LECTURES ON ALGEBRAIC GEOMETRY BY SHOU-WU ZHANG

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Contents

1. Lecture 1: July 29

1.1. The language of category theory. Modern mathematics usually uses category theory extensively. Roughly speaking, a category consists of two parts: objects, and morphisms. We usually the same name for a category and its objects, for example the objects in the category **Sets** are sets, and the objects in the category $\mathbf{Vect}_{\mathbb{R}}$ are $\mathbb{R}$-vector spaces. Note that usually the objects in a category don't form a set. The morphisms in a category are relations between objects. For example, the morphisms in the category **Sets** are the maps between sets, and the morphisms in the category $\mathbf{Vect}_{\mathbb{R}}$ are $\mathbb{R}$-linear maps between $\mathbb{R}$-vector spaces. We require that if A, B are objects in a category, then the morphisms $\mathrm{Hom}(A, B)$ from A to B form a set.

Usually, the interesting part in a category is morphisms between objects, not object itself. For example, in $\mathbf{Vect}_{\mathbb{R}}$ an object (i.e. a $\mathbb{R}$-vector space) is completely determined, up to isomorphism, by its dimension, however, linear maps between two $\mathbb{R}$-vector spaces are much richer.

Sometimes we can recover object from its morphisms in a category, for example, in $\mathbf{Vect}_{\mathbb{R}}$, we can recover any $\mathbb{R}$-vector space V by $V = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, V)$ (this depends on the special object $\mathbb{R}$ in the category $\mathbf{Vect}_{\mathbb{R}}$). But in general this is not easy to do.

Exercise 1.1. Study category theory. (Give precise definition of a category, functor, and natural transform, etc.)

1.2. Rings and modules. Rings and modules are generalizations of fields and vector spaces.

In this course, a ring R is always commutative and with a unity ($1 \in R$). For example, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$... are rings. If R is a ring, the polynomial ring $R[X]$ is a ring.

If R is a ring, a set $I \subset R$ is called an ideal of R if I is closed under addition and $R \cdot I \subset R$. If R is a ring and I is an ideal of R, then R/I is also a ring.

Let R be a ring. An R-module M is an abelian group endowed with multiplication-by-R map $\cdot : R \times M \to M$ such that

$$r \cdot (m + m') = r \cdot m + r \cdot m', \qquad (r + r') \cdot m = r \cdot m + r' \cdot m, \qquad (rr') \cdot m = r \cdot (r' \cdot m)$$

for any $r, r' \in R$ and $m, m' \in M$. For example, if $R = \mathbb{Z}$ then R-modules are just abelian groups.

Similar to subspace and quotient space of a vector space, we can define submodule and quotient module of an R-module.

If M_1 and M_2 are R-modules, then $\mathrm{Hom}_R(M_1, M_2)$ is the set of maps $\varphi : M_1 \to M_2$ such that

$$\varphi(m + m') = \varphi(m) + \varphi(m'), \quad \varphi(rm) = r\varphi(m), \quad \forall m, m' \in M_1, \forall r \in R.$$

If $\varphi \in \mathrm{Hom}_R(M_1, M_2)$, then we can define $\ker(\varphi)$, $\mathrm{Im}(\varphi)$ and $\mathrm{coker}(\varphi)$, they are all R-modules.

1.3. Restriction of scalar and base change. Let $\varphi : A \to B$ be a ring homomorphism. Let N be a B-module. Then N can be viewed as A-module via φ, denoted by N_A.

Conversely, if M is an A-module, how to construct a B-module from it? There are two possible ways:

- "base change": $M_B := B \otimes_A M$, where the tensor product $B \otimes_A M$ is defined by

$$B \otimes_A M := \frac{\{\text{formal } A\text{-linear combination of } b \otimes m, b \in B, m \in M\}}{\left\langle \begin{array}{l|l} (b + b') \otimes m - b \otimes m - b' \otimes m, & \\ b \otimes (m + m') - b \otimes m - b \otimes m', & b, b' \in B, m, m' \in M, a \in A \\ ab \otimes m - a \cdot (b \otimes m), & \\ b \otimes am - a \cdot (b \otimes m) & \end{array} \right\rangle}.$$

- "induced module": $M^B := \mathrm{Hom}_A(B, M)$.

Hence we obtain functors

$$\begin{aligned} B\text{-}\mathbf{Mod} &\to A\text{-}\mathbf{Mod}, \quad N \mapsto N_A, \\ A\text{-}\mathbf{Mod} &\to B\text{-}\mathbf{Mod}, \quad M \mapsto M_B, \\ A\text{-}\mathbf{Mod} &\to B\text{-}\mathbf{Mod}, \quad M \mapsto M^B. \end{aligned}$$

Exercise 1.2. Prove that for any A-module M and B-module N, there are *natural* isomorphisms:

$$\mathrm{Hom}_A(M, N_A) \cong \mathrm{Hom}_B(M_B, N),$$
$$\mathrm{Hom}_A(N_A, M) \cong \mathrm{Hom}_B(N, M^B),$$

here you should know the meaning of "natural" after working on Exercise 1.1.

Consequently, the functors $(\ )_B$ and $(\ )^B$ are uniquely determined (up to isomorphism) by these properties.

The above exercise actually tells us that the functor $(\ )_B$ is left adjoint to $(\ )_A$, and the functor $(\ )^B$ is right adjoint to $(\ )_A$.

Example 1.1. Let $A = \mathbb{R}$ and $B = \mathbb{C}$. If V is a $\mathbb{R}$-vector space, then $V_{\mathbb{C}} = V \oplus iV$. Also, $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, V) = V \oplus iV$.

The phenomenon in the above example holds as long as B is free of finite rank as A-module. In general $(\ )_B \neq (\ )^B$, for example if $A = \mathbb{R}$ and $B = \mathbb{R}[X]$.

1.4. **Exact sequence.**

1.4.1. Let $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ be a sequence of homomorphisms of R-modules. It's called *exact* at M_2 if $\mathrm{Im}(\alpha) = \ker(\beta)$.

A *short exact sequence* of R-modules is a sequence of R-modules of following form

$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$$

such that it's exact at M_1, M_2 and M_3, namely α is injective, β is surjective, and $\mathrm{Im}(\alpha) = \ker(\beta)$.

Example 1.2. The following sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & a & \longmapsto & (a, 0) & & & & \\ & & & & (a, b) & \longmapsto & b & & \end{array}$$

is a short exact sequence.

If R is a field, then all short exact sequences of R-modules are isomorphic to the above form. However this is not true if R is a general ring.

1.4.2. Suppose a functor between categories of modules is given. A natural question is that does it preserve short exact sequences? If it's true, we call such functor *exact*.

In general the answer is no. For example, if N is an R-module, then the functor $- \otimes_R N$ is only right exact, i.e. if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact, then

$$M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0$$

is exact, but the map $M_1 \otimes_R N \to M_2 \otimes_R N$ is not necessarily injective. The functor $\mathrm{Hom}_R(N, -)$ is only left exact, i.e.

$$0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3)$$

is exact, but the map $\mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3)$ is not necessarily surjective. The functor $\mathrm{Hom}_R(-, N)$ is only left exact (note: this is a *contravariant* functor), i.e.

$$0 \to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$$

is exact, but the map $\mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ is not necessarily surjective.

Exercise 1.3. For each above functor, find an example such that the mentioned map is not injective (resp. surjective).

Definition 1.3. Let N be an R-module.

(i) N is called a *projective* R-module if the functor $\mathrm{Hom}_R(N, -)$ is exact.
(ii) N is called a *injective* R-module if the functor $\mathrm{Hom}_R(-, N)$ is exact.
(iii) N is called a *flat* R-module if the functor $- \otimes_R N$ is exact.

Theorem 1.4. *N is a projective R-module if and only it's a direct summand of a free R-module, i.e. there exists an R-module N' and a free R-module F such that $N \oplus N' \cong F$.*

Sketch of proof. "$\Leftarrow$": it's easy to see that a free R-module is projective. From this it's easy to see that a direct summand of a free R-module is also projective. $\square$

Exercise 1.4. Prove the "$\Rightarrow$" part. (Hint: for any R-module N, we can always find a free R-module F with a surjective R-module homomorphism $F \twoheadrightarrow N$.)

Exercise 1.5. N is an injective R-module if and only if for any ideal I of R, the natural map $N = \mathrm{Hom}_R(R, N) \to \mathrm{Hom}_A(I, N)$ is surjective. (This is called Baer's criterion.)

Exercise 1.6. N is a flat R-module if and only if for any ideal I of R, the natural map $I \otimes_R N \to N$ is injective (equivalently, $I \otimes_A N = I \cdot N \subset N$).

Exercise 1.7. Classify these modules when R is a PID.

1.5. **Noetherian condition.** Let R be a ring. An R-module M is called *Noetherian* R-module if for any ascending R-submodules $M_1 \subset M_2 \subset M_3 \subset \cdots$ of M, there exists an integer N such that for any $n \geq N$, $M_n = M_N$. The ring R is called *Noetherian ring* if R itself is a Noetherian R-module.

Example 1.5. A field is a Noetherian ring. The polynomial ring $\mathbb{R}[X]$ is a Noetherian ring. More generally, a PID is a Noetherian ring.

Theorem 1.6 (Hilbert basis theorem)**.** *If R is a Noetherian ring, then the polynomial ring $R[X]$ is also a Noetherian ring.*

Exercise 1.8. Prove the above theorem.

Exercise 1.9. Let R be a ring. Prove that the following conditions are equivalent:

(i) R is Noetherian ring.
(ii) Every ideal of R is finitely generated as R-module.
(iii) Every R-submodule of any finitely generated R-module is finitely generated.

Exercise 1.10. Let k be a field. Prove that the rational function field $k(X)$ is *not* a finitely generated k-algebra.

Exercise 1.11. Let R be a Noetherian ring, M be a finitely generated R-module. Let $\varphi : M \to M$ be a surjective R-module homomorphism. Prove that φ is an isomorphism.

Exercise 1.12. Let $A \hookrightarrow B \hookrightarrow C$ be injective ring homomorphisms of Noetherian rings. Assume that C is finitely generated A-algebra, C is finite B-algebra (i.e. C is finitely generated as a B-module). Prove that B is finitely generated A-algebra.

1.6. **Grothendieck group.** Let R be a Noetherian ring. Define

$$\mathbb{K}'(R) := \frac{\left\{ \begin{array}{l} \text{formal } \mathbb{Z}\text{-linear combination of isomorphism classes} \\ [M] \text{ of finitely generated } R\text{-modules} \end{array} \right\}}{\left\langle [M_2] - [M_1] - [M_3] \;\middle|\; \begin{array}{l} 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ is a short exact sequence} \\ \text{of finitely generated } R\text{-modules} \end{array} \right\rangle}$$

and

$$\mathbb{K}(R) := \frac{\left\{ \begin{array}{l} \text{formal } \mathbb{Z}\text{-linear combination of isomorphism classes} \\ [M] \text{ of finitely generated projective } R\text{-modules} \end{array} \right\}}{\left\langle [M_2] - [M_1] - [M_3] \;\middle|\; \begin{array}{l} 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ is a short exact sequence} \\ \text{of finitely generated projective } R\text{-modules} \end{array} \right\rangle}.$$

Prove that $(\mathbb{K}(R), \oplus, \otimes)$ is a ring, and $(\mathbb{K}'(R), \oplus)$ with $\otimes : \mathbb{K}(R) \times \mathbb{K}'(R) \to \mathbb{K}'(R)$ gives a $\mathbb{K}(R)$-module structure of $\mathbb{K}'(R)$. Give an example that in general $\mathbb{K}'(R)$ is *not* a ring.

Question. How to describe $\mathbb{K}(R)$ and $\mathbb{K}'(R)$?

Prove that if $R = k$ is a field, then $\mathbb{K}(R) = \mathbb{K}'(R) \cong \mathbb{Z}$ given by $[V] \mapsto \dim_k V$. What if R is a PID?

If $A \to B$ is a ring homomorphism between Noetherian rings, prove that there is a well-defined natural map $\mathbb{K}(A) \to \mathbb{K}(B)$ given by $[M] \mapsto [M \otimes_A B]$. If moreover B is a finite A-algebra, prove that there is a well-defined natural map $\mathbb{K}'(B) \to \mathbb{K}'(A)$ given by $[N] \mapsto [N_A]$.

2. Lecture 2: July 31

2.1. **From equation to algebra and from algebra to geometry.** Suppose $f(X) \in \mathbb{Q}[X]$ is a polynomial and we want to solve $f(x) = 0$ over a field $L/\mathbb{Q}$. Then this is equivalent to find all ring homomorphism $\mathrm{Hom}(\mathbb{Q}[X]/f(X), L)$. On the other hand, all the solutions of $f(x) = 0$ form a geometric object (i.e. a space) and we can study the functions on it.

2.2. **Ringed space.** A *ringed space* $(X, \mathscr{O}_X)$ consists of a topological space X and a *sheaf of rings* $\mathscr{O}_X$ on X, that is,

- for any open set $U \subset X$, a ring $\mathscr{O}_X(U)$ is given (regarded as "space of functions on U"),
- for any open sets $V \subset U$ in X, a ring homomorphism $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ is given, denoted by r_{VU} or simply denoted by $|_V$ (regarded as "the restriction map from U to V"),

such that

- (presheaf property) $r_{UU} = \mathrm{id}$ and if $W \subset V \subset U$, then $r_{WV} \circ r_{VU} = r_{WU}$,
- (sheaf property) for any open set U of X and any open cover $\{U_i\}_{i\in I}$ of U, the following sequence is an exact sequence of abelian groups:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathscr{O}_X(U) \longrightarrow \prod_{i\in I} \mathscr{O}_X(U_i) \longrightarrow \prod_{i,j\in I} \mathscr{O}_X(U_i \cap U_j), \\ f \longmapsto (x|_{U_i})_{i\in I}, \\ (f_i)_{i\in I} \longmapsto (f_i|_{U_i\cap U_j} - f_j|_{U_i\cap U_j})_{i,j\in I} \end{array} \tag{2.1}$$

namely, if for every $i \in I$, $f_i \in \mathscr{O}_X(U_i)$ is given, such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ holds for any $i, j \in I$, then there exists a unique $f \in \mathscr{O}_X(X)$ such that $f|_{U_i} = f_i$.

The presheaf property means that $\mathscr{O}_X : \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Rings}$ is a contravariant functor from $\mathbf{Top}_X$ to $\mathbf{Rings}$, here the category $\mathbf{Top}_X$ consists of objects all open subsets of X, and the set of morphisms from V to U is empty if $V \not\subset U$, and consists of the inclusion map from V to U if $V \subset U$.

Example 2.1. Suppose X is a manifold (resp. smooth manifold, complex manifold...), then we can consider the sheaf of continuous functions (resp. C^r-functions, smooth functions, analytic functions...) on it, namely, $\mathscr{O}_X(U)$ consists of the space of continuous functions (resp. C^r-functions, smooth functions, analytic functions...) on U. Then $(X, \mathscr{O}_X)$ is a ringed space.

Similarly we can define sheaf of abelian groups, sheaf of modules... on X.

A *morphism* of ringed spaces $f = (f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$, here $f_*\mathscr{O}_X$ is a sheaf on Y, called the *direct image* of $\mathscr{O}_X$, and is defined by $(f_*\mathscr{O}_X)(V) := \mathscr{O}_X(f^{-1}(V))$ for any open subset V of X.

Exercise 2.1. Study ringed space. (Definition and examples, etc.)

2.3. **Spectrum of a ring.** Let R be a commutative ring. We are going to define $X = \mathrm{Spec}(R) = (X, \mathscr{O}_X)$ called the spectrum of R, which is a ringed space such that $\mathscr{O}_X(X) = R$ naturally.

2.3.1. Firstly we define $X := \{\text{prime ideal } \mathfrak{p} \text{ of } R\}$ as a set. Recall that $\mathfrak{p}$ is a prime ideal of R means that $\mathfrak{p}$ is an ideal such that $R/\mathfrak{p}$ is an integral domain (which means that $R/\mathfrak{p}$ is a subring of a field). We follow the usual convention that the zero ring 0 is not an integral domain nor a field.

Exercise 2.2. Find a set-theoretic criterion of a ring R being an integral domain.

2.3.2. We define the topology on X via giving all closed subsets of it: the closed subset of X is of form $Z(I) := \{\mathfrak{p} \in X \mid \mathfrak{p} \supset I\}$. Note that this indeed satisfies the axiom of closed sets: $Z(R) = \varnothing$, $Z(0) = X$, $Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$ and $\bigcap_{\lambda\in\Lambda} Z(I_\lambda) = Z(\sum_{\lambda\in\Lambda} I_\lambda)$.

Note that $Z(I) = \bigcap_{f\in I} Z(f)$. Hence if we define $U(I) := X \setminus Z(I)$ and $U(f) := X \setminus Z(f)$, then they are open subsets of X, any open subset of X is of form $U(I)$, and $\{U(f) \mid f \in R\}$ is a topological basis of X, since $U(I) = \bigcup_{f\in I} U(f)$. The open subset of form $U(f)$ is called a *principal open subset* of X.

2.3.3. Let $\phi : A \to B$ be a ring homomorphism. Then we can define $\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by $\mathfrak{P} \mapsto \phi^*(\mathfrak{P}) := \phi^{-1}(\mathfrak{P})$. Note that the natural map $A/\phi^{-1}(\mathfrak{P}) \to B/\mathfrak{P}$ is injective, hence $A/\phi^{-1}(\mathfrak{P})$ is also an integral domain and $\phi^{-1}(\mathfrak{P}) \in \mathrm{Spec}(A)$.

Exercise 2.3. Prove that ϕ^* is continuous.

