g(d)^p = \int_{[\overline{M}_g(\mathbf{P}^4,d)^p]^{virt}_{loc}} 1 \in \mathbb{Q}$

Not $f : C \to X$ quintic.



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LG theory of $K_{\mathbb{P}^4}$

Theorem (Chang - L)

The two sets of invariants are equivalent

$$N_g(d) = (-1)^{d+g+1} N_g(d)^p.$$

Up shot:

- $N_g(d)$ are virtual counting of maps to the quintic X;
  - counting $[f : C \to X \subset \mathbb{P}^4]$
- $N_g(d)^p$ is a virtual counting of fields on curves:

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  $$([f : \mathcal{C} \to \mathbb{P}^4], \rho \in H^0(\mathcal{C}, f^*\mathscr{O}(5) \otimes \omega_{\mathcal{C}}));$$
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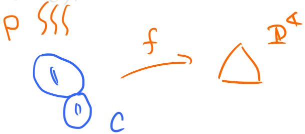

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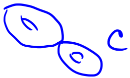
$\{\{\{ \varphi_1, \dots, \varphi_5$

$f = [\varphi_1, \dots, \varphi_5] : C \to \mathbb{P}^4$ or



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$\varphi_1, \cdots, \varphi_5, \rho$ 6 fields

$\mathcal{C}$

LG theory of $K_{\mathbb{P}^4}$

$N_g(d)^p$ is a virtual counting of fields because it v. counts

$$(\mathcal{C}, \mathcal{L}, \varphi_1, \cdots, \varphi_5, \rho);$$

they are fields taking values in $K_{\mathbb{P}^4} = (\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*$ because

- $\varphi_1 \in H^0(\mathcal{L})$ and $\rho \in H^0(\mathcal{L}^{-5} \otimes \omega_{\mathcal{C}})$, (compare) $\mathbb{C}^*$ acts on $\mathbb{C}^5$ and $\mathbb{C}$ of weights 1 and -5;
- $(\varphi_1, \cdots, \varphi_5)$ never zero and ρ arbitrary, (compare) $(\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*$;
- the line bundle $\mathcal{L}$ is up to scaling, (compare) quotient by $\mathbb{C}^*$.

$N_g(d)^p$ is a field theory taking values in $K_{\mathbb{P}^4} = (\mathbb{C}^5 - 0) \times \mathbb{C}/\mathbb{C}^*$.

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The other side wall crossing: LG theory of $\mathbb{C}^5/\mathbb{Z}_5$

- It originated by Witten's class;
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- $\varphi_i \in H^0(\mathcal{C}, \mathcal{L})$, $\rho \in H^0(\mathcal{C}, \mathcal{L}^{-5} \otimes \omega_{\mathcal{C}}(\Sigma^{\mathcal{C}}))$
- φ_i arbitrary, ρ nowhere vanishing.
  - (compare) $\mathbb{C}^5/\mathbb{Z}_5 = \mathbb{C}^5 \times (\mathbb{C} - 0)/\mathbb{C}^*$.

$\Sigma^{\mathcal{C}}$ are marked points,
$\mathcal{C}$ twisted curves.

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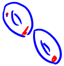
Theorem (Chang - Li - L)

The FJRW invariants can be constructed using cosection localized virtual cycles of the moduli of spin fields:

$$\overline{M}_{g,\gamma}(\mathbf{w}_5, \mathbb{Z}_5)^{5p} = \{ \big(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \varphi_1, \cdots, \varphi_5, \rho\big) \,|\, \dots \} / \sim$$

555 fields $\varphi_1, \dots, \varphi_5, \rho$. 6 fields



Cosection technique

- The construction of the two theories
  1. *the GW invariants of stable maps with p-fields*
  2. *the FJRW invariants of* $(\mathbf{w}_5, \mathbb{Z}_5)$

  both rely on the construction of cosection localized virtual cyels;

Theorem (Kiem - L)

A DM stack M with a perfect obstruction theory, and a cosection $\sigma : Ob_M \to \mathcal{O}_M$ provides us a cosection localized virtual cycle (letting $D(\sigma) = \{\sigma = 0\}$)

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Remark

1. The cosection localized virtual cycles allows one to construct invariants of non-compact moduli spaces;
2. The cosections used in the GW with p-fields and FJRW are induced by the same equivariant LG function

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Cosection technique

The fields: $\xi = (\mathcal{C}, \mathcal{L}, \varphi_1, \cdots, \varphi_5, \rho) \in H^0(\mathcal{L})^{\oplus 5} \oplus H^0(\mathcal{L}^{\otimes -5} \otimes \omega_{\mathcal{C}})$