Example 2.2. Let I be an ideal of A and consider the natural surjective ring homomorphism $\phi : A \twoheadrightarrow A/I$. Then $\phi^* : \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is injective and $\mathrm{Spec}(A/I) \cong \{\mathfrak{p} \in \mathrm{Spec}(A) \mid I \subset \mathfrak{p}\} = Z(I)$. This means that a closed subset in a spectrum has the structure of a spectrum.

Example 2.3. Let $f \in A$ be an element and consider the localization $A_f := A[X]/(Xf-1)$, or simply denoted by $A[\frac{1}{f}]$. Let $\phi : A \to A_f$ be the natural map. Then $\phi^* : \mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ is injective and $\mathrm{Spec}(A_f) \cong \{\mathfrak{p} \in \mathrm{Spec}(A) \mid f \notin \mathfrak{p}\} = U(f)$. This means that a principal open subset in a spectrum has the structure of a spectrum.

In general, let S be a multiplicative subset of A, then we have the notion of localization $S^{-1}A$. Let $\phi : A \to S^{-1}A$ be the natural map, then $\phi^* : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is injective. The $\mathrm{Spec}(S^{-1}A)$ may be regarded as "$\bigcap_{f\in S} U(f)$".

Exercise 2.4. Prove that $S^{-1}A$ is a flat A-module. You may use Exercise 1.6.

In particular, if $\mathfrak{p} \in \mathrm{Spec}(A)$, consider the localization $A_\mathfrak{p} := (A \setminus \mathfrak{p})^{-1}A$. Then we have the injection $\mathrm{Spec}(A_\mathfrak{p}) \to \mathrm{Spec}(A)$, the $\mathrm{Spec}(A_\mathfrak{p})$ may be regarded as "$\bigcap_{U\ni\mathfrak{p}} U$", called the "infinitesimal neighborhood of $\mathfrak{p}$".

We can also consider the ring homomorphisms $A \twoheadrightarrow A/\mathfrak{p}$ and $A/\mathfrak{p} \hookrightarrow k(\mathfrak{p}) := \mathrm{Frac}(A/\mathfrak{p})$, and we obtain injective maps between spectra

$$\mathrm{Spec}(k(\mathfrak{p})) \to \mathrm{Spec}(A/\mathfrak{p}) \to \mathrm{Spec}(A).$$

Note that $\mathrm{Spec}(k(\mathfrak{p}))$ only consists of one point, and $\mathrm{Spec}(A/\mathfrak{p})$ may be regarded as a closed subset of $\mathrm{Spec}(A)$. The image of $\mathrm{Spec}(k(\mathfrak{p}))$ in $\mathrm{Spec}(A)$ is just the point $\mathfrak{p}$ itself.

Exercise 2.5. Prove that the closure of $\mathrm{Spec}(k(\mathfrak{p}))$ in $\mathrm{Spec}(A)$ is equal to $\mathrm{Spec}(A/\mathfrak{p})$. The point $\mathrm{Spec}(k(\mathfrak{p}))$ is usually called "the generic point of $\mathrm{Spec}(A/\mathfrak{p})$".

This means that in general, one-point set in $\mathrm{Spec}(A)$ is not closed, hence $\mathrm{Spec}(A)$ is not Hausdorff in general. Nevertheless, $\mathrm{Spec}(A)$ is T_0, namely for any $\mathfrak{p} \neq \mathfrak{q}$ in $\mathrm{Spec}(A)$, there exists an open subset U which contains exactly one of $\mathfrak{p}$ and $\mathfrak{q}$.

At first it looks like that point which is not closed makes things complicated, but in the following lectures we will see that it actually makes things simpler.

We have another fact that $\mathrm{Spec}(A)$ is (quasi-)compact, i.e. if $\bigcap_{\lambda\in\Lambda} Z(I_\lambda) = \varnothing$, then there exists a finite subset $\Lambda' \subset \Lambda$ such that $\bigcap_{\lambda\in\Lambda'} Z(I_\lambda) = \varnothing$. In fact, the condition implies that $\sum_{\lambda\in\Lambda} I_\lambda = A$, otherwise it is contained in a maximal ideal $\mathfrak{m}$ of A and we have $\mathfrak{m} \in \bigcap_{\lambda\in\Lambda} Z(I_\lambda)$, a contradiction. Therefore we can choose a finite subset $\Lambda' \subset \Lambda$ such that there exists $a_\lambda \in I_\lambda$ for $\lambda \in \Lambda'$ such that $\sum_{\lambda\in\Lambda'} a_\lambda = 1 \in A$, now it's easy to see that $\bigcap_{\lambda\in\Lambda'} Z(I_\lambda) = \varnothing$.

2.3.4. Now we define the sheaf of rings $\mathscr{O}_X$ of the ringed space $\mathrm{Spec}(R) = (X, \mathscr{O}_X)$. It is natural to define it by the following way:

(i) on principal open subsets, $\mathscr{O}_X(U(f)) := R_f$;

(ii) for general open set U, define $\mathscr{O}_X(U)$ by the sheaf property (2.1).

The idea itself is good, but however it's difficult to check that the (ii) is well-defined (e.g. $\mathscr{O}_X(U)$ is a ring, independent of the choice of covering, for general open sets – not necessarily principle open subsets – the (2.1) also holds). Here we replace (ii) by another definition.

First we introduce the concept of the stalk. Let $\mathfrak{p} \in \mathrm{Spec}(R)$, define the *stalk* of $\mathscr{O}_X$ at $\mathfrak{p}$ to be

$$\mathscr{O}_{X,\mathfrak{p}} := \varinjlim_{U\ni\mathfrak{p}} \mathscr{O}_X(U).$$

This may be regarded as a generalization of the space of Taylor expansions around $x \in X$ if X is a manifold and $\mathscr{O}_X$ is the sheaf of smooth functions on X. Since the principal open subsets form a topological bases of X, and by (i) we define $\mathscr{O}_X(U(f)) := R_f$, we have

$$\mathscr{O}_{X,\mathfrak{p}} = \varinjlim_{f\notin\mathfrak{p}} R_f = R_\mathfrak{p}.$$

Now we replace (ii) by the following

(ii)′ for general open set U, define $\mathscr{O}_X(U)$ by

$$\mathscr{O}_X(U) := \left\{ (g_\mathfrak{p})_\mathfrak{p} \in \prod_{\mathfrak{p}\in U} R_\mathfrak{p} \;\middle|\; \begin{array}{l} \text{there exists a cover of } U \text{ by principal open subsets} \\ U = \bigcup_{i\in I} U(f_i) \text{ and } g_i \in R_{f_i} \text{ for each } i \in I\text{, such that} \\ \text{the image of } g_i \text{ in } R_\mathfrak{p} \text{ is } g_\mathfrak{p} \text{ for any } i \in I \text{ and } \mathfrak{p} \in U(f_i) \end{array} \right\}. \tag{2.2}$$

It turns out that this definition is compatible with the previous definition (i) when U is already a principal open subset, and this definition satisfies the sheaf property (2.1). For example, we have the following result:

Theorem 2.4. *We have $\mathscr{O}_X(X) = R$.*

Sketch of proof. For any non-zero element f of R, the $\mathrm{Ann}(f) := \{g \in R \mid gf = 0\}$ is an ideal of R and is not equal to R (since $1 \notin \mathrm{Ann}(f)$), hence it is contained in some maximal ideal $\mathfrak{m}$ of R and the image of f in $R_{\mathfrak{m}}$ is non-zero. Therefore the natural map $R \to \prod_{\mathfrak{p}\in X} R_{\mathfrak{p}}$ is injective. □

Exercise 2.6. Complete the above proof by proving that the image of $R \to \prod_{\mathfrak{p}\in X} R_{\mathfrak{p}}$ is equal to $\mathscr{O}_X(X)$.

3. Lecture 3: August 1

Let $\phi : A \to B$ be a ring homomorphism, we are going to define the morphism of ringed spaces $\mathrm{Spec}(\phi) : Y \to X$ where $Y := \mathrm{Spec}(B)$ and $X := \mathrm{Spec}(A)$. Recall that we have defined the map $f := \phi^* : Y \to X$ of topological spaces by $\mathfrak{P} \mapsto \phi^{-1}(\mathfrak{P})$. To define the morphism of sheaves $f^\# : \mathscr{O}_X \to f_*\mathscr{O}_Y$, we define the ring homomorphism on sections of principal open subsets $A_f = \mathscr{O}_X(U(f)) \to B_{\phi(f)} = \mathscr{O}_Y(U(\phi(f)))$ for every $f \in A$ (note that $f^{-1}(U(f)) = U(\phi(f))$) to be the natural map induced by $\phi : A_f \to B_{\phi(f)}$ and glue them to general open subsets.

Therefore we constructed the following contravariant functor

$$\mathrm{Spec} : \mathbf{Rings}^{\mathrm{op}} \to \mathbf{RingedSpaces}.$$

We introduce the notion of a *locally ringed space*, which is a ringed space $(X, \mathscr{O}_X)$ such that for every $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. We have already known that for a ring A, the $\mathrm{Spec}(A)$ is a locally ringed space. A morphism between locally ringed spaces is a morphism of ringed spaces $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ such that for every $x \in X$, the ring homomorphism $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ induced by $f^\#$ is a local homomorphism, i.e. the preimage of the maximal ideal is a maximal ideal.

Theorem 3.1. *The functor*

$$\mathrm{Spec} : \mathbf{Rings}^{\mathrm{op}} \to \mathbf{LocallyRingedSpaces}$$

is fully faithful, i.e. for any rings A and B, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Rings}}(A, B) \cong \mathrm{Hom}_{\mathbf{LocallyRingedSpaces}}(\mathrm{Spec}(B), \mathrm{Spec}(A)),$$

with the inverse map given by $(f, f^\#) \mapsto f^\#(\mathrm{Spec}(A))$.

Note that if we only consider **RingedSpaces** then this is faithful but not full, see for example [3], Ch. II, Example 2.3.2.

3.1. Examples of affine schemes.

Definition 3.2. An *affine scheme* is a locally ringed space which is isomorphic to $\mathrm{Spec}(A)$ for some ring A.

Example 3.3. If k is a field, then $\mathrm{Spec}(k)$ consists of only one point (0), and $\mathscr{O}_{\mathrm{Spec}(k)}((0)) = k$.

Example 3.4. The $\mathrm{Spec}(\mathbb{Z})$ consists of the point (0) and (p) for each prime p. Its closed subset are $\varnothing$, $\mathrm{Spec}(\mathbb{Z})$ and any finite subsets which does not contain (0). If S is any non-zero integer, we have $\mathscr{O}_{\mathrm{Spec}(\mathbb{Z})}(\mathrm{Spec}(\mathbb{Z}) \setminus \{(p) : p \mid S\}) = \mathbb{Z}[\frac{1}{S}]$.

Example 3.5. Similarly, if k is a field, the $\mathrm{Spec}(k[X])$ consists of the point (0) and $(f(X))$ for each irreducible polynomial $f(X)$ of $k[X]$. Its closed subset are $\varnothing$, $\mathrm{Spec}(k[X])$ and any finite subsets which does not contain (0). If Z is any such finite subset, let $U = \mathrm{Spec}(k[X])\setminus Z$ and let $f(X)$ be the product of a generator of the non-zero prime ideals contained in Z, then we have $\mathscr{O}_{\mathrm{Spec}(k[X])}(U) = k[X,T]/(Tf(X)-1)$, or simply written as $k[X, 1/f(X)]$.

In particular, if k is algebraically closed, then $\mathrm{Spec}(k[X]) \cong k \sqcup \{\eta\}$ as a set, where an element $a \in k$ corresponds to a maximal ideal $(X-a)$ which is a closed point, and η (called "generic point") corresponds to (0), whose closure is $\mathrm{Spec}(k[X])$.

Example 3.6. Affine space over k of dimension n, which is $\mathbb{A}^n_k := \mathrm{Spec}(k[X_1, \cdots, X_n])$.

Theorem 3.7 (Hilbert's Nullstellensatz)**.** *If k is an algebraically closed field, then the set of all closed points of $\mathbb{A}^n_k$ is equal to k^n: for $(a_1, \cdots, a_n) \in k^n$, it gives a maximal ideal $(X_1 - a_1, \cdots, X_n - a_n)$ of $k[X_1, \cdots, X_n]$.*

Exercise 3.1. Prove the above theorem using Exercise 1.12. (Hint: if $\mathfrak{m}$ is a maximal ideal of $k[X_1, \cdots, X_n]$, consider $k \hookrightarrow k' \hookrightarrow k[X_1, \cdots, X_n]/\mathfrak{m}$ where k' is the maximal pure transcendental extension of k contained in $k[X_1, \cdots, X_n]/\mathfrak{m}$.)

3.2. **Dimension and Noetherian condition.** We want to define the notion of dimension such that $\dim \mathbb{A}_k^n = n$. We note that $\mathbb{A}_k^n \supsetneqq Z(X_1) \supsetneqq Z(X_1, X_2) \supsetneqq \cdots \supsetneqq Z(X_1, \cdots, X_n)$ which is a descending chain of length n, and $Z(X_1, \cdots, X_i) \cong \mathbb{A}_k^{n-i}$.

Definition 3.8. A topological space X is called Noetherian topological space if any descending sequence of closed subsets stabilizes.

Exercise 3.2. If A is a Noetherian ring, show that $\operatorname{Spec}(A)$ is a Noetherian topological spaces.

The converse is not true, for example if $A = k[[X]][X^{1/n} \mid n \geq 1]$, then A is not Noetherian, while $\operatorname{Spec}(A) = \{(0), \mathfrak{m}_A\}$ where $\mathfrak{m}_A = (X^{1/n} \mid n \geq 1)$ is the maximal ideal of A.

Definition 3.9. A closed subset Y of a topological space X is called *irreducible* if there are no non-empty subspaces $Y_1 \subsetneqq Y$ and $Y_2 \subsetneqq Y$ such that $Y_1 \cup Y_2 = Y$.

Theorem 3.10. *If X is a Noetherian topological space, then any non-empty closed subset is a finite union of irreducible subsets.*

Idea of proof. Consider the set $S := \{\text{closed subset of } X \text{ which is not a finite union of irreducible subsets}\}$. If $S \neq \varnothing$ then by the Noetherian property of X, there exists a minimal element of S. Deduce contradiction from this. $\square$

Definition 3.11. If X is a Noetherian topological space, $X = \bigcup_{i \in I} X_i$ is a finite union of irreducible subsets, such that $X_i \not\subset X_j$ whenever $i \neq j$, then each X_i is called an *irreducible component* of X.

Definition 3.12. The *dimension* of a Noetherian topological space X is the maximal number n such that there exists a chain of irreducible subsets $Y_0 \supsetneqq Y_1 \supsetneqq \cdots \supsetneqq Y_n$ in X.

Note that the dimension of a Noetherian topological space can be infinity.

Example 3.13. $\dim \mathbb{A}_k^n = n$.

Exercise 3.3. Prove that in $\operatorname{Spec}(A)$, $Z(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal, here $\sqrt{I} := \{a \in A \mid a^n \in I \text{ for some } n \geq 1\}$.

Recall the Krull dimension $\dim A$ of a ring A is the maximal number n (can be infinity) such that there exists a chain of prime ideals $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n$ in A.

Exercise 3.4. If A is a Noetherian ring, then A has only finitely many minimal prime ideals, and the intersection of them is $\sqrt{0}$, the nilradical.

In classical algebraic geometry, the dimension of an affine variety can be described by the dimension of the tangent space at a point, or by the transcendental degree $\operatorname{tr}\deg(A/k)$ (roughly speaking, the number of free variable of A over k) of the ring of regular functions A of it.

More precisely, for a k-algebra A, the transcendental degree $\operatorname{tr}\deg(A/k)$ is the maximal number n such that there exists an inclusion $k(X_1, \cdots, X_n) \hookrightarrow K$, or equivalently, the maximal number of algebraic independent elements of A over k.

Theorem 3.14. *If A is a finitely generated k-algebra, then* $\dim \operatorname{Spec}(A) = \dim A = \operatorname{tr}\deg(A/k)$.

Exercise 3.5. Read the proof of the above theorem.

Exercise 3.6. Let B be a Noetherian integral domain, A be a finite integral extension of B. Prove that A is a field if and only if B is a field.

Exercise 3.7. Let A be an integral extension of finite type over $\mathbb{Z}$. If $0 \neq f \in A$ is not invertible, then $\dim(A/fA) = \dim A - 1$.

3.3. **Sheaf of modules.** If A is a ring, M is an A-module, we can define a sheaf $\widetilde{M}$ of $\mathscr{O}_X$-module over $X = \mathrm{Spec}(A)$ (i.e. a sheaf of abelian groups over X such that for each open subset $U \subset X$, $\widetilde{M}(U)$ is a $\mathscr{O}_X(U)$-module and satisfies some compatibility property) similar to the definition of $\mathscr{O}_X$ by defining $\widetilde{M}_\mathfrak{p} := M_\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(A)$ and using (2.2).

Exercise 3.8. Write down the definition explicitly and prove that on a principal open subset $U(f)$, we have $\widetilde{M}(U(f)) = M_f$.

Therefore we obtain a functor

$$\widetilde{\ } : A\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}$$

which is fully faithful. A natural question is to determine the image of it.

Glueing of sheaves. If $X = \bigcup_{i\in I} X_i$ is an open cover of a topological space, for $i \in I$ a sheaf $\mathscr{F}_i$ on X_i is given, and for $i, j \in I$ an isomorphism of sheaves $\iota_{ij} : \mathscr{F}_i|_{X_i\cap X_j} \xrightarrow{\sim} \mathscr{F}_j|_{X_i\cap X_j}$ is given, such that $\iota_{ii} = \mathrm{id}$ and $\iota_{jk}|_{X_i\cap X_j\cap X_k} \circ \iota_{ij}|_{X_i\cap X_j\cap X_k} = \iota_{ik}|_{X_i\cap X_j\cap X_k}$. Then we can define a unique sheaf $\mathscr{F}$ over X and isomorphism $\mathscr{F}|_{X_i} \xrightarrow{\sim} \mathscr{F}_i$.

Definition 3.15. A sheaf $\mathscr{F}$ on $X = \mathrm{Spec}(A)$ is called *quasi-coherent* if there is a cover of X by principal open subsets $X = \bigcup_f U(f)$ such that $\mathscr{F}$ restrict to $U(f)$ is isomorphic to $\widetilde{M_f}$ for some A_f-module M_f.

Theorem 3.16. *The image of the functor* $\widetilde{\ }$ *is equal to the category of quasi-coherent sheaves.*

Exercise 3.9. Read the proof of it.

3.4. **Grothendieck group of coherent sheaves.** Let A be a Noetherian ring.

Definition 3.17. A sheaf $\mathscr{F}$ on $X = \mathrm{Spec}(A)$ is called *coherent* if $\mathscr{F} \cong \widetilde{M}$ for some finitely generated A-module M.

Exercise 3.10. If $\mathscr{F}$ is a quasi-coherent sheaf on $X = \mathrm{Spec}(A)$ then $\mathscr{F}$ is coherent if and only if there is a cover of X by principal open subsets $X = \bigcup_f U(f)$ such that $\mathscr{F}$ restrict to $U(f)$ is isomorphic to $\widetilde{M_f}$ for some finitely generated A_f-module M_f.

We define $\mathbb{K}'(\mathrm{Spec}(A))$ to be the Grothendieck group of coherent sheaves on $\mathrm{Spec}(A)$, similar to the definition in §1.6.

Here we need to define the notion of short exact sequence of sheaves over a topological space X: the following sequence

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

is called a short exact sequence, here α and β are morphisms of sheaves, if for any $x \in X$, the induced sequence $0 \to \mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x \to 0$ is a short exact sequence of abelian groups. Equivalently, for any open subset U of X, the induced sequence $0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is exact, and for any open subset U of X and any $g \in \mathscr{H}(U)$, there exists an open cover $\{U_i\}_{i\in I}$ of U and for every $i \in I$, there exists $f_i \in \mathscr{G}(U_i)$ such that $g|_{U_i} = \beta(U_i)(f_i)$.

It turns out that $\mathbb{K}'(\mathrm{Spec}(A)) = \mathbb{K}'(A)$ defined in §1.6.

Similarly we define $\mathbb{K}(\mathrm{Spec}(A))$ to be the Grothendieck group of locally free coherent sheaves on $\mathrm{Spec}(A)$. Here a coherent sheaf $\mathscr{F}$ on $\mathrm{Spec}(A)$ is called *locally free* if for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathscr{F}_\mathfrak{p}$ is a free $\mathscr{O}_{X,\mathfrak{p}}$-module of finite rank. The locally free coherent sheaf on $\mathrm{Spec}(A)$ is a generalization of the vector bundle on a manifold.

Theorem 3.18. *We have* $\mathbb{K}(\mathrm{Spec}(A)) = \mathbb{K}(A)$.

This is by the following

Exercise 3.11. Let A be a Noetherian ring. Prove that if M is a finitely generated A-module, then M is a projective A-module if and only if M is locally free, which means that for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module.

Remark 3.19. In the general scheme theory there is no the notion of "projective", we use "locally free" instead.

4. Lecture 4: August 5

Recall that last lecture we defined affine schemes as locally ringed spaces. They are building blocks of general schemes.

Definition 4.1. A *scheme* is a locally ringed space which is locally isomorphic to affine schemes.

Weekly Exercise 4.1. Read the precise definition.

Today we are going to discuss projective schemes and coherent sheaves on it.

4.1. Graded rings and modules.

Definition 4.2. A *graded ring* A is a ring with $A = \bigoplus_{d=0}^{\infty} A_d$ where A_d are abelian groups, such that for any $m, n \geq 0$ we have $A_m \cdot A_n \subset A_{m+n}$.

This implies that

- A_0 is a subring of A;
- A is an A_0-algebra;
- each A_d is an A_0-module;
- for any $n \geq 0$, the $\bigoplus_{d=n}^{\infty} A_d$ is an ideal of A.

Denote by $A_+ := \bigoplus_{d=1}^{\infty} A_d$. If a is a non-zero element in A_d for some $d \geq 0$, it is called a *homogeneous element of degree* d, and denote $\deg(a) = d$. Denote by $\widetilde{A} := \bigcup_{d=0}^{\infty} A_d$ the set of all homogeneous elements in A, and denote by $\widetilde{A}_+ := \bigcup_{d=1}^{\infty} A_d$ the set of all homogeneous elements in A_+.

Example 4.3. Let R be a ring, then $A = R[X_1, \cdots, X_n]$ is a graded ring with A_d consists of homogeneous polynomial of degree d.

Definition 4.4. Let A be a graded ring. A *graded A-module* is an A-module M with $M = \bigoplus_{d=N}^{\infty} M_d$ for some $N \in \mathbb{Z}$, where M_d are abelian groups, such that for any $m \geq 0$ and $n \geq N$ we have $A_m \cdot M_n \subset M_{m+n}$.

Definition 4.5. If M is a graded A-module, for $m \in \mathbb{Z}$, we define the graded A-module $M(m)$ to be M as an A-module, with the graded piece defined by $M(m)_d := M_{d+m}$ for any d.

Consider the category of graded rings. If $\varphi : A \to B$ is a homomorphism of graded rings (i.e. φ is a ring homomorphism whose image is a graded subring of B, namely $\mathrm{Im}(\varphi) = \bigoplus_{d=0}^{\infty} C_d$ for some abelian groups C_d such that $C_d = \mathrm{Im}(\varphi) \cap B_d$ for any $d \geq 0$), then its kernel $I := \ker(\varphi)$ is a *homogeneous ideal* (i.e. $I = \bigoplus_{d=0}^{\infty} I_d$ with $I_d = I \cap A_d$).

4.2. Projective spectrum for a graded ring.

The goal in this section is to define a scheme $\mathrm{Proj}(A)$ if A is a graded ring.

Firstly, as a set, $\mathrm{Proj}(A) := \{\text{homogeneous prime ideal of } A \text{ not containing } A_+\}$. Its all closed sets are given by $Z(I) := \{\mathfrak{p} \in \mathrm{Proj}(A) \mid \mathfrak{p} \supset I\}$ for any homogeneous ideal I. If $I = \bigoplus_{d=0}^{\infty} I_d$ is a homogeneous ideal, denote by $\widetilde{I} := \bigcup_{d=0}^{\infty} I_d$ and $\widetilde{I}_+ := \bigcup_{d=1}^{\infty} I_d$, then $Z(I) = \bigcap_{f \in \widetilde{I}} Z(f) = \bigcap_{f \in \widetilde{I}_+} Z(f)$. They satisfy the axiom of closed sets, e.g. $Z(f_1 f_2) = Z(f_1) \cup Z(f_2)$, etc. The closed set of form $Z(f)$ for some $f \in \widetilde{A}_+$ is called a *principal closed subset* of $\mathrm{Proj}(A)$.

For $f \in \widetilde{A}_+$, define $D_+(f) := \mathrm{Proj}(A) \setminus Z(f)$, called a *principal open subset* of $\mathrm{Proj}(A)$. Then $\{D_+(f) \mid f \in \widetilde{A}_+\}$ if a topological basis of $\mathrm{Proj}(A)$.

Proposition 4.6. *$D_+(f)$ is affine.*

In fact, we can consider $A[\frac{1}{f}]$ which is also a "graded ring" by declaring $\deg(\frac{1}{f}) := -\deg(f)$ (note that this does not satisfy our original definition of a graded ring, since there are elements with negative degree). Consider its subring $A_{(f)} := A[\frac{1}{f}]_{\deg=0}$. Define the map $D_+(f) \to \mathrm{Spec}(A_{(f)})$ by $\mathfrak{p} \mapsto \mathfrak{p}[\frac{1}{f}] \cap A_{(f)}$.

Daily Exercise 4.2. (i) Prove that this is an isomorphism of sets.

(ii) If $f, g \in \widetilde{A}_+$ with $\deg(f) = \deg(g)$, then the diagram

$$\begin{tikzcd} D_+(gf) \arrow[r, hook] \arrow[d, "\cong"] & D_+(f) \arrow[d, "\cong"] \\ \mathrm{Spec}(A_{(gf)}) \arrow[r, "(*)"] & \mathrm{Spec}(A_{(f)}) \end{tikzcd}$$

commutes, and the image of $(*)$ is equal to $\mathrm{Spec}(A_{(f)}[\frac{g}{f}])$.

Therefore this map is also a homeomorphism of topological spaces, and which allows us to define the structure sheaf $\mathscr{O}_X$ of $X = \mathrm{Proj}(A)$ by glueing the structure sheaf on $\mathrm{Spec}(A_{(f)})$. The obtained $X = \mathrm{Proj}(A) = (X, \mathscr{O}_X)$ is a locally ringed space which is locally isomorphic to affine schemes, hence it is a scheme.

4.2.1. Now we give some examples of morphisms involving projective spectrum.

Example 4.7. Let A be a graded ring. Then the natural ring homomorphism $A_0 \to A$ gives the map $\mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$, $\mathfrak{p} \mapsto \mathfrak{p} \cap A_0$.

Daily Exercise 4.3. Prove that this gives a scheme morphism.

Example 4.8. Let A, B be two graded rings and let $\varphi : A \to B$ be a ring homomorphism such that $\varphi(A_d) \subset B_d$ for any $d \geq 0$. In general this map does not induce morphisms between schemes $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$, since for $\mathfrak{P} \in \mathrm{Proj}(B)$, $\varphi^{-1}(\mathfrak{P}) \not\supset A_+$ is not always true. Nevertheless, for $f \in \widetilde{A}_+$, there is a natural isomorphism $D_+(\varphi(f)) \to D_+(f)$, hence we can define

$$\bigcup_{f \in \widetilde{A}_+} D_+(\varphi(f)) \to \mathrm{Proj}(A)$$

which maps an open subset of $\mathrm{Proj}(B)$ to $\mathrm{Proj}(A)$.

4.2.2. Now we give some examples of projective spectrum.

Example 4.9. Let R be a ring, define the graded ring $A = R$ with $A_d = A$ if $d = 0$, $A_d = 0$ if $d \geq 1$. Then $\mathrm{Proj}(A) = \varnothing$.

In general, $\mathrm{Proj}(A) = \varnothing$ if and only if A_+ is nilpotent.

Example 4.10. Let k be a field. Then $\mathrm{Proj}(k[T]) = \{0\}$.

Daily Exercise 4.4. If R is a ring, prove that $\mathrm{Proj}(R[T]) = \mathrm{Spec}(R)$.

Example 4.11. Study $\mathrm{Proj}(\mathbb{C}[T_0, \cdots, T_n])$. Consider $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times = \{\text{lines in } \mathbb{C}^{n+1}\}$. We can define the injective map

$$\begin{aligned} \mathbb{CP}^n &\to \mathrm{Proj}(\mathbb{C}[T_0, \cdots, T_n]), \\ [a_0 : \cdots : a_n] &\mapsto (a_i T_j - a_j T_i \mid 0 \leq i, j \leq n), \end{aligned}$$

whose image is equal to the closed points of $\mathrm{Proj}(\mathbb{C}[T_0, \cdots, T_n])$. This also holds if we replace $\mathbb{C}$ by an algebraically closed field.

This motivates us to define the concept of projective space:

Definition 4.12. If R is a ring, define $\mathbb{P}^n_R := \mathrm{Proj}(R[X_0, \cdots, X_n])$, called the projective space over R of (relative) dimension n.

Note that if $A = R[X_0, \cdots, X_n]$, then the ideal A_+ is generated by $X_0, \cdots, X_n$. Hence $\mathbb{P}^n_R = \bigcup_{i=0}^n D_+(X_i)$. Each $D_+(X_i)$ is isomorphic to $\mathrm{Spec}(R[\frac{X_j}{X_i} \mid j \neq i]) \cong \mathbb{A}^n_R$, which means that the project space of dimension n can be covered by $n+1$ affine spaces of dimension n. We call $[X_0 : \cdots : X_n]$ in $\mathbb{P}^n_R$ the homogeneous coordinate, and call $(\frac{X_j}{X_i} \mid j \neq i)$ the affine coordinate on $D_+(X_i)$.

Warning. $\mathrm{Proj}(A)$ is *not* (quasi)-compact in general.

4.3. Quasi-coherent sheaf on a projective spectrum.

Definition 4.13. Let X be a scheme. A *quasi-coherent* sheaf $\mathscr{F}$ on X is an $\mathscr{O}_X$-module on X such that there exists an open affine cover $X = \bigcup_{i \in I} X_i$ of X such that for each $i \in I$ the restriction of $\mathscr{F}$ to X_i is isomorphic to $\widetilde{M_i}$ for some $\mathscr{O}_X(X_i)$-module M_i.

For a graded module M over a graded ring A, we can construct a quasi-coherent sheaf $\widetilde{M}$ on $X = \mathrm{Proj}(A)$ similar to the construction of the structure sheaf $\mathscr{O}_X$ of X, namely, we define $\widetilde{M}|_{D_+(f)} := \widetilde{M_{(f)}}$ for $f \in \widetilde{A}_+$ and glue them together, where $M_{(f)} := M[\frac{1}{f}]_{\deg = 0}$ is an $A_{(f)}$-module.

4.3.1. *The most important example.* Let A be a graded ring, $X = \mathrm{Proj}(A)$. If d is an integer, we define $\mathscr{O}_X(d) := \widetilde{A(d)}$.

Proposition 4.14. *Suppose that as an A_0-algebra, A is generated by elements of A_1 (or we simply write $A = A_0[A_1]$). Then*

(i) *For any $d \in \mathbb{Z}$, $\mathscr{O}_X(d)$ is an invertible sheaf (i.e. locally free of rank 1).*

(ii) *If M and N are graded A-modules, then $\widetilde{M \otimes N} = \widetilde{M} \otimes \widetilde{N}$. In particular, $\widetilde{M(d)} = \widetilde{M} \otimes \mathscr{O}_X(d)$ and $\mathscr{O}_X(d_1) \otimes \mathscr{O}_X(d_2) = \mathscr{O}_X(d_1 + d_2)$.*

Proof. (i) We only need to prove that for any non-zero element $f \in A_1$, $\mathscr{O}_X(d)|_{D_+(f)} = \widetilde{A(d)_{(f)}}$ is locally free of rank 1. In fact, we have $A(d)_{(f)} = A(d)[\frac{1}{f}]_{\deg=0} = A[\frac{1}{f}]_{\deg=d} = A[\frac{1}{f}]_{\deg=0} \cdot f^d$ which is a free $A_{(f)} = A[\frac{1}{f}]_{\deg=0}$-module of rank 1.

(ii) Similarly, for any non-zero element $f \in A_1$, we have $(M \otimes_A N)_{(f)} = (M \otimes_A N)[\frac{1}{f}]_{\deg=0} = (M[\frac{1}{f}] \otimes_{A[\frac{1}{f}]} N[\frac{1}{f}])_{\deg=0} = M_{(f)} \otimes_{A_{(f)}} N_{(f)}$ since $\deg(f) = 1$. $\square$

4.4. **Noetherian condition of graded ring.**

Proposition 4.15. *If A is a Noetherian graded ring, then* $\mathrm{Proj}(A)$ *is a Noetherian topological space.*

Proposition 4.16. *If A is a graded ring, then A is Noetherian if and only if A_0 is Noetherian and A is a finitely generated A_0-algebra.*

Sketch of proof. "$\Leftarrow$": Hilbert basis theorem.

"$\Rightarrow$": In this case we know that $A_0 = A/A_+$ is Noetherian. Also, A_+ is a finitely generated ideal, let $x_1, \cdots, x_n$ be a set of generators of A_+. $\square$

Daily Exercise 4.5. Complete the above proof by proving that $A = A_0[x_1, \cdots, x_n]$ if $x_1, \cdots, x_n$ is a set of generators of A_+.

Weekly Exercise 4.6. Let A be a Noetherian graded ring with $A_0 = k$ be a field. Let M be a finitely generated graded A-module. For simplicity, assume $M_d = 0$ for $d < 0$. Write $A = k[x_1, \cdots, x_n]$ where for each i, x_i is a homogeneous element of A of degree d_i. Define the formal power series

$$Q_M(T) := \sum_{d=0}^{\infty} (\dim_k M_d) \cdot T^d \in \mathbb{Z}[[T]].$$

(1) Prove that

$$Q_M(T) = \frac{R_M(T)}{\prod_{i=1}^{n}(1 - T^{d_i})}$$

for some polynomial $R_M(T) \in \mathbb{Z}[T]$.

(2) If for any i we have $d_i = 1$, then there exists $P(T) \in \mathbb{Q}[T]$ such that for $d \gg 0$ we have $\dim_k M_d = P(d)$.

5. Lecture 5: August 7

5.1. **Quasi-coherent sheaf on a projective spectrum (continued).** Recall that last talk we defined the projective spectrum of a graded ring and quasi-coherent sheaf over a projective spectrum given by a graded module.

Note that in general, *not* all (quasi-)coherent sheaves over a project spectrum come from a graded modules. For example, if $A = \bigoplus_{n=0}^{\infty} A_n$ is a graded ring, for an integer $d \geq 2$, define the graded ring $A' := \bigoplus_{n=0}^{\infty} A'_n$ with $A'_n := A_{nd}$. Then we have $\mathrm{Proj}(A) \cong \mathrm{Proj}(A')$ as scheme, and the $\mathscr{O}_{\mathrm{Proj}(A)}(d)$ over $\mathrm{Proj}(A)$ is isomorphic to $\mathscr{O}_{\mathrm{Proj}(A')}(1)$ over $\mathrm{Proj}(A')$. The sheaf $\mathscr{O}_{\mathrm{Proj}(A)}(1)$ does *not* come from a graded A'-module.