The rel-obstruction space at ξ:

$$(\dot\varphi, \dot\rho) \in Ob|_\xi = H^1(\mathcal{L})^{\oplus 5} \oplus H^1(\mathcal{L}^{\otimes -5} \otimes \omega_{\mathcal{C}})$$

The cosection $\sigma|_\xi : Ob|_\xi \longrightarrow \mathbb{C}$:

$$\sigma|_\xi(\dot\varphi, \dot\rho) = \dot\rho \sum x_i^5 + \rho \sum 5\varphi_i^4 \cdot \dot\varphi_i \in H^1(\omega_{\mathcal{C}}) \cong \mathbb{C}.$$

Compare with

$$\delta\big(p \cdot (x_1^5 + \cdots + x_5^5)\big) = \dot\rho \cdot \sum x_i^5 + \rho \sum 5x_i^4 \cdot \dot x_i$$

Next step is to geometrically realizing the
wall crossing
of these two field theories envisioned by Witten

We define

$$\text{An MSP field} = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi_1, \cdots, \varphi_5, \rho, \nu_1, \nu_2)$$

Mixed Spin-P fields

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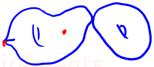
$$\text{An MSP field} = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi_1, \cdots, \varphi_5, \rho, \nu_1, \nu_2)$$

where

1. $(\Sigma^{\mathcal{C}}, \mathcal{C})$ is a pointed twisted curve,
2. $\mathcal{L}$ and $\mathcal{N}$ are line bundles, $\mathcal{L}$ as before, $\mathcal{N}$ is new;
3. $\varphi_i \in H^0(\mathcal{C}, \mathcal{L})$, $\rho \in H^0(\mathcal{C}, \mathcal{L}^{\otimes -5} \otimes \omega_{\mathcal{C}}^{\log})$, as before;
4. $\nu_1 \in H^0(\mathcal{L} \otimes \mathcal{N})$, $\nu_2 \in H^0(\mathcal{N})$;
5. plus combined GIT like stability requirements.

$\Sigma^{\mathcal{C}} \subset \mathcal{C}$



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Moduli of MSP fields

Theorem

The moduli $\mathcal{W}_{g,\gamma,d}$ of stable MSP-fields of

1. genus $g = g(C)$;
2. monodromy $\gamma = (\gamma_1, \cdots, \gamma_\ell)$ of $\mathcal{L}$ along $\Sigma^{\mathcal{C}}$, and
3. degrees $d = (d_0, d_\infty)$ (of $\mathcal{L}$ and $\mathcal{N}$)

is a separated DM stack, locally of finite type.

Moduli of MSP fields

Theorem

The moduli $\mathcal{W}_{g,\gamma,d}$ is a $\mathbb{C}^*$ stack, via

$$(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)^t = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu^t)$$

where $\nu^t = (t\nu_1, \nu_2)$.

Moduli of MSP fields

Theorem

The moduli $\mathcal{W}_{g,\gamma,d}$ has a $\mathbb{C}^*$ equivariant perfect obstruction theory, an equivariant cosection of its obstruction sheaf, thus an equivariant cosection localized virtual cycle

$$[\mathcal{W}_{g,\gamma,d}]^{virt}_{loc} \in A^{\mathbb{C}^*}_* \mathcal{W}^-_{g,\gamma,d}.$$

where $\mathcal{W}^-_{g,\gamma,d} = (\sigma = 0)$.

- A technical Lemma: $(\sigma = 0)$ is compact.

Polynomial relations

How to play with this cycle

$$[\mathcal{W}_{g,\gamma,d}]^{virt}_{loc} \in A^{\mathbb{C}^*}_* \mathcal{W}^-_{g,\gamma,d}$$

equivariant class.

Taking

1. $\gamma = \emptyset$ (no marked points),
2. $(d_0, d_\infty) = (d, 0)$,

then

$$[\mathcal{W}_{g,d}]^{virt}_{\sigma} \in H^{\mathbb{C}^*}_{2(d+1-g)}(\mathcal{W}^-_{g,d}, \mathbb{Q}).$$

Polynomial relations

How to play with this cycle

$$[\mathcal{W}_{g,\gamma,d}]^{virt}_{loc} \in A^{\mathbb{C}^*}_* \mathcal{W}^-_{g,\gamma,d}$$

Taking

1. $\gamma = \emptyset$ (no marked points),
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$$\left[\mathcal{W}_{g,d}\right]^{virt}_{\sigma} \in H^{\mathbb{C}^*}_{2(d+1-g)}(\mathcal{W}^-_{g,d}, \mathbb{Q}).$$