In the remaining part of this section, we assume A is a Noetherian graded ring such that $A = A_0[A_1]$. We are going to recover the graded ring A (resp. the graded A-module M) from $(\mathrm{Proj}(A), \mathscr{O}_{\mathrm{Proj}(A)}(1))$ (resp. $\widetilde{M}$).

Recall that for an affine scheme and a (quasi-)coherent sheaf over it, it can be recovered by its global section. In the projective case, the question is how to recover the module and its graded piece.

5.1.1. Let $\mathscr{F}$ be a quasi-coherent sheaf on $X = \mathrm{Proj}(A)$. Denote $\mathscr{O} := \mathscr{O}_X$. For $n \in \mathbb{Z}$, define $\mathscr{F}(n) := \mathscr{F} \otimes \mathscr{O}(n)$. Consider $M_n := \Gamma(X, \mathscr{F}(n))$ the global section of $\mathscr{F}(n)$, then it is an A_0-module. For simplicity, assume that $M_n = 0$ for $n < 0$. Let $M := \bigoplus_{n=0}^{\infty} M_n$ which is a graded A-module.

Theorem 5.1. *We have the natural isomorphism $\widetilde{M} \cong \mathscr{F}$.*

Example 5.2. Let R be a ring, $A = R[X_0, \cdots, X_n]$, and $X = \mathrm{Proj}(A) = \mathbb{P}^n_R$. First we determine $\Gamma(X, \mathscr{O}(d))$. Note that $X = \bigcup_{i=0}^n D_+(X_i)$ is a finite cover of X by principal open subsets, and we have $D_+(X_i) \cong \mathrm{Spec}(A_{(X_i)})$ with $A_{(X_i)} = R[\frac{X_j}{X_i} \mid j \neq i]$. We have $\mathscr{O}(d)|_{D_+(X_i)} \cong \widetilde{A_{(X_i)} X_i^d}$ over $\mathrm{Spec}(A_{(X_i)})$, and $A_{(X_i)} X_i^d$ is a free $A_{(X_i)}$-module of rank 1. Therefore

$$\Gamma(D_+(X_i), \mathscr{O}(d)) = A_{(X_i)} X_i^d = A\left[\frac{1}{X_i}\right]_{\deg=0} X^d$$
$$= \left\{\text{degree } d \text{ homogeneous polynomial in } A\left[\frac{1}{X_i}\right] = R\left[X_0, \cdots, X_n, \frac{1}{X_i}\right]\right\}.$$

The $\Gamma(X, \mathscr{O}(d))$ is the glueing of $\Gamma(D_+(X_i), \mathscr{O}(d))$, $i = 0, \cdots, n$, hence we obtain

$$\Gamma(X, \mathscr{O}(d)) = \{\text{degree } d \text{ homogeneous polynomial in } A = R[X_0, \cdots, X_n]\} = A_d.$$

Therefore $\bigoplus_{d=0}^{\infty} \Gamma(X, \mathscr{O}(d)) = \bigoplus_{d=0}^{\infty} A_d = A = R[X_0, \cdots, X_n]$.

Sketch of proof of Theorem 5.1. We may write X as a finite union of principal open subsets $X = \bigcup_x D_+(x)$ such that every x is in A_1. We have $\widetilde{M}|_{D_+(x)} \cong \widetilde{M_{(x)}}$ on $D_+(x) \cong \mathrm{Spec}(A_{(x)})$. On the other hand, since $\mathscr{F}$ is quasi-coherent, we have $\mathscr{F}|_{D_+(x)} \cong \widetilde{N_{(x)}}$ where $N_{(x)} := \Gamma(D_+(x), \mathscr{F})$. Now we only need to construct the natural isomorphism $M_{(x)} \cong \Gamma(D_+(x), \mathscr{F})$ as $A_{(x)}$-modules.

Note that $M_{(x)} = \sum_{n=0}^{\infty} M_n \cdot x^{-n}$ and $\Gamma(D_+(x), \mathscr{F}(n)) = \Gamma(D_+(x), \mathscr{F}) \cdot x^n$. Hence the natural restriction map $\Gamma(X, \mathscr{F}(n)) \to \Gamma(D_+(x), \mathscr{F}(n))$ induces the natural map $M_n \cdot x^{-n} \to \Gamma(D_+(x), \mathscr{F})$ and summing up together we obtain $M_{(x)} \to \Gamma(D_+(x), \mathscr{F})$. It remains to prove that this map is an isomorphism by using Lemma 5.3:

Injectivity. For any $n \geq 0$ and any $s \in \Gamma(X, \mathscr{F}(n))$, if $sx^{-n} = 0$ in $\Gamma(D_+(x), \mathscr{F})$ then there exists $m \geq 0$ such that $x^m s = 0$ in $\Gamma(X, \mathscr{F}(n+m))$.

Surjectivity. For any $s \in \Gamma(D_+(x), \mathscr{F})$, there exists $n \geq 0$ such that $x^n s \in \Gamma(D_+(x), \mathscr{F}(n))$ can be extended to a section of $\Gamma(X, \mathscr{F}(n))$. □

Lemma 5.3 ("Ugly lemma"). *Let X be a Noetherian separated scheme. Let $\mathscr{L}$ be an invertible sheaf over X and $\mathscr{F}$ be a quasi-coherent sheaf over X. Fix an element $\ell \in \Gamma(X, \mathscr{L})$. Let X_ℓ be the* non-zero *locus of ℓ, i.e. consists of $x \in X$ such that the image of $\ell_x \in \mathscr{L}_x \cong \mathscr{O}_{X,x}$ in $k_x := \mathscr{O}_{X,x}/\mathfrak{m}_x$ is non-zero.*

(1) If $s \in \Gamma(X, \mathscr{F})$ is such that $s|_{X_\ell} = 0$, then there exists $n \geq 0$ such that $s \otimes \ell^n = 0$ in $\Gamma(X, \mathscr{F} \otimes \mathscr{L}^n)$.

(2) For an element $s \in \Gamma(X_\ell, \mathscr{F})$, there exists $n \geq 0$ and $t \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^n)$ such that $t|_{X_\ell} = s \otimes (\ell|_{X_\ell})^n$.

Daily Exercise 5.1. Read the definition of a separated scheme. Prove that if X is a separated scheme and if U and V are affine open subsets of X, then $U \cap V$ is also an affine open subset of X.

Proof of Lemma 5.3. Let $X = \bigcup_i \mathrm{Spec}(A_i)$ be a finite cover of X by affine schemes such that for each i, $\mathscr{L}|_{\mathrm{Spec}(A_i)}$ is free of rank 1, say $\mathscr{L}|_{\mathrm{Spec}(A_i)} \cong \widetilde{A_i \cdot \ell_i}$. For each i, there exists a unique $a_i \in A_i$ such that $\ell|_{\mathrm{Spec}(A_i)} = a_i \cdot \ell_i$, and we have $X_\ell \cap \mathrm{Spec}(A_i) = \mathrm{Spec}(A_i[\frac{1}{a_i}])$, hence $X_\ell = \bigcup_i \mathrm{Spec}(A_i[\frac{1}{a_i}])$. Say $\mathscr{F}|_{\mathrm{Spec}(A_i)} \cong \widetilde{M_i}$ for some A_i-module M_i.

(1) In this case for each i, the image of $s_i := s|_{\mathrm{Spec}(A_i)} \in M_i$ in $M_i[\frac{1}{a_i}]$ is zero, hence there exists some $n_i \geq 0$ such that $a_i^{n_i} s_i = 0$. Take $n = \max_i(n_i)$, then it's easy to see that $s \otimes \ell^n = 0$.

(2) For each i, the $s|_{\mathrm{Spec}(A_i[\frac{1}{a_i}])} \in M_i[\frac{1}{a_i}]$ is of form $t_i/a_i^{n_i}$ for some $t_i \in M_i$ and $n_i \geq 0$. Take $n = \max_i(n_i)$, then $t_i a_i^{n-n_i} \otimes \ell_i^n \in \Gamma(\mathrm{Spec}(A_i), \mathscr{F} \otimes \mathscr{L}^n)$ and they can be glued together to obtain an element $t \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^n)$, which satisfies the desired condition. □

5.1.2. *Question.* If M is a graded A-module such that $\widetilde{M}$ is a coherent sheaf over $\mathrm{Proj}(A)$. Is M a finitely generated A-module?

The following is a weak answer to the above question:

Theorem 5.4. *If $\mathscr{F}$ is a coherent sheaf over $X = \mathrm{Proj}(A)$, then there exists $n \geq 0$ such that $\mathscr{F}(n)$ is generated by global sections, i.e. for any $x \in X$, the image of the map $\Gamma(X, \mathscr{F}(n)) \to \mathscr{F}(n)_x$ generates $\mathscr{F}(n)_x$ as an $\mathscr{O}_{X,x}$-module.*

Idea of proof. For $x \in A_1$, the $\mathscr{F}(D_+(x))$ is generated by finitely many elements as $A_{(x)}$-module. Lift them to global sections by Lemma 5.3. $\square$

5.1.3. *Ample line bundle.*

Definition 5.5. An invertible sheaf $\mathscr{L}$ over a scheme X is called *ample* if for any coherent sheaf $\mathscr{F}$ over X, there exists $n \geq 0$ such that $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections.

There are a few important results regarding ample line bundle in algebraic geometry:

- A scheme is quasi-projective (i.e. admits an embedding to a projective space) if and only if there exists an ample line bundle over it.
- A scheme is quasi-affine (i.e. admits an open embedding to an affine space) if and only if its structure sheaf if ample.
- The pullback of an ample line bundle under an affine morphism (i.e. preimage of an affine open subset is affine) is also ample.

Weekly Exercise 5.2. Read the definition of ample line bundle and its application to project embeddings.

5.2. **Čech cohomology.** Let X be a topological space and $\mathscr{F}$ be a sheaf of abelian groups over X. Let $\mathscr{U}$ be an open cover of X. Assume that $\mathscr{U} = \{U_i\}_{i \in I}$ with I a total order set. if $i_0 < \cdots < i_p$ are elements of I, let $U_{i_0 \cdots i_p} := \bigcap_{i=0}^{p} U_{i_k}$.

Define $C^p(\mathscr{U}, \mathscr{F}) := \prod_{i_0 < \cdots < i_p} \Gamma(U_{i_0 \cdots i_p}, \mathscr{F})$ if $p \geq 0$, and $C^p(\mathscr{U}, \mathscr{F}) := 0$ if $p < 0$. Define the map $d^p : C^p(\mathscr{U}, \mathscr{F}) \to C^{p+1}(\mathscr{U}, \mathscr{F})$ by

$$d^p : \prod_{i_0 < \cdots < i_p} \Gamma(U_{i_0 \cdots i_p}, \mathscr{F}) \to \prod_{i_0 < \cdots < i_{p+1}} \Gamma(U_{i_0 \cdots i_{p+1}}, \mathscr{F}),$$
$$(s_{i_0 \cdots i_p})_{i_0 < \cdots < i_p} \mapsto \left(\sum_{k=0}^{p+1} (-1)^k s_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}} \right)_{i_0 < \cdots < i_{p+1}}$$

if $p \geq 0$, and $d^p := 0$ if $p < 0$. Here $\widehat{i_k}$ means omitting the term i_k.

Daily Exercise 5.3. Prove that $d^{p+1} \circ d^p = 0$ for any $p \in \mathbb{Z}$, and $\ker(d^0) = \Gamma(X, \mathscr{F})$.

Therefore the maps $d^\bullet$ make $C^\bullet(\mathscr{U}, \mathscr{F})$ a cochain complex of abelian groups. We define $H^p_{\mathscr{U}}(X, \mathscr{F}) := H^p(C^\bullet(\mathscr{U}, \mathscr{F})) := \ker(d^p)/\operatorname{Im}(d^{p-1})$ for any $p \geq 0$. Then $H^0_{\mathscr{U}}(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

Example 5.6. Let $X = \mathbb{RP}^1$ with an open cover $\mathscr{U} = \{\{x \neq \infty\}, \{x \neq 0\}\}$. Consider the constant sheaf $\underline{\mathbb{Z}}$ over X. Then we have $H^0_{\mathscr{U}}(X, \underline{\mathbb{Z}}) = H^1_{\mathscr{U}}(X, \underline{\mathbb{Z}}) = \mathbb{Z}$.

If $\mathscr{V}$ is a refinement of $\mathscr{U}$, then there is a natural morphism $C^\bullet(\mathscr{U}, \mathscr{F}) \to C^\bullet(\mathscr{V}, \mathscr{F})$ between cochain complexes, and we obtain natural maps $H^p_{\mathscr{U}}(X, \mathscr{F}) \to H^p_{\mathscr{V}}(X, \mathscr{F})$ between cohomology groups.

Definition 5.7. The *Čech cohomology* of $\mathscr{F}$ over X of dimension p is

$$\check{H}^p(X, \mathscr{F}) := \varinjlim_{\mathscr{U}} H^p_{\mathscr{U}}(X, \mathscr{F}).$$

Theorem 5.8. *Let X be a Noetherian separated scheme and $\mathscr{F}$ be a quasi-coherent sheaf over X.*

(1) *Let $\mathscr{U}$ be a finite affine open cover of X. Then for any p, $H^p_{\mathscr{U}}(X, \mathscr{F})$ is independent of the choice of the affine open cover $\mathscr{U}$. (Therefore this is equal to the Čech cohomology $\check{H}^p(X, \mathscr{F})$.)*

(2) *If X is an affine scheme then $\check{H}^p(X, \mathscr{F}) = 0$ for all $p \geq 1$.*

(3) *In general $\check{H}^p(X, \mathscr{F}) = 0$ for $p \gg 0$ independent of $\mathscr{F}$.*

(4) *Let $\mathscr{U}$ be a finite affine open cover of X. Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves over X. Then it induces, for each $p \geq 0$, a natural (i.e. functorial) map $H^p_{\mathscr{U}}(X, \mathscr{F}_3) \to H^{p+1}_{\mathscr{U}}(X, \mathscr{F}_1)$, which fits into the following natural (i.e. functorial) long exact sequences*

of cohomology groups:

$$\begin{tikzcd} 0 \arrow[r] & H^0_{\mathscr{U}}(X,\mathscr{F}_1) \arrow[r] & H^0_{\mathscr{U}}(X,\mathscr{F}_2) \arrow[r] & H^0_{\mathscr{U}}(X,\mathscr{F}_3) \arrow[dll, out=0, in=180] \\ & H^1_{\mathscr{U}}(X,\mathscr{F}_1) \arrow[r] & H^1_{\mathscr{U}}(X,\mathscr{F}_2) \arrow[r] & H^1_{\mathscr{U}}(X,\mathscr{F}_3) \arrow[dll, out=0, in=180] \\ & H^2_{\mathscr{U}}(X,\mathscr{F}_1) \arrow[r] & H^2_{\mathscr{U}}(X,\mathscr{F}_2) \arrow[r] & H^2_{\mathscr{U}}(X,\mathscr{F}_3) \arrow[r] & \cdots. \end{tikzcd}$$

(Therefore, by taking direct limit $\varinjlim_{\mathscr{U}}$, we obtain the same result for Čech cohomology groups $\check{H}^p(X,-)$.)

(5) *If A is a Noetherian graded ring, $X = \mathrm{Proj}(A)$ and $\mathscr{F}$ is a coherent sheaf over X, then for each $p \geq 0$, $\check{H}^p(X,\mathscr{F})$ is a finitely generated A_0-module.*

Therefore, if A is a Noetherian graded ring, then for a coherent sheaf $\mathscr{F}$ over $\mathrm{Proj}(A)$, we can define $\chi(\mathscr{F}) := \sum_{p\geq 0}(-1)^p[\check{H}^p(X,\mathscr{F})] \in \mathbb{K}'(A_0)$, which induces a well-defined homomorphism of abelian groups $\chi : \mathbb{K}'(\mathrm{Proj}(A)) \to \mathbb{K}'(A_0)$.

6. Lecture 6: August 8

Recall that for a Noetherian separated scheme X, we defined functors

$$\check{H}^i(X,-) : (\text{quasi-coherent sheaves on } X) \to \mathbf{Ab}$$

for $i \geq 0$. In this talk we are going to prove Theorem 5.8.

We may also consider the relative version of sheaf cohomology: if $f : X \to Y$ is a morphism between Noetherian separated schemes, then it induces a functor $f_* : \mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_Y\text{-}\mathbf{Mod}$ by $f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$ for open subset V of Y. We want to define $R^if_* : \mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_Y\text{-}\mathbf{Mod}$ for each $i \geq 0$, such that

- $R^0f_* = f_*$;
- a short exact sequence of sheaves induce long exact sequence of R^if_*'s similar to Theorem 5.8(4);
- if f is an affine morphism (i.e. the preimage of an affine open subset of Y is an affine open subset of X), then $R^if_* = 0$ for any $i \geq 1$.

Such a cohomology theory is unique if it exists.

Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two above morphisms. Then we have $(gf)_* = g_*f_*$. But the $R^i(gf)_*$, R^ig_* and R^if_* don't satisfy the similar relation in general. To describe the correct relation for them, we'd better to use K-groups.

6.1. Čech complex resolution.

Same as §5.2, let X be a topological space and $\mathscr{F}$ be a sheaf of abelian groups over X. Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open cover of X with I a total order set. If $i_0 < \cdots < i_p$ are elements of I, denote $j_{i_0\cdots i_p} : U_{i_0\cdots i_p} \hookrightarrow X$ the natural inclusion. We define a sheaf $\mathscr{C}^p(\mathscr{U},\mathscr{F})$ on X by the following way:

$$\mathscr{C}^p(\mathscr{U},\mathscr{F}) := \prod_{i_0<\cdots<i_p} (j_{i_0\cdots i_p})_*(\mathscr{F}|_{U_{i_0\cdots i_p}}),$$

or equivalently, if U is an open subset of X,

$$\Gamma(U,\mathscr{C}^p(\mathscr{U},\mathscr{F})) := \prod_{i_0<\cdots<i_p} \Gamma(U\cap U_{i_0\cdots i_p},\mathscr{F}) = C^p(\mathscr{U}\cap U,\mathscr{F}|_U).$$

Here $\mathscr{U}\cap U := \{U_i\cap U\}_{i\in I}$ is an open cover of U. In particular, we have $\Gamma(X,\mathscr{C}^p(\mathscr{U},\mathscr{F})) = C^p(\mathscr{U},\mathscr{F})$. Therefore, we may use the map $d^p : C^p(\mathscr{U},\mathscr{F}) \to C^{p+1}(\mathscr{U},\mathscr{F})$ defined in §5.2 to define the morphism $d^p : \mathscr{C}^p(\mathscr{U},\mathscr{F}) \to \mathscr{C}^{p+1}(\mathscr{U},\mathscr{F})$ of sheaves on X.

Daily Exercise 6.1. Prove that $d^{p+1}\circ d^p = 0$ for any $p \in \mathbb{Z}$, and $\ker(d^0) = \mathscr{F}$.

Therefore the morphisms $d^\bullet$ makes $\mathscr{C}^\bullet(\mathscr{U},\mathscr{F})$ a cochain complex of sheaves over X.

Theorem 6.1. *The complex $\mathscr{C}^\bullet(\mathscr{U},\mathscr{F})$ is exact except at degree* 0.

To prove this we need to introduce some concepts in homological algebra.

6.1.1. *Homotopy of morphisms of complexes.* Let $\mathscr{A}$ be an abelian category (e.g. the category of R-modules for a ring R, or the category of sheaves of abelian groups over a topological space X). A cochain complex (or simply called a complex) $C^\bullet = (C^\bullet, d_C^\bullet)$ is a sequence $(C^p)_{p\in\mathbb{Z}}$ of objects in $\mathscr{A}$ and a sequence $(d_C^p : C^p \to C^{p+1})_{p\in\mathbb{Z}}$ of morphisms in $\mathscr{A}$ such that $d^{p+1} \circ d^p = 0$ for any $p \in \mathbb{Z}$. Define the cohomology of a complex $C^\bullet$ at p to be $H^p(C^\bullet) := \ker(d_C^p)/\operatorname{Im}(d_C^{p-1})$. For simplicity, in the following, unless stated explicitly, we only consider complexes $C^\bullet$ such that $C^p = 0$ for $p < 0$.

A chain map $f^\bullet : C^\bullet \to D^\bullet$ is a sequence $(f^p : C^p \to D^p)_{p\in\mathbb{Z}}$ of morphisms in $\mathscr{A}$ such that $d_D^p \circ f^p = f^{p+1} \circ d_C^p$ for any $p \in \mathbb{Z}$:

$$\begin{tikzcd} \cdots \arrow[r] & C^p \arrow[r, "d_C^p"] \arrow[d, "f^p"'] \arrow[dr, phantom, "\circlearrowleft"] & C^{p+1} \arrow[r] \arrow[d, "f^{p+1}"] & \cdots \\ \cdots \arrow[r] & D^p \arrow[r, "d_D^p"'] & D^{p+1} \arrow[r] & \cdots . \end{tikzcd}$$

The chain map $f^\bullet$ induces maps of cohomology objects $H^p(f^\bullet) : H^p(C^\bullet) \to H^p(D^\bullet)$ for any $p \in \mathbb{Z}$. The chain map $f^\bullet$ it is called homotopic to zero, denoted by $f^\bullet \sim 0$, if there exists a sequence $(h^p : C^p \to D^{p-1})_{p\in\mathbb{Z}}$ of morphisms such that $f^p = h^{p+1} \circ d_C^p + d_D^{p-1} \circ h^p$ for any $p \in \mathbb{Z}$:

$$\begin{tikzcd} & \cdots \arrow[r] & C^p \arrow[r, "d_C^p"] \arrow[dl, "h^p"'] \arrow[d, "f^p"] & C^{p+1} \arrow[r] \arrow[dl, "h^{p+1}"] & \cdots \\ \cdots \arrow[r] & D^{p-1} \arrow[r, "d_D^{p-1}"'] & D^p \arrow[r] & \cdots . \end{tikzcd}$$

Two chain maps $f^\bullet$ and $g^\bullet$ are called homotopic to each other, denoted by $f^\bullet \sim g^\bullet$, if $f^\bullet - g^\bullet \sim 0$.

Daily Exercise 6.2. If $f^\bullet \sim g^\bullet$, prove that they induce the same maps of cohomology objects. (For simplicity you can assume that $\mathscr{A}$ is the category of R-modules.)

6.1.2. *Proof of Theorem 6.1.* Define the complex $\mathscr{D}^\bullet(\mathscr{U},\mathscr{F})$ by $\mathscr{D}^p(\mathscr{U},\mathscr{F}) := \mathscr{C}^p(\mathscr{U},\mathscr{F})$ for $p \geq 0$, and $\mathscr{D}^{-1}(\mathscr{U},\mathscr{F}) := \mathscr{F}$ with the morphism $\mathscr{D}^{-1}(\mathscr{U},\mathscr{F}) \to \mathscr{D}^0(\mathscr{U},\mathscr{F})$ defined to be the natural morphism $\mathscr{F} \to \prod_{i\in I}(j_i)_*(\mathscr{F}|_{U_i})$. Then the Theorem 6.1 is equivalent to that the complex $\mathscr{D}^\bullet(\mathscr{U},\mathscr{F})$ is exact. This is equivalent to that the complex $\mathscr{D}^\bullet(\mathscr{U},\mathscr{F})_x$ is exact for any $x \in X$. We prove this by constructing a homotopy $h_x^\bullet$ connecting $\mathrm{id} \sim 0 : \mathscr{D}^\bullet(\mathscr{U},\mathscr{F})_x \to \mathscr{D}^\bullet(\mathscr{U},\mathscr{F})_x$. In fact, if we fix a U_j containing x for some $j \in I$, then for $p \geq 1$ we can define

$$\begin{aligned} h_x^p : \mathscr{D}^p(\mathscr{U},\mathscr{F})_x &\to \mathscr{D}^{p-1}(\mathscr{U},\mathscr{F})_x, \\ (s_{i_0\cdots i_p})_{i_0<\cdots<i_p} &\mapsto (s_{ji_0\cdots i_{p-1}})_{i_0<\cdots<i_{p-1}}, \end{aligned}$$

here $s_{ji_0\cdots i_{p-1}} := 0$ if j is equal to one of $i_0, \cdots, i_{p-1}$, and $s_{ji_0\cdots i_{p-1}} := (-1)^{k-1} s_{i_0\cdots i_{k-1} j i_k \cdots i_{p-1}}$ if $i_{k-1} < j$ and $j < i_k$. For $p = 0$ we define

$$\begin{aligned} h_x^0 : \mathscr{D}^0(\mathscr{U},\mathscr{F})_x &\to \mathscr{F}_x, \\ (s_i)_{i\in I} &\mapsto s_j. \end{aligned}$$

Daily Exercise 6.3. Prove that the $h_x^\bullet$ is a homotopy connecting $\mathrm{id} \sim 0 : \mathscr{D}^\bullet(\mathscr{U},\mathscr{F})_x \to \mathscr{D}^\bullet(\mathscr{U},\mathscr{F})_x$.

6.1.3. In the following we let X be a Noetherian separated scheme.

Proof of Theorem 5.8(2). We prove that if X is affine then for any open cover $\mathscr{U}$ of X, $H^p_{\mathscr{U}}(X,\mathscr{F}) = 0$ for any $p \geq 1$, and in particular, $\check{H}^p(X,\mathscr{F}) = 0$ for any $p \geq 1$.

This is a corollary of Theorem 6.1, note that the functor

$$\Gamma(X,-) : (\text{quasi-coherent sheaves on } X) \to \mathbf{Ab}$$

is exact (since X is affine), and $\Gamma(X,\mathscr{C}^p(\mathscr{U},\mathscr{F})) = C^p(\mathscr{U},\mathscr{F})$. □

Proof of Theorem 5.8(4). Let $\mathscr{U}$ be a finite affine open cover of X. Then it's easy to see that for any $p \geq 0$, the functor $\mathscr{C}^p(\mathscr{U},-)$ is an exact functor (note that a sequence of quasi-coherent sheaves is exact if and only if for an *affine* cover $\{U_i\}$, the sequence of their sections on U_i is exact for all i). Hence the short exact sequence

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

of quasi-coherent sheaves on X induces a short exact sequence

$$0 \to \mathscr{C}^\bullet(\mathscr{U}, \mathscr{F}_1) \to \mathscr{C}^\bullet(\mathscr{U}, \mathscr{F}_2) \to \mathscr{C}^\bullet(\mathscr{U}, \mathscr{F}_3) \to 0$$

of complexes of quasi-coherent sheaves on X. On the other hand, if U is an *affine* open subset, then $\Gamma(U, -)$ is an exact functor, therefore it's easy to see that by taking global section of the above short exact sequence, we obtain a short exact sequence

$$0 \to C^\bullet(\mathscr{U}, \mathscr{F}_1) \to C^\bullet(\mathscr{U}, \mathscr{F}_2) \to C^\bullet(\mathscr{U}, \mathscr{F}_3) \to 0$$

of complexes of abelian groups. Now the following "long snake lemma" gives the long exact sequence of $H^p_{\mathscr{U}}(X, -)$. □

Proposition 6.2 ("Long snake lemma"). *Let $\mathscr{A}$ be an abelian category and let $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ be a short exact sequence of complexes of objects in $\mathscr{A}$. Then it induces, for each $p \geq 0$, a natural (i.e. functorial) map $H^p(Z^\bullet) \to H^{p+1}(X^\bullet)$, which fits into the following natural (i.e. functorial) long exact sequences of cohomology groups:*

$$\begin{tikzcd}
0 \arrow[r] & H^0(X^\bullet) \arrow[r] & H^0(Y^\bullet) \arrow[r] & H^0(Z^\bullet) \arrow[dll] \\
 & H^1(X^\bullet) \arrow[r] & H^1(Y^\bullet) \arrow[r] & H^1(Z^\bullet) \arrow[dll] \\
 & H^2(X^\bullet) \arrow[r] & H^2(Y^\bullet) \arrow[r] & H^2(Z^\bullet) \arrow[r] & \cdots.
\end{tikzcd}$$

Proposition 6.3 (Snake lemma). *Let $\mathscr{A}$ be an abelian category and let*

$$\begin{tikzcd}
 & X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z'
\end{tikzcd}$$

be a commutative diagram of exact sequences in $\mathscr{A}$. Then it induces a natural morphism $\ker(h) \to \operatorname{coker}(f)$ which fits into the following long exact sequence:

$$\ker(f) \to \ker(g) \to \ker(h) \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h).$$

Moreover, if $X \to Y$ is injective, then $\ker(f) \to \ker(g)$ is injective, also, if $Y' \to Z'$ is surjective, then $\operatorname{coker}(g) \to \operatorname{coker}(h)$ is surjective.

Daily Exercise 6.4. Use snake lemma to prove the "long snake lemma". For simplicity you may assume $\mathscr{A}$ is the category of R-modules.

Proof of Theorem 5.8(1). When $p = 0$ this holds, since it is equal to $\Gamma(X, \mathscr{F})$.

Let $\mathscr{W}$ be an auxiliary finite affine open cover of X. Then there is a natural injective map $\mathscr{F} \to \mathscr{C}^0(\mathscr{W}, \mathscr{F})$ of quasi-coherent sheaves on X. Let $\mathscr{G} := \operatorname{coker}(\mathscr{F} \to \mathscr{C}^0(\mathscr{W}, \mathscr{F}))$, so that

$$0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{W}, \mathscr{F}) \to \mathscr{G} \to 0$$

is a short exact sequence of quasi-coherent sheaves on X. The Theorem 5.8(2) (as well as the definition of $\mathscr{C}^0(\mathscr{W}, \mathscr{F})$) tells us that $H^p_{\mathscr{U}}(X, \mathscr{C}^0(\mathscr{W}, \mathscr{F})) = 0$ for any $p \geq 1$, hence by Theorem 5.8(4) we obtain the exact sequence

$$0 \to \Gamma(X, \mathscr{F}) \to C^0(\mathscr{W}, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to H^1_{\mathscr{U}}(X, \mathscr{F}) \to 0,$$

as well as $H^{p-1}_{\mathscr{U}}(X, \mathscr{G}) \xrightarrow{\sim} H^p_{\mathscr{U}}(X, \mathscr{F})$ for $p \geq 2$. Therefore $H^1_{\mathscr{U}}(X, \mathscr{F}) = \operatorname{coker}\left(C^0(\mathscr{W}, \mathscr{F}) \to \Gamma(X, \mathscr{G})\right)$ is independent of the choice of $\mathscr{U}$, and for $p \geq 2$, by induction on p we obtain that $H^p_{\mathscr{U}}(X, \mathscr{F})$ is independent of the choice of $\mathscr{U}$, since $H^{p-1}_{\mathscr{U}}(X, \mathscr{G})$ is independent of the choice of $\mathscr{U}$. □

Proof of Theorem 5.8(3). Let $\mathscr{U}$ be a finite affine open cover of X. Then $\mathscr{C}^p(\mathscr{U}, \mathscr{F}) = 0$ for $p \geq \#\mathscr{U}$, in particular $C^p(\mathscr{U}, \mathscr{F}) = 0$ for $p \geq \#\mathscr{U}$. Therefore it follows from Theorem 5.8(1). □

7. Lecture 7: August 12

7.1. Cohomology groups on project schemes.

7.1.1. Let R be a ring, $A = R[X_0, \cdots, X_n]$, $X = \mathrm{Proj}(A) = \mathbb{P}^n_R$. Recall that for $d \in \mathbb{Z}$ we have defined sheaf $\mathscr{O}(d) = \mathscr{O}_X(d)$ on X, by $\mathscr{O}(d) := \widetilde{A(d)}$, with $\mathscr{O}(d)|_{D_+(X_i)} = \widetilde{A[\frac{1}{X_i}]}_{\deg=d}$.

Theorem 7.1. *We have*

$$H^p(X, \mathscr{O}(d)) = \begin{cases} R[X_0, \cdots, X_n]_{\deg=d}, & \text{if } p = 0, \\ \left(X_0^{-1} \cdots X_n^{-1} \cdot R[X_0^{-1}, \cdots, X_n^{-1}]\right)_{\deg=d}, & \text{if } p = n, \\ 0, & \text{otherwise.} \end{cases}$$

Here we define that $\deg(X_i^{-1}) = -1$ *for* $0 \le i \le n$. *Moreover there is a perfect duality*

$$H^i(X, \mathscr{O}(d)) \times H^{n-i}(X, \mathscr{O}(-1-n-d)) \to H^n(X, \mathscr{O}(-1-n)) \cong R.$$

Proof. It's already known when $p = 0$. When $p = n$, we take $\mathscr{U} = \{D_+(X_i)\}_{i=0}^n$ to be a finite affine open cover of X. Then we have

$$H^n(X, \mathscr{O}(d)) = \frac{C^n(\mathscr{U}, \mathscr{O}(d))}{\mathrm{Im}\left(C^{n-1}(\mathscr{U}, \mathscr{O}(d)) \xrightarrow{d^{n-1}} C^n(\mathscr{U}, \mathscr{O}(d))\right)},$$

where

$$\begin{aligned} C^{n-1}(\mathscr{U}, \mathscr{O}(d)) &= \bigoplus_{i=0}^n \Gamma\left(D_+(X_0 \cdots \widehat{X_j} \cdots X_n), \mathscr{O}(d)\right) \\ &= \bigoplus_{i=0}^n \left\{ f \in R[X_0^{\pm 1}, \cdots, X_n^{\pm 1}]_{\deg=d} \;\middle|\; \text{the degree of } X_i \text{ in } f \text{ is} \ge 0 \right\}, \end{aligned}$$

and

$$C^n(\mathscr{U}, \mathscr{O}(d)) = \Gamma\left(D_+(X_0 \cdots X_n), \mathscr{O}(d)\right) = R[X_0^{\pm 1}, \cdots, X_n^{\pm 1}]_{\deg=d},$$

and the map d^{n-1} is defined by $(f_i)_{i=0}^n \mapsto \sum_{i=0}^n (-1)^i f_i$. Now it's easy to see that

$$\mathrm{Im}(d^{n-1}) = \left\{ f \in R[X_0^{\pm 1}, \cdots, X_n^{\pm 1}]_{\deg=d} \;\middle|\; \begin{array}{l} \text{for any monomial in } f, \text{ there exists } i \\ \text{such that the degree of } X_i \text{ in it is} \ge 0 \end{array} \right\},$$

therefore $H^n(X, \mathscr{O}(d)) = \left(X_0^{-1} \cdots X_n^{-1} \cdot R[X_0^{-1}, \cdots, X_n^{-1}]\right)_{\deg=d}$.

Thus we have proved the duality for $i = 0, n$.