Polynomial relations

$$\left[\mathcal{W}_{g,d}\right]^{virt}_{\sigma} \in H^{\mathbb{C}^*}_{2(d+1-g)}(\mathcal{W}^{-}_{g,d}, \mathbb{Q}).$$

when $d + 1 - g > 0$

$$\left(u^{d+1-g} \cdot \left[\mathcal{W}_{g,d}\right]^{virt}_{\sigma}\right)_0 = 0.$$

Polynomial relations

Let F_Γ be the connected components of $\left(\mathcal{W}^-_{g,d}\right)^{\mathbb{C}^*}$;

$$\sum_\Gamma \left[u^{d+1-g} \cdot \frac{[F_\Gamma]^{virt}_{\sigma_\Gamma}}{e(N_{F_\Gamma})} \right]_0 = 0.$$

- for cosection localized version, proved by Chang-Kiem-L.

Polynomial relations

$$\sum_{\Gamma} \left[u^{d+1-g} \cdot \frac{[F_\Gamma]^{virt}_{\sigma_\Gamma}}{e(N_{F_\Gamma})} \right]_0 = 0.$$

is a polynomial relation among (after proving a vanishing result),

1. GW invariants of the quintic Calabi-Yau $N_g(d)$;
2. FJRW invariants of $(\mathbf{w}_5, \mathbb{Z}_5)$ with insertions $-\frac{2}{5}$;
3. Hodge integrals of $\overline{M}_{g',n'}$ involving ψ classes (calculable).

Polynomial relations

Application I

Letting $d_\infty = 0$, the relations provide an effective algorithm to evaluate the GW invariants $N_g(d)$ provided the following are known

1. FJRW invariants of insertions $-\frac{2}{5}$ and genus $g' \le g$;
2. $N_{g'}(d')$ for (g', d') such that $g' < g$, and $d' \le d$;
3. $N_g(d')$ for $d' \le g$.

Polynomial relations

Application II

Letting $d_0 = 0$, the relations provide an relations indexed by $d_\infty > g - 1$ among FJRW invariants with insertions $-\frac{2}{5}$.

Forward looking

Conjecture

These relations, indexed by (d_0, d_∞) (with $d_0 + d_\infty + 1 - g > 0$), provide an effective algorithm to determine all genus GW invariants and FJRW invariants of insertions $-\frac{2}{5}$.

Example III: GW technique to AG

Conjecture: Any smooth projective complex K3 surface S contains infinitely many rational curves.

This is motivated by Lang's conjecture:

Lang Conjecture: Let X be a general type complex manifold. Then the union of the images of holomorphic $u : \mathbb{C} \to X$ lies in a finite union of proper subvarieties of X.

Example III: GW technique to AG

Key to the existence of rational curves:
A class $\alpha \neq 0 \in H^2(S,\mathbb{Z})$ is Hodge (i.e. $\in H^{1,1}(S,\mathbb{C}) \cap H^2(S,\mathbb{Q})$) is necessary and sufficient for the existence of a union of rational curves C_i so that $\sum [C_i] = \alpha$.

Example: Say we can have a family S_t, $t \in$ disk,

- $\alpha \in H^{1,1}(S_0,\mathbb{C}) \cap H^2(S_0,\mathbb{Q})$ so that S_0 has $C_0 \cong \mathbb{CP}^1 \subset S_0$ with $[C_0] = \alpha$;
- in case $\alpha \notin H^{1,1}(S_t,\mathbb{C})$ for general t, then $\mathbb{CP}^1 \cong C_0 \to S_0$ can not be extended to holomorphic $u_t : \mathbb{CP}^1 \to S_t$.

Example III: GW technique to AG

- We will consider polarized K3 surfaces (S, H), $c_1(H) > 0$;
- we can group them according to $H^2 = 2d$:

$$\mathcal{M}_{2d} = \{(S, H) \mid H^2 = 2d\}.$$

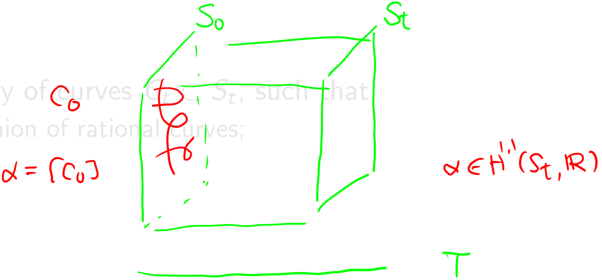
- each $\mathcal{M}_{2d}$ is smooth, of dimension 19;
- each $\mathcal{M}_{2d}$ is defined over $\mathbb{Z}$. (defined by equation with coefficients in $\mathbb{Z}$.)
- to show that (S, H) contains infinitely many rational curves, it suffices to show that
  - for any N, there is a rational curve $R \subset S$ so that $[R] \cdot H \geq N$.
- we define $\rho(S) = \dim H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Q})$, called the rank of the Picard group of S.