For $1 \le p \le n-1$ we use induction on n. Consider $\mathscr{F} := \bigoplus_{d \in \mathbb{Z}} \mathscr{O}(d)$ which is a quasi-coherent sheaf over X. Take $X_n \in \Gamma(X, \mathscr{O}(1))$, then we can define a multiply-by-X_n map $\mathscr{F}(-1) \xrightarrow{\times X_n} \mathscr{F}$ which fits into the following short exact sequence:

$$0 \to \mathscr{F}(-1) \xrightarrow{\times X_n} \mathscr{F} \to \mathscr{F}/X_n\mathscr{F} \to 0.$$

If we let $\alpha_n : Z(X_n) \hookrightarrow \mathbb{P}^n_R$ to be the natural closed embedding, and let $\mathscr{F}_n := \bigoplus_{d \in \mathbb{Z}} \mathscr{O}_{Z(X_n)}(d)$, then we have $Z(X_n) \cong \mathbb{P}^{n-1}_R$ and $\mathscr{F}/X_n\mathscr{F} = (\alpha_n)_*\mathscr{F}_n$, i.e.

$$0 \to \mathscr{F}(-1) \xrightarrow{\times X_n} \mathscr{F} \to (\alpha_n)_*\mathscr{F}_n \to 0$$

is exact. It induces a long exact sequence of cohomology groups:

$$\cdots \to H^{p-1}(X, (\alpha_n)_*\mathscr{F}_n) \to H^p(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^p(X, \mathscr{F}) \to H^p(X, (\alpha_n)_*\mathscr{F}_n) \to \cdots.$$

Note that $H^p(X, (\alpha_n)_*\mathscr{F}_n) = H^p(Z(X_n), \mathscr{F}_n)$ which is zero when $1 \le p \le n-2$ by induction hypothesis, the map $H^p(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^p(X, \mathscr{F})$ is injective if $2 \le p \le n-1$, and is surjective if $1 \le p \le n-2$. We claim this it is also injective if $p = 1$ and is surjective if $p = n-1$. When $p = 1$, we have the following long exact sequence:

$$0 \to A(-1) \xrightarrow{\times X_n} A \to A/X_nA \to H^1(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^1(X, \mathscr{F}),$$

note that $A \to A/X_nA$ is surjective, hence $H^1(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^1(X, \mathscr{F})$ is injective. When $p = n-1$, we have the following long exact sequence:

$$H^{n-1}(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^{n-1}(X, \mathscr{F}) \to H^{n-1}(Z(X_n), \mathscr{F}_n) \xrightarrow{\delta} H^n(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^n(X, \mathscr{F}) \to 0.$$

We may use the duality for $i = 0$ for the last three terms to show that δ is in fact injective. Hence $H^{n-1}(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^{n-1}(X, \mathscr{F})$ is surjective.

In conclusion we know that when $1 \le p \le n-1$, the map $H^p(X, \mathscr{F}(-1)) \xrightarrow{\times X_n} H^p(X, \mathscr{F})$ is an isomorphism, hence $H^p(X, \mathscr{F}) = H^p(X, \mathscr{F})[\frac{1}{X_n}] = H^p(D_+(X_n), \mathscr{F}) = 0$, since $D_+(X_n)$ is affine. $\square$

Daily Exercise 7.1. Read and complete the unclear parts of the proof.

Remark 7.2. By Serre duality, we actually have

$$H^n(X, \mathscr{F})^\vee := \operatorname{Hom}(H^n(X, \mathscr{F}), R) \cong \operatorname{Hom}(\mathscr{F}, \mathscr{O}(-d-n-1))$$

for any coherent sheaf $\mathscr{F}$ on X.

7.1.2. Now we state an application of the above result to general graded rings. Let A be a Noetherian graded ring satisfying $A = A_0[A_1]$. Let $X = \operatorname{Proj}(A)$. Let $\mathscr{F}$ be a coherent sheaf on X.

Theorem 7.3 (Serre)**.** (1) *For any $p \ge 0$, $H^p(X, \mathscr{F})$ is a finitely generated A_0-module, i.e. Theorem 5.8(5) holds.*

(2) *There exists some $m \gg 0$ such that $H^p(X, \mathscr{F}(m)) = 0$ for any $p \ge 1$.*

Proof. First we may reduce it to projective space case. Note that since A is Noetherian and $A = A_0[A_1]$, if we write $R = A_0$, then A may be realized as a quotient of $R[X_0, \cdots, X_n]$ for some n by a graded ideal I. The map $R[X_0, \cdots, X_n] \twoheadrightarrow A$ gives a closed embedding $i : X \hookrightarrow \mathbb{P}^n_R$ with $X \cong Z(I)$. This is an affine map which satisfies $i^*\mathscr{O}_{\mathbb{P}^n_R}(1) = \mathscr{O}_X(1)$. Therefore $H^p(X, \mathscr{F}(d)) = H^p(\mathbb{P}^n_R, (i_*\mathscr{F})(d))$ for any $d \in \mathbb{Z}$ and any $p \ge 0$. Hence in the following we may assume $X = \mathbb{P}^n_R$.

(1) By Theorem 5.4 we know that there exists some $d \gg 0$ such that $\mathscr{F}(d)$ is generated by global sections. We can choose finitely many of these sections, since A is Noetherian. Note that $\Gamma(X, \mathscr{F}(d)) = \operatorname{Hom}(\mathscr{O}_X, \mathscr{F}(d))$, these sections gives a surjective morphism of sheaves $\mathscr{O}_X^N \twoheadrightarrow \mathscr{F}(d)$, hence $\mathscr{O}_X(-d)^N \twoheadrightarrow \mathscr{F}$. Let $\mathscr{G}$ be the kernel of $\mathscr{O}_X(-d)^N \twoheadrightarrow \mathscr{F}$, then it is also a coherent sheaf and we obtain a short exact sequence

$$0 \to \mathscr{G} \to \mathscr{O}_X(-d)^N \to \mathscr{F} \to 0$$

of coherent sheaves. Taking long exact sequence of cohomology groups we obtain the map $H^p(X, \mathscr{F}) \to H^{p+1}(X, \mathscr{G})$, which is an isomorphism for $0 \le p \le n-2$, injective of cokernel a finitely generated R-module for $p = n-1$. Also, $H^n(X, \mathscr{F})$ is a quotient of a finitely generated R-module, and $H^p(X, \mathscr{F}) = 0$ for $p > n$. The induction on p implies that $H^p(X, \mathscr{F})$ is a finitely generated R-module for any p.

(2) Similarly we can find $m \gg d$ such that $H^p(X, \mathscr{F}(m)) \xrightarrow{\sim} H^{p+1}(X, \mathscr{G}(m))$ for any $p \ge 1$. Since $H^p(X, \mathscr{F}(m)) = 0$ for $p > n$, an induction on p implies that we can find $m \gg d$ such that $H^p(X, \mathscr{F}(m)) = 0$ for $p \ge 1$. $\square$

Consider $\Gamma_*(X, \mathscr{F}) := \bigoplus_{n \ge 0} \Gamma(X, \mathscr{F}(n))$ which is a graded A-module. By the same proof we may obtain that

Proposition 7.4. (1) *The $\Gamma_*(X, \mathscr{F})$ is a finitely generated graded A-module.*

(2) *We have $\widetilde{\Gamma_*(X, \mathscr{F})} \cong \mathscr{F}$.*

(3) *If M is a finite generated graded A-module, $\mathscr{F} := \widetilde{M}$, then there is a natural map $M \to \Gamma_*(X, \mathscr{F})$, which induces $M_n \to \Gamma(X, \mathscr{F}(n))$, and which is an isomorphism when $n \gg 0$.*

In the proof of (3), the following lemma may be useful:

Lemma 7.5. *If M is a finitely generated graded A-module, then $\widetilde{M} = 0$ if and only if $M_n = 0$ for $n \gg 0$.*

Remark 7.6. If $A_0 = k$ is a field, M is a finitely generated graded A-module, then there exists a polynomial $P(T) \in \mathbb{Q}[T]$ such that for $n \gg 0$ the $\dim_k M_n = \dim_k \Gamma(X, \mathscr{F}(n)) = P(n)$ holds. In fact we may define

$$\chi(\mathscr{F}(n)) := \sum_{i \ge 0} (-1)^i \dim_k H^i(X, \mathscr{F}(n)),$$

then $\chi(\mathscr{F}(n)) = P(n)$ holds for any $n \in \mathbb{Z}$. This may be viewed as a consequence of Riemann-Roch Theorem.

Weekly Exercise 7.2. Study Serre duality.

8. Lecture 8: August 14

8.1. **Scheme of dimension one.** Let X be a Noetherian scheme of dimension 1 which is integral and regular. Here we say

- a scheme X is *integral* if for any open subset U of X, $\mathscr{O}_X(U)$ is an integral domain, or equivalently, X is irreducible as a topological space, and admits a cover by affine open subsets which are spectrums of integral domains;
- a scheme X is *regular* if for any $x \in X$, $\dim_{k_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathscr{O}_{X,x}$.

Then for any affine open subset U of X, $\mathscr{O}_X(U)$ is a *Dedekind domain*, i.e. a Noetherian integrally closed domain of Krull dimension 1. The X has only one generic point η, and $K := \mathscr{O}_{X,\eta}$ is a field, called the *function field* of X; in fact $K = \mathrm{Frac}(\mathscr{O}_X(U))$ for any affine open subset U of X. The other points x of X are closed points, and $\mathscr{O}_{X,x}$ is a *discrete valuation ring*, i.e. it is endowed with a surjective map $v_x : \mathscr{O}_{X,x} \twoheadrightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ satisfying

- $v_x(a) = +\infty$ if and only if $a = 0$;
- $v_x(ab) = v_x(a) + v_x(b)$, $v_x(a+b) \geq \min\{v_x(a), v_x(b)\}$;
- $\mathfrak{m}_x = \{a \in \mathscr{O}_{X,x} \mid v_x(a) \geq 1\}$.

We have $K = \mathrm{Frac}(\mathscr{O}_{X,x})$ and the v_x extends to a discrete valuation $v_x : K \twoheadrightarrow \mathbb{Z} \cup \{+\infty\}$ of K, which makes K a *discrete valuation field.* Roughly speaking, if we view elements of $\mathscr{O}_X(U)$ "functions on U", we can talk about "order of zero of a function at a closed point $x \in U$".

Assume that X is separated, then we may define the injection

$$\{\text{closed points of } X\} \hookrightarrow \{\text{non-trivial discrete valuations on } K\},$$
$$x \mapsto v_x,$$

here a discrete valuation $v : K \to \mathbb{Z} \cup \{+\infty\}$ is *non-trivial* means that it is surjective (or equivalently, before normalization, its image is strictly larger than $\{0, +\infty\}$).

In the following we fix a field k and let X be a Noetherian separated scheme of dimension 1 which is finite type over k, integral and regular. Then there is the structure morphism $X \to \mathrm{Spec}(k)$ and the corresponding $k \hookrightarrow \mathscr{O}_X(X)$. The $\mathscr{O}_X(X)$ is a finitely generated integral k-algebra. We assume that k is the maximal subfield in $\mathscr{O}_X(X)$.

Questions. (1) How to "complete" X?

(2) How to describe the structure of $\mathbb{K}(X)$ and $\mathbb{K}'(X)$? How to define the Euler characteristic map $\chi : \mathbb{K}'(X) \to \mathbb{K}(k)$?

First we consider the case that $X = \mathrm{Spec}(A)$ is affine, here A is a finite generated k-algebra which is a Dedekind domain, such that k is the maximal subfield of A. Then there is a structure theorem of finite generated A-modules:

Theorem 8.1. *If M is a finitely generated A-module, then*

$$M \cong A^r \oplus I \oplus \bigoplus_{i=1}^{n} A/\mathfrak{p}_i^{n_i}$$

for some $r \geq 0$, I ideal of A, $n \geq 0$, $n_i \geq 1$ and $\mathfrak{p}_i$ non-zero prime ideal of A.

Daily Exercise 8.1. Prove it. (Recall that if A is a Dedekind domain and I_1, I_2 ideals of A, then $I_1 \oplus I_2 \cong A \oplus I_1 I_2$.)

Corollary 8.2. *We have $\mathbb{K}(A) = \mathbb{K}'(A)$.*

Proof. Note that $0 \to \mathfrak{p}_i^{n_i} \to A \to A/\mathfrak{p}_i^{n_i} \to 0$ is exact, $[A/\mathfrak{p}_i^{n_i}] \in \mathbb{K}'(A)$ is equal to $[A] - [\mathfrak{p}_i^{n_i}] \in \mathbb{K}(A)$. $\square$

We define two invariants on $\mathbb{K}(A)$. Let $K = \mathrm{Frac}(A)$. The first one is

$$c_0 : \mathbb{K}(A) \to \mathbb{Z},$$
$$[M] \mapsto \mathrm{rank}\, M := \dim_K(M \otimes K).$$

This is also called "rank". The second one is $c_1 : \mathbb{K}(A) \to \mathrm{Pic}(A)$ (which is also called "det"),

$$\mathrm{Pic}(A) := \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{invertible projective } A\text{-modules} \end{array} \right\} = \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{non-zero ideals of } A \end{array} \right\}$$

is the class group of A. For a class $[M] \in \mathbb{K}(A)$, consider the natural map $M \to M \otimes K$. Denote $n := \dim_K(M \otimes K)$. Define $\det(M \otimes K) := \wedge^n(M \otimes K) = K \cdot (e_1 \wedge \cdots \wedge e_n)$, here $e_1, \cdots, e_n$ is any

K-basis of $M \otimes K$. Then we define $c_1([M])$ to be the class of $\det M$, here $\det M := \{m_1 \wedge \cdots \wedge m_n \mid m_1, \cdots, m_n \in M\} \subset \det(M \otimes K)$.

Recall that $\mathbb{K}(A)$ is a ring with addition $\oplus$ and multiplication $\otimes$. We define the ring structure on $\mathbb{Z} \oplus \mathrm{Pic}(A)$ such that the multiplication satisfies "$\mathrm{Pic}(A)^2 = 0$", namely $(a, [I]) \cdot (b, [J]) := (ab, a[J] + b[I])$.

Theorem 8.3. *The map* $(c_0, c_1) : \mathbb{K}(A) \to \mathbb{Z} \oplus \mathrm{Pic}(A)$ *is a ring isomorphism.*

Daily Exercise 8.2. Prove it.

8.2. **Algebraic curve.**

8.2.1. In the following we assume that X is a Noetherian separated scheme of dimension 1 which is finite type over a field k, geometric integral (i.e. integral after base change to $\overline{k}$) and regular.

Let η be the unique generic point of X, and denote by $k(X) := K = \mathscr{O}_{X,\eta}$ the function field of X. Then it is a transcendental extension of k of transcendental degree 1, and $K \otimes_k \overline{k}$ is still a field.

8.2.2. Conversely, if such a field K is given, can we find an algebraic curve C such that $k(C) = K$?

Theorem 8.4. *There exists a projective curve C/k, unique up to canonical isomorphism, such that $k(C) = K$. If X is an above scheme such that $k(X) = K$, then there exists a unique open embedding $X \hookrightarrow C$ which induces $k(C) = k(X) = K$. We have*

$$\{\textit{closed points of } C\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \textit{non-trivial discrete valuations on } K \\ \textit{which is trivial on } k \end{array} \right\}.$$

It's tempting to imagine

$$\{\text{points of } C\} \xleftrightarrow{1:1} \{\text{all valuations on } K\},$$

define the open sets to be finite intersections of sets of form $\{v \mid v(f) \geq 0\}$ for some $f \in K^\times$, and for each v, define the stalk $\mathscr{O}_v := \{f \in K \mid v(f) \geq 0\}$, and define the section $\mathscr{O}(U) := \bigcap_{v \in U} \mathscr{O}_v$. The question is to how to prove that it forms a scheme.

We use the following idea instead: find an element $f \in K$ which gives a finite separable morphism $C \xrightarrow{f} \mathbb{P}^1_k$.

Theorem 8.5. *There exists an element $f \in K$ which is transcendental over k and such that $K/k(f)$ is a finite separable extension.*

Recall that an algebraic extension L/K is called *separable* if any one of the following equivalent conditions hold:

- for any $\alpha \in L$, let f be the minimal polynomial of α over K, then $f'(\alpha) \neq 0$ (or equivalently, $f' \not\equiv 0$);
- $\Omega_{L/K} = 0$, here for a ring A and an A-algebra B, $\Omega_{B/A}$ is the B-module generated by formal symbols $\mathrm{d}b$, $b \in B$, such that $\mathrm{d}(b + b') = \mathrm{d}b + \mathrm{d}b'$, $\mathrm{d}(bb') = b\mathrm{d}b' + b'\mathrm{d}b$, and $\mathrm{d}a = 0$ for any $a \in A$.

If L/K is a finite extension, then they are also equivalent to

- $\mathrm{Tr}_{L/K}(L) = K$;
- $L \otimes_K \overline{K} \cong \overline{K}^{[L:K]}$.

For example, if K is of characteristic zero, then L/K is always separable. On the other hand, $\mathbb{F}_p(t^{1/p})/\mathbb{F}_p(t)$ is not separable. In general, if K is of characteristic p and $L = K(\alpha)$, let $f(X)$ be the minimal polynomial of α over K, then L is inseparable if and only if f is a polynomial in X^p.

Sketch of proof of Theorem 8.5. Choose $f \in K$ transcendental such that the inseparable degree $[K : k(f)^s]$ is minimal, where $k(f)^s$ is the maximal separable extension of $k(f)$ contained in K.

If it is > 1, then there exists $\alpha \in K$ such that $g'(\alpha) = 0$, here $g(X) \in k(f)[X]$ is the minimal polynomial of α over $k(f)$. After clearing denominators, we may assume that Then $g(X)$ is a polynomial in X^p. After clearing denominators, we may assume that $g(X) = h(f, X^p)$ for some $h \in k[X, Y]$ irreducible. We claim that h is not a polynomial in f^p. Otherwise over $\bar{k}$, $h(f, X^p) = F(f, X)^p$. Thus in $\bar{k} \otimes_k K$, we have a nilpotent element $F(f, \alpha)$. Thus f is separable over $k(\alpha)$, and we have $k(f)^s \subset k(\alpha)^s$, and

$$[K : k(\alpha)^s] < [K : k(f)^s].$$

Contradiction. ∎

Therefore we obtain a finite separable field extension $k(t) \hookrightarrow K$, $t \mapsto f$. Note that if $K = k(t)$ is the rational function field over k of one variable, then we can choose $C = \mathbb{P}^1_k$ in Theorem 8.4 and it satisfies all the assertions in this theorem. Note that $\mathbb{P}^1_k = \operatorname{Spec}(k[t]) \cup \operatorname{Spec}(k[1/t])$ is an affine cover.

Now let A and B be the normalization (i.e. integral closure) of $k[t]$ and $k[1/t]$ in K, they are Dedekind domains. We obtain finite separable morphisms $\operatorname{Spec}(A) \to \operatorname{Spec}(k[t])$ and $\operatorname{Spec}(B) \to \operatorname{Spec}(k[1/t])$. We can glue $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ together to obtain a curve C with a finite separable morphism $C \xrightarrow{f} \mathbb{P}^1_k$. We may prove that it satisfies all the assertions in Theorem 8.4.

To prove that C is projective, we claim that $\mathscr{L} := f^*\mathscr{O}_{\mathbb{P}^1_k}(1)$ is ample. In fact, for any coherent sheaf $\mathscr{F}$ on C, the $f_*\mathscr{F}$ is also coherent, and we have $\Gamma(C, \mathscr{F} \otimes \mathscr{L}^n) = \Gamma(\mathbb{P}^1_k, (f_*\mathscr{F})(n))$, since $\mathscr{O}_{\mathbb{P}^1_k}(1)$ is ample, the $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for $n \gg 0$, hence $\mathscr{L}$ is ample.

Remark 8.6. We can also construct C by the following way: note that $\mathbb{P}^1_k = \operatorname{Proj}(k[X_0, X_1])$, we embed $k[X_0, X_1]$ into $K[X_0]$ by $X_1 \mapsto f \cdot X_0$. Let A be the integral closure of $k[X_0, X_1]$ in $K[X_0]$, which is also a graded ring. Then we construct $C := \operatorname{Proj}(A)$. This also proves that C is projective.

8.2.3. Now we consider the structure of $\mathbb{K}(C)$ and $\mathbb{K}'(C)$ for a projective curve C. First we claim that $\mathbb{K}(C) = \mathbb{K}'(C)$. In fact, if $\mathscr{F}$ is a coherent sheaf over C, we define

$$\mathscr{F}_{\text{tors}} := \ker(\mathscr{F} \to \mathscr{F} \otimes_\eta K)$$

to be the torsion subsheaf of $\mathscr{F}$. Equivalently, $\mathscr{F}_{\text{tors}}(U) := \{s \in \mathscr{F}(U) \mid s_\eta = 0\}$ for any open subset U of C. Then we have a short exact sequence

$$0 \to \mathscr{F}_{\text{tors}} \to \mathscr{F} \to \mathscr{F}/\mathscr{F}_{\text{tors}} \to 0$$

with $\mathscr{F}/\mathscr{F}_{\text{tors}}$ torsion-free sheaf.

Exercise 8.3. The $\mathscr{F}/\mathscr{F}_{\text{tors}}$ is locally free.

Therefore $[\mathscr{F}/\mathscr{F}_{\text{tors}}] \in \mathbb{K}(C)$. On the other hand, the $\mathscr{F}_{\text{tors}}$ is of form

$$\sum_{\substack{x \in C \\ \text{closed point}}} (i_x)_* \widetilde{M_x}$$

for some M_x finitely generated torsion $\mathscr{O}_{C,x}$-module, which must be of the form $\bigoplus_{i=1}^n \mathscr{O}_{C,x}/\mathfrak{m}_x^{n_i}$ since $\mathscr{O}_{C,x}$ is a discrete valuation ring. This means that $[\mathscr{F}_{\text{tors}}] \in \mathbb{K}(C)$. So $[\mathscr{F}] = [\mathscr{F}_{\text{tors}}] + [\mathscr{F}/\mathscr{F}_{\text{tors}}] \in \mathbb{K}(C)$.

8.2.4. We define the group of divisors of C

$$\operatorname{Div}(C) := \bigoplus_{\substack{x \in C \\ \text{closed point}}} \mathbb{Z} \cdot x$$

which consists of formal $\mathbb{Z}$-linear combinations of closed points of C. For a closed point x of C, there is a short exact sequence

$$0 \to \mathscr{I}_x \to \mathscr{O}_C \to (i_x)_*\mathscr{O}_x \to 0,$$

here $\mathscr{O}_x$ is the structure sheaf of $\operatorname{Spec}(k_x)$, $\mathscr{I}_x$ is a sheaf of ideals, locally generated by π_x a uniformizer of $\mathscr{O}_{C,x}$. The $\mathscr{I}_x$ is also denoted by $\mathscr{O}_C(-x)$, which is an invertible sheaf. In general, if $D = \sum_x n_x \cdot x$ is a divisor of C, then we may define the sheaf $\mathscr{O}_C(D)$ to be the subsheaf of the sheaf $\mathscr{K}$ of rational functions on C by

$$\mathscr{O}_C(D)(U) := \{f \in \mathscr{K}(U) \mid v_x(f) \geq -n_x \text{ for all } x \in U\}.$$

Similar to $\mathscr{I}_x$, For an integer $n \geq 1$, the $\mathscr{O}_C(-nx)$ fits into the following short exact sequence

$$0 \to \mathscr{O}_C(-nx) \to \mathscr{O}_C \to (i_x)_*(\mathscr{O}_{C,x}/\mathfrak{m}_x^n) \to 0,$$

which means that $[(i_x)_*(\mathscr{O}_{C,x}/\mathfrak{m}_x^n)] = [\mathscr{O}_C] - [\mathscr{O}_C(-nx)]$ in $\mathbb{K}'(C)$.

Similar to the affine case, we define

$$\begin{aligned} c_0 : \mathbb{K}(C) &\to \mathbb{Z}, \\ [\mathscr{F}] &\mapsto \dim(\mathscr{F} \otimes_\eta K), \end{aligned}$$

and

$$\begin{aligned} c_1 : \mathbb{K}(C) &\to \operatorname{Pic}(C), \\ [\mathscr{F}] &\mapsto \det(\mathscr{F}) := \wedge^{\dim(\mathscr{F} \otimes_\eta K)}(\mathscr{F}), \end{aligned}$$

here $\operatorname{Pic}(C)$ is the group of isomorphism classes of invertible sheaves on C.

Theorem 8.7. *The map* $(c_0, c_1) : \mathbb{K}(C) \to \mathbb{Z} \oplus \operatorname{Pic}(C)$ *is an isomorphism of rings.*

Daily Exercise 8.4. Prove it.

9. Lecture 9, August 15

9.1. **Riemann-Roch theorem.** Still let C be the projective curve in the last talk. A divisor of C is called a *principal divisor* if it is of form $\operatorname{div}(f) := \sum_{x \in C} v_x(f) \cdot x$ for some $f \in K^\times$. The divisor class group $\operatorname{Cl}(C)$ of C is the $\operatorname{Div}(C)$ modulo the principal divisors. We have the isomorphism $\operatorname{Cl}(C) \cong \operatorname{Pic}(C)$ induced by the natural map $\operatorname{Div}(C) \to \{\text{invertible sheaves on } C\}$, $D \mapsto \mathscr{O}_C(D)$.

If $\mathscr{L}$ is an invertible sheaf on C, then we have $c_0(\mathscr{L}) = 1$ and $c_1(\mathscr{L}) = [\mathscr{L}]$. Therefore the preimage of $(0, [\mathscr{L}]) \in \mathbb{Z} \oplus \operatorname{Pic}(C)$ under the map (c_0, c_1) is $[\mathscr{L}] - [\mathscr{O}_C]$.

We define the degree map

$$\begin{aligned} \deg : \operatorname{Div}(C) &\to \mathbb{Z}, \\ \sum_x n_x \cdot x &\mapsto \sum_x n_x \cdot \deg x, \end{aligned}$$

where $\deg x := [k_x : k]$. It's known that a principal divisor have degree zero, hence the degree map factors through $\operatorname{Cl}(C)$. If $[\mathscr{F}] \in \mathbb{K}'(C)$, define $\deg([\mathscr{F}]) := \deg(c_1([\mathscr{F}]))$. If $[\mathscr{F}]$ is of form $[\mathscr{F}] = [\mathscr{O}_C] \cdot \operatorname{rank} \mathscr{F} + \sum_x n_x [\mathscr{O}_C / \pi_x \mathscr{O}_{C,x}]$, then we have $\deg([\mathscr{F}]) = \sum_x n_x \cdot \deg x$.

Theorem 9.1 (Riemann-Roch)**.** *Let*

$$\begin{aligned} \chi : \mathbb{K}'(C) &\to \mathbb{K}'(k) = \mathbb{Z}, \\ [\mathscr{F}] &\mapsto \dim_k H^0(C, \mathscr{F}) - \dim_k H^1(C, \mathscr{F}). \end{aligned}$$

Then $\chi([\mathscr{F}]) = \chi([\mathscr{O}_C]) \cdot \operatorname{rank} \mathscr{F} + \deg([\mathscr{F}])$.

Proof. This is by the above discussion and $\chi([\mathscr{O}_C / \pi_x \mathscr{O}_{C,x}]) = \deg x$. □

Definition 9.2. The (arithmetic) genus of C is $g(C) := \dim_k H^1(C, \mathscr{O}_C)$.

Example 9.3. If $C = \mathbb{P}^1_k$ then $g(C) = 0$.

9.2. **Serre duality.** We define the *canonical bundle* $\Omega^1_{C/k}$ on C, or called sheaf of differential forms, to be $\Omega^1_{C/k}|_{\operatorname{Spec}(A_i)} := \widetilde{\Omega^1_{A_i/k}}$, if $C = \bigcup_i \operatorname{Spec}(A_i)$ is an affine open cover of C.

Theorem 9.4. $\Omega^1_{C/k}$ *is locally free of rank one.*

Proof. Consider $C \xrightarrow{f} \mathbb{P}^1_k$ in the last talk. It induces the short exact sequence $0 \to \Omega^1_{k(t)/k} \otimes K \to \Omega^1_{K/k} \to \Omega^1_{K/k(t)} \to 0$. Note that $\Omega^1_{K/k(t)} = 0$, so we have $\Omega^1_{C/k} \otimes K = \Omega^1_{K/k} = K \cdot \mathrm{d}f$. On the other hand, $\Omega^1_{C/k}$ is torsion-free, hence it is locally free of rank one. □

This means that $\Omega^1_{C/k}$ is an invertible sheaf, and $\chi(\Omega^1_{C/k}) = 1 - g + \deg \Omega^1_{C/k}$.

Example 9.5. Let $C = \mathbb{P}^1_k$. Then we have $\mathbb{Z} \cong \operatorname{Pic}(\mathbb{P}^1_k)$, $n \mapsto [\mathscr{O}(n)]$. Now it's easy to see that $\Omega^1_{\mathbb{P}^1_k/k} \cong \mathscr{O}(-2)$ and $H^1(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k/k}) \cong k$.

In the following we fix an isomorphism $H^1(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k/k}) \cong k$.

Theorem 9.6 (Serre duality)**.** *On C, we have $\dim H^1(C, \Omega^1_{C/k}) = 1$. Moreover for any coherent sheaf $\mathscr{F}$ on C, we have a canonical perfect pairing*

$$\operatorname{Hom}(\mathscr{F}, \Omega^1_{C/k}) \times H^1(C, \mathscr{F}) \to H^1(C, \Omega^1_{C/k}) \cong k.$$

Sketch of proof. **Step 0.** Reduce to the case that $\mathscr{F}$ is torsion-free. In this case we have $\operatorname{Hom}(\mathscr{F}, \Omega^1_{C/k}) = H^0(C, \mathscr{F}^\vee \otimes \Omega^1_{C/k})$.

Step 1. We prove that it holds for $C = \mathbb{P}^1_k$. By induction we only need to prove it for $\mathscr{F}$ invertible sheaf case. The invertible sheaf on $\mathbb{P}^1_k$ is of form $\mathscr{O}(n)$, $n \in \mathbb{Z}$, it's easy to prove that the theorem holds for it.

Step 2. Consider $C \xrightarrow{f} \mathbb{P}^1_k$ in the last talk. Then we have

$$H^0(C, \mathscr{F}^\vee \otimes \Omega^1_{C/k}) \cong H^0(\mathbb{P}^1_k, f_*(\mathscr{F}^\vee \otimes \Omega^1_{C/k})), \qquad H^1(C, \mathscr{F}) \cong H^1(\mathbb{P}^1_k, f_*\mathscr{F}).$$

Apply Serre duality for $\mathbb{P}^1_k$, we are reduced to construct a perfect pairing of sheaves on $\mathbb{P}^1_k$:

$$f_*(\mathscr{F}^\vee \otimes \Omega^1_{C/k}) \otimes f_*\mathscr{F} \longrightarrow \Omega^1_{\mathbb{P}^1_k/k}.$$

First we define this map at the generic point of $\mathbb{P}^1_k$ using identity $\Omega^1_{k(C)/k} = k(C) \cdot \Omega^1_{k(\mathbb{P}^1)/k}$ and trace map $\mathrm{Tr}_{k(C)/k(\mathbb{P}^1)} : k(C) \to k(\mathbb{P}^1)$. To show that this induces a perfect pairing as above, we only need to work on local case: let $x \in C$ with image $y \in \mathbb{P}^1$ with local rings B and A respectively. Then we may assume that $\mathscr{F}_x = B^n$. Thus we are reduce to the case where $\mathscr{F}_x = B$. It remains to prove that the trace map induces a perfect pairing of A-modules:

$$\Omega_{B/k} \otimes_A B \longrightarrow \Omega_{A/k}.$$

Let t be a local parameter of $\mathbb{P}^1$ at y. Then $\Omega_{A/k} = A\,\mathrm{d}t$ and $\Omega_{B/k} = D\,\mathrm{d}t$ for a fractional ideal D of B in $k(C)$. Thus the above pairing is

$$D \otimes_A B \longrightarrow A, \qquad (x, y) \mapsto \mathrm{tr}(xy).$$

To show this pairing is perfect, it suffices to show that D is equal to the dual $B^\vee$ of B under the trace pairing. Now we use two exact sequences.

The first one is

$$0 \to \mathfrak{d}_{B/A} \to B \xrightarrow{d} \Omega_{B/A} \to 0,$$

where $\mathfrak{d}_{B/A} \subset B$ is the relative different of B/A, namely, the inverse of $B^\vee$. To see the exactness, we check that if $B = A[\alpha]$ for some $\alpha \in B$, let f be the minimal polynomial of α over A, then we have $\mathfrak{d}_{B/A} = f'(\alpha) \cdot B$.

On the other hand, we have an exact sequence

$$0 \longrightarrow \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow \Omega_{B/A} \longrightarrow 0.$$

From these two exact sequence, we get an identity of two ideals of B:

$$\mathfrak{d}_{B/A} = \Omega_{A/k} \otimes_A \Omega^{-1}_{B/k}.$$

This identity is equivalent to the perfectness of the pairing

$$\Omega_{B/k} \otimes_A B \longrightarrow \Omega_{A/k}.$$

□

Daily Exercise 9.1. Fill in the details and complete the proof.

The Serre duality has the following consequences. Take $\mathscr{F} = \Omega^1_{C/k}$ we obtain $H^1(C, \Omega^1_{C/k}) \cong H^0(C, \mathscr{O}_C)^\vee$ which is of dimension 1. Take $\mathscr{F} = \mathscr{O}_C$ we obtain $H^1(C, \mathscr{O}_C) \cong H^0(C, \Omega^1_{C/k})^\vee$ which is of dimension g. Therefore take $\mathscr{F} = \Omega^1_{C/k}$ in Riemann-Roch theorem we obtain $\deg \Omega^1_{C/k} = 2g - 2$.

Another application is Hurwitz genus formula. Let $f : X \to Y$ be a finite separable morphism between smooth projective curves over k. Then we have

$$0 \to f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0,$$

and the $\Omega^1_{X/Y}$ is a torsion sheaf on X with $c_1(\Omega^1_{X/Y}) = \sum_{x \in X} \mathrm{ord}_x \left(\frac{\mathrm{d}\pi_{f(x)}}{\mathrm{d}\pi_x}\right) \cdot x$. Therefore we have $\deg \Omega^1_{X/k} = \deg \Omega^1_{Y/k} \cdot \deg f + \deg \Omega^1_{X/Y}$, that is,

$$2g(X) - 2 = (2g(Y) - 2) \cdot \deg f + \sum_{x \in X} \mathrm{ord}_x \left(\frac{\mathrm{d}\pi_{f(x)}}{\mathrm{d}\pi_x}\right) \cdot \deg x.$$

For $x \in X$, we define the *ramification index* of f at x to be $e_x := \mathrm{ord}_{\pi_x}(\pi_{f(x)})$. The x is called *tamely ramified* if e_x is non-zero in k_x (in our case it is equivalent to say that $\mathrm{char}(k) \nmid e_x$). A direct calculation shows that

$$\mathrm{ord}_x \left(\frac{\mathrm{d}\pi_{f(x)}}{\mathrm{d}\pi_x}\right) \geq e_x - 1,$$

and if x is tamely ramified, then "=" holds.

The Riemann-Roch theorem allows us to do the classification of curves. For example, if C is a curve of genus 0 with $C(k) \neq \varnothing$, fix a point $p \in C(k)$ and take $\mathscr{L} = \mathscr{O}_C(p)$, then $\dim_k H^0(C, \mathscr{L}) = 2$. The global sections of $\mathscr{L}$ allows us to construct an isomorphism $C \cong \mathbb{P}^1_k$. In general, if C is a curve of genus 0, then we can take $\mathscr{L} = (\Omega^1_{C/k})^{-1}$, it allows us to find a closed embedding of C into $\mathbb{P}^2_k$.

Remark 9.7. If C is a smooth projective curve of genus g, $\mathscr{L}$ is an invertible sheaf on C. If $\deg \mathscr{L} \geq 2g-1$, then $H^1(C, \mathscr{L}) = 0$ and $\dim_k H^0(C, \mathscr{L}) = 1 - g + \deg \mathscr{L}$. If $\deg \mathscr{L} \geq 2g$ then it is generated by global sections. If $\deg \mathscr{L} \geq 2g + 1$ then it is "very ample", i.e. gives an embedding of C into a projective space.