Extension Problem

- a family of polarized K3 surface (S_t, H_t), $t \in T$ (a parameter space);
- $C_0 \subset S_0$ a union of rational curves;
- $\alpha = [C_0] = m[H_t] \in H^2(S, \mathbb{Z})$; (a multiple of polarization);

We like to show

- exists a family of curves $C_t \subset S_t$, such that
  - C_t are union of rational curves;
  - $C_0 = C_0$.



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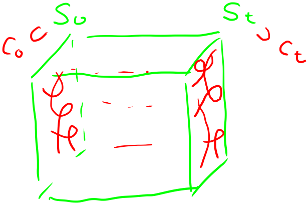


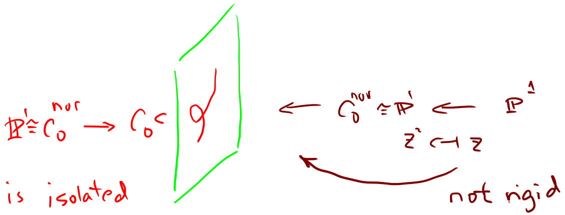
Extension Problem

Use moduli of genus 0 stable maps

- represent $C_0 \subset S_0$ as the image of $[u_0] \in \overline{M}_0(S_0, \alpha)$.

Extension Lemma (Ran, Bogomolov-Tschinkel, –)

Suppose $[u_0] \in \overline{M}_0(S_0, \alpha)$ is isolated, then u_0 extends to $u_t \in \overline{M}_0(S_t, \alpha)$ for general $t \in T$.



Extension Problem

Definition: We say a map $[u] \in \overline{M}_0(S, \alpha)$ rigid if $[u]$ is an isolated point in $\overline{M}_0(S, \alpha)$.

Extension principle: In case (a genus zero stable map) $u : C \to S$ is rigid, then u extends to nearby K3 surfaces as long as the class $u_*[C] \in H^2(S, \mathbb{Z})$ remains ample.

The existence theorem

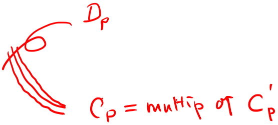
Theorem (Bogomolov - Hassett - Tschikel, L - Liedtke)

Let (X, H) be a polarized complex K3 surface such that $\rho(X)$ is odd. Then X contains infinitely many rational curves.

Outline of proof

- We only need to prove the Theorem for (X, H) defined over a number field K;
- say $K = \mathbb{Q}$, we get a family $X_\mathfrak{p}$ for every prime $p \in \mathbb{Z}$, X is the generic member of this family;
- $\forall \mathfrak{p}$, exists $D_\mathfrak{p} \subset X_\mathfrak{p}$, $D_\mathfrak{p} \notin \mathbb{Z}H$,
    - we have $\sup D_\mathfrak{p} \cdot H \to \infty$;
- pick $C_\mathfrak{p} \subset X_\mathfrak{p}$ union of rationals, $D_\mathfrak{p} + C_\mathfrak{p} \in |n_\mathfrak{p} H|$

Difficulty: $D_\mathfrak{p} + C_\mathfrak{p}$ may not be representable as the image of a rigid genus zero stable map.

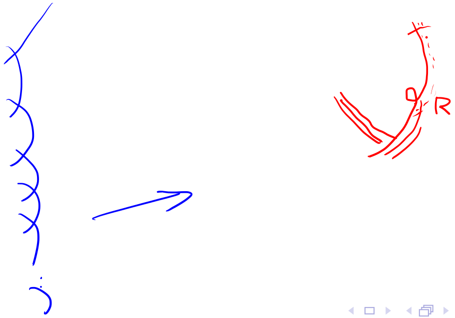


The existence theorem

Solution: Suppose we can find a nodal rational curves $R \subset X$, of class kH for some k, then for some large m we can represent

$$C_{\mathfrak{p}} + D_{\mathfrak{p}} + mR,$$

which is a class in $(n + mk)H$, by a rigid genus zero stable map.



The existence theorem

End of the proof: In general, X may not contain any nodal rational curve in $|kH|$. However, we know a small deformation of X in $\mathcal{M}_{2d}$ contains nodal rational curves in $|kH|$. Using this, plus some further algebraic geometry argument, we can complete the proof.

Thank you!