10. Lecture 10: August 19

10.1. **Review and generalizations.** Recall that we have learned the following key concepts in algebraic geometry:

10.1.1. *Affine scheme.* If a ring A is given, we can define a locally ringed space $\mathrm{Spec}(A)$. The functor

$$\mathbf{Rings}^{\mathrm{op}} \to \mathbf{LocallyRingedSpaces}, \qquad A \mapsto \mathrm{Spec}(A)$$

is fully faithful. If A is a ring, $X = \mathrm{Spec}(A)$, then the functor

$$A\text{-}\mathbf{Mod} \to \{\text{quasi-coherent } \mathscr{O}_X\text{-modules}\}, \qquad M \mapsto \widetilde{M}$$

is an equivalence of category. If moreover A is a Noetherian ring, then the functor

$$\{\text{finitely generated } A\text{-modules}\} \to \{\text{coherent } \mathscr{O}_X\text{-modules}\}, \qquad M \mapsto \widetilde{M}$$

is an equivalence of category.

10.1.2. *Scheme.* A scheme is a locally ringed space which is locally isomorphic to $\mathrm{Spec}(A)$ for some ring A. There are some important concepts for a scheme:

- A scheme X is called *separated* if the diagonal map $X \to X \times X$ is a closed immersion.
- Noetherian condition.
- Quasi-coherent and coherent sheaves on a scheme.
- K-groups for a scheme: if X is a scheme, $\mathbb{K}'(X)$ is the Grothendieck group generated by the isomorphism classes of coherent sheaves on X modulo short exact sequence. The $\mathbb{K}(X)$ is the Grothendieck group generated by the isomorphism classes of locally free coherent sheaves on X modulo short exact sequence.
- Pushforward and pullback: if $f : X \to Y$ is a morphism of schemes, then there is
$$f_* : \{\text{abelian sheaves on } X\} \to \{\text{abelian sheaves on } Y\}, \qquad \mathscr{F} \mapsto f_*\mathscr{F},$$
with $f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$. The f_* maps $\mathscr{O}_X$-modules to $\mathscr{O}_Y$-modules. There is also
$$f^{-1} : \{\text{abelian sheaves on } Y\} \to \{\text{abelian sheaves on } X\},$$
which satisfies $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$ for $x \in X$. There is also
$$f^* : \mathscr{O}_Y\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}, \qquad \mathscr{G} \mapsto f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$
It satisfies $(f^*\mathscr{G})_x = \mathscr{G}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$ for $x \in X$. We have the following adjoint property: if $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, Y, respectively, then there is a natural isomorphism
$$\mathrm{Hom}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}(\mathscr{G}, f_*\mathscr{F}),$$
and if $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-module and $\mathscr{O}_Y$-module, respectively, then there is a natural isomorphism
$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F}).$$

Question. Do they induce morphisms of K-groups?

The pullback f^* induces $\mathbb{K}(Y) \to \mathbb{K}(X)$, $[\mathscr{F}] \mapsto [f^*\mathscr{F}]$ by its behavior on stalks. However it does not induce $\mathbb{K}'(Y) \to \mathbb{K}'(X)$ unless $\mathscr{O}_X$ is flat over $f^{-1}\mathscr{O}_Y$. In general the pushforward f_* does not define maps on $\mathbb{K}$ or $\mathbb{K}'$. If f is an affine map, and is finite or proper, then f_* induces maps on $\mathbb{K}$ and $\mathbb{K}'$.

Exercise 10.1. Let U be an open subset of a scheme X, denote $i : U \hookrightarrow X$ the natural inclusion. Let $\mathscr{F}$ be a coherent sheaf on U. Then there exists a coherent sheaf $\mathscr{G}$ on X such that $i^*\mathscr{G} = \mathscr{F}$.

10.1.3. *Cohomology.* We have two cohomology theories:

Absolute theory. Let X be a scheme. Then there is a series of functors

$$\{\text{quasi-coherent sheaves on } X\} \to \{H^0(X, \mathscr{O}_X)\text{-modules}\}, \qquad \mathscr{F} \mapsto H^i(X, \mathscr{F})$$

for $i \geq 0$, which satisfies

- $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$;
- if U is an affine scheme, $f : U \to X$ is a morphism, $\mathscr{G}$ is a quasi-coherent sheaf on U, then $H^i(X, f_*\mathscr{G}) = 0$ for all $i \geq 1$;
- short exact sequence of sheaves induces long exact sequence of cohomology groups (see Theorem 5.8(4)).

These properties uniquely determine these cohomology group functors.

Relative theory. If $f : X \to Y$ is a morphism of schemes, then there is also a series of functors

$$\{\text{quasi-coherent sheaves on } X\} \to \{\text{quasi-coherent sheaves on } Y\}, \qquad \mathscr{F} \mapsto R^i f_* \mathscr{F},$$

which can be defined by $R^i f_* \mathscr{F}(U) := H^i(f^{-1}(U), \mathscr{F})$ for any affine open subset U of X. They satisfies

- $R^0 f_* \mathscr{F} = f_* \mathscr{F}$;
- if f is an affine map, then $R^i f_* = 0$ for all $i \geq 1$;
- short exact sequence of sheaves induces long exact sequence of cohomology objects, i.e. if $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is a short exact sequence of quasi-coherent sheaves over X, then for each $p \geq 0$, there is a natural (i.e. functorial) morphism $R^p f_* \mathscr{F}_3 \to R^{p+1} f_* \mathscr{F}_1$ of sheaves, which fits into the following natural (i.e. functorial) long exact sequences of cohomology objects:

$$\begin{tikzcd}
0 \arrow[r] & R^0 f_* \mathscr{F}_1 \arrow[r] & R^0 f_* \mathscr{F}_2 \arrow[r] & R^0 f_* \mathscr{F}_3 \arrow[dll, rounded corners] \\
 & R^1 f_* \mathscr{F}_1 \arrow[r] & R^1 f_* \mathscr{F}_2 \arrow[r] & R^1 f_* \mathscr{F}_3 \arrow[dll, rounded corners] \\
 & R^2 f_* \mathscr{F}_1 \arrow[r] & R^2 f_* \mathscr{F}_2 \arrow[r] & R^2 f_* \mathscr{F}_3 \arrow[r] & \cdots .
\end{tikzcd}$$

Again, these properties uniquely determine these functors.

The property of sheaf cohomology (Theorem 5.8) tells us that if X and Y are Noetherian and separated, then we have $R^i f_* = 0$ for $i \gg 0$. This means that under this situation, if for all coherent sheaf $\mathscr{F}$ and all $i \geq 0$, the $R^i f_* \mathscr{F}$ is also a coherent sheaf, then we may define $f_* : \mathbb{K}'(X) \to \mathbb{K}'(Y)$ by $[\mathscr{F}] \mapsto f_*[\mathscr{F}] := \sum_{i \geq 0} (-1)^i [R^i f_* \mathscr{F}]$.

Remark 10.1. It's tempted to define $f^* : \mathbb{K}'(Y) \to \mathbb{K}'(X)$ by $[\mathscr{F}] \mapsto \sum_{i \geq 0} (-1)^i [\mathscr{T}or_{f^{-1}\mathscr{O}_Y}(\mathscr{O}_X, f^{-1}\mathscr{F})]$, however, this is not a finite sum in general.

Note that if $\mathscr{E}$ is locally free, then $f_*[f^*\mathscr{E} \otimes \mathscr{F}] = \mathscr{E} \otimes f_*[\mathscr{F}]$, hence $f_* \mathbb{K}'(X)$ is a $\mathbb{K}(Y)$-submodule of $\mathbb{K}'(Y)$.

10.1.4. *Projective scheme.* Recall that if A is a graded ring, then it defines a scheme $\mathrm{Proj}(A)$ endowed with a quasi-coherent sheaf $\mathscr{O}(1)$. If A is Noetherian, then $\mathscr{O}(1)$ is ample. If A and B are graded rings such that $(\mathrm{Proj}(A), \mathscr{O}_{\mathrm{Proj}(A)}(1)) \cong (\mathrm{Proj}(B), \mathscr{O}_{\mathrm{Proj}(B)}(1))$, then $A_n \cong B_n$ for $n \gg 0$.

Recall that we have constructed a functor

$$\{\text{graded } A\text{-module}\} \to \{\text{quasi-coherent } \mathscr{O}_X\text{-module}\}, \qquad M \mapsto \widetilde{M}.$$

If A is Noetherian, M is a finite generated graded A-module, then $\widetilde{M} = 0$ if and only if $M_n = 0$ for $n \gg 0$. On the other hand we have a functor

$$\{\text{quasi-coherent } \mathscr{O}_X\text{-module}\} \to \{\text{graded } A\text{-module}\}, \qquad \mathscr{F} \mapsto \Gamma_*(X, \mathscr{F}) := \bigoplus_{n \geq 0} \Gamma(X, \mathscr{F}(n)).$$

If $\mathscr{F}$ is a coherent sheaf, then $H^i(X, \mathscr{F})$ is a finitely generated $H^0(X, \mathscr{O}_X)$-module for any $i \geq 0$, hence we may define $\mathbb{K}'(X) \to \mathbb{K}'(H^0(X, \mathscr{O}_X))$, $[\mathscr{F}] \mapsto \sum_{i \geq 0} (-1)^i [H^i(X, \mathscr{F})]$. There exists some m depending on X and $\mathscr{F}$ such that $H^i(X, \mathscr{F}(n)) = 0$ for any $i \geq 1$ and $n \geq m$.

10.1.5. *Curves.* Let k be a field, K/k be a finitely generated field extension of transcendental degree 1 such that $K \otimes_k \overline{k}$ is still a field (or we say K is geometric integral over k). Then $\Omega^1_{K/k}$ is a free K-module of rank 1, and there exists $t \in K$ which is transcendental over k, such that $K/k(t)$ is a finite separated algebraic extension. The $\mathbb{P}^1_k$ has function field $k(t)$, and we want to construct $C \xrightarrow{f} \mathbb{P}^1_k$ which has function field K. It's constructed by giving its affine cover, namely, $C = \mathrm{Spec}(A) \cup \mathrm{Spec}(B)$ where A and B are integral closure of $k[t]$ and $k[t^{-1}]$ in K, respectively. The C satisfies that

- C is smooth projective curve over k;
- the closed points of C corresponds to all k-valuations of K.

Recall that we have $\mathbb{K}(C) = \mathbb{K}'(C) \xrightarrow[\cong]{(c_0, c_1)} \mathbb{Z} \oplus \mathrm{Pic}(C)$. There is a map $\deg : \mathrm{Div}(C) \to \mathbb{Z}$ which factors through $\mathrm{Cl}(C) \cong \mathrm{Pic}(C)$. Define the map $\chi : \mathbb{K}'(C) \to \mathbb{Z}$, $[\mathscr{F}] \mapsto \sum_{i \geq 0} (-1)^i \dim_k H^i(C, \mathscr{F})$.

Then the Riemann-Roch theorem says that the following diagram

$$\begin{tikzcd}
\mathbb{K}'(C) \arrow[r, "(c_0,c_1)", "\cong"'] \arrow[dr, "\chi"'] & \mathbb{Z} \oplus \mathrm{Pic}(C) \arrow[d, "(1-g,\deg)"] \\
& \mathbb{Z}
\end{tikzcd}$$

commutes. The kernel of the map $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ is denoted by $\mathrm{Pic}^0(C)$ (and is represented by $\mathrm{Jac}(C)$ the Jacobian of C). We define the genus g of C to be $g = g(C) := \dim_k H^1(C, \mathscr{O}_C)$.

The Serre duality says that $\dim_k H^1(C, \Omega^1_{C/k}) = 1$ and for any coherent sheaf $\mathscr{F}$ on C, there is a perfect pairing $\mathrm{Hom}(\mathscr{F}, \Omega^1_{C/k}) \times H^1(C, \mathscr{F}) \to H^1(C, \Omega^1_{C/k})$. The key point of the proof is to consider $C \xrightarrow{f} \mathbb{P}^1_k$ and construct a perfect pairing $f_*\mathscr{O}_C \times f_*\Omega^1_{C/k} \to \Omega^1_{\mathbb{P}^1_k/k}$.

The Riemann-Roch theorem and Serre duality gives some other expressions on g.

10.2. **Preliminaries to Grothendieck-Riemann-Roch theorem.** This week we are going to introduce Grothendieck-Riemann-Roch theorem. Reference: [2]. In the following a scheme is always Noetherian, separated, finite type over a field k or $\mathbb{Z}$.

Theorem 10.2. *Let X be a Noetherian separated regular scheme. Then $\mathbb{K}(X) = \mathbb{K}'(X)$.*

Recall that a scheme is *regular* if for any $x \in X$ we have $\dim_{k_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathscr{O}_{X,x}$.

Proof. Let $X = \bigcup_i U_i$ be a cover of X by finitely many affine open subsets. We may assume that for each i, the $X \setminus U_i$ is a union of codimension one integral subschemes ("effect Cartier divisor" D_i). Let $\mathscr{F}$ be a coherent sheaf on X. We only need to find a finite length resolution of $\mathscr{F}$ by locally free coherent sheaves.

Let $U_i = \mathrm{Spec}(A_i)$ and let $\mathscr{F}|_{U_i} = \widetilde{M_i}$ for some finitely generated A_i-module M_i. Let $\{m_{ij}\}_{i=1}^{n_i}$ be a set of generators of M_i. We may write $U_i = X_{s_i}$ the non-zero locus of s_i for some $s_i \in \Gamma(X, \mathscr{O}_X(D_i))$. Then by Lemma 5.3, we may find a (common) $\ell_i \gg 0$ such that $s_i^{\ell_i} m_{ij}$ extends to a section of $\mathscr{F} \otimes \mathscr{O}_X(\ell_i D_i)$ for any $1 \leq j \leq n_i$. Therefore we defined a morphism $\mathscr{O}_X(\ell_i D_i)^{\oplus n_i} \to \mathscr{F}$ which is surjective on U_i. Hence we can define $\mathscr{E}_0 := \bigoplus_i \mathscr{O}_X(\ell_i D_i)^{\oplus n_i}$ with surjection $\mathscr{E}_0 \twoheadrightarrow \mathscr{F}$. Repeat this process on its kernel, we obtain a resolution of $\mathscr{F}$ by locally free coherent sheaves

$$\cdots \to \mathscr{E}_2 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

A theorem in commutative algebra tells us that $\ker(\mathscr{E}_{n-1} \to \mathscr{E}_{n-2})$ is a locally free sheaf, here $n = \dim X$. Hence we can find a finite length resolution of $\mathscr{F}$ by locally free coherent sheaves. $\square$

Let $\mathscr{E}$ be a locally free coherent sheaf of rank $n+1$ on X. Let $\mathrm{Sym}^*\mathscr{E} := \bigoplus_{d\geq 0} \mathrm{Sym}^d \mathscr{E}$ be the sheaf of symmetric algebra of $\mathscr{E}$, which is a sheaf of graded $\mathscr{O}_X$-algebras on X. Define a scheme $\mathbb{P}(\mathscr{E}) := \mathrm{Proj}(\mathrm{Sym}^*\mathscr{E})$ with a morphism $\pi : \mathbb{P}(\mathscr{E}) \to X$ by the following way: if $X = \bigcup_i U_i$ is an affine open cover, $U_i = \mathrm{Spec}(A_i)$, define $\mathrm{Proj}(\mathrm{Sym}^*(\mathscr{E})(U_i)) \to U_i$ (note that $\mathrm{Sym}^*(\mathscr{E})(U_i)$ is a graded ring with subring of degree 0 equals A_i), and glue them together. For each $d \in \mathbb{Z}$ there is a sheaf $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(d)$ on it, defined by the glueing of $\mathscr{O}(d)$ on each $\mathrm{Proj}(\mathrm{Sym}^*(\mathscr{E})(U_i))$.

Remark 10.3. $\mathbb{P}(\mathscr{E})$ is the moduli space of quotient line bundles of $\mathscr{E}$.

Theorem 10.4. *We have*
(i) $\pi_*[\mathscr{O}_{\mathbb{P}(\mathscr{E})}(i)] = [\mathrm{Sym}^i \mathscr{E}]$ *for* $i \geq 0$,
(ii) $\pi_*[\mathscr{O}_{\mathbb{P}(\mathscr{E})}(i)] = 0$ *for* $-1-n < i < 0$.

Daily Exercise 10.2. Prove it.

Exercise 10.3. Can you calculate $\pi_*[\mathscr{O}_{\mathbb{P}(\mathscr{E})}(i)]$ for $i \leq -1-n$? You may need to study relative Serre duality.

11. Lecture 11, August 21

We usually write $\mathbb{K}(X)$ as $\mathbb{K}^0(X)$, write $\mathbb{K}'(X)$ as $\mathbb{K}_0(X)$, because "$\mathbb{K}(X)$ is contravariant" and "$\mathbb{K}'(X)$ is covariant".

11.1. **Preliminaries to Grothendieck-Riemann-Roch theorem (continued).**

Theorem 11.1. *Let X be a Noetherian scheme.*

(1) *Let $i : Y \hookrightarrow X$ be a closed subscheme, $U := X \setminus Y$, $j : U \hookrightarrow X$ be an open subscheme, then*

$$\mathbb{K}'(Y) \xrightarrow{i_*} \mathbb{K}'(X) \xrightarrow{j^*} \mathbb{K}'(U) \to 0$$

is exact.

(2) *Let $\pi : Y \to X$ be an* affine bundle *(i.e. locally of form* $\mathrm{Spec}(A[X_1, \cdots, X_n]) \to \mathrm{Spec}(A)$*), then $\pi^* : \mathbb{K}'(X) \to \mathbb{K}'(Y)$ is an isomorphism.*

(3) *Let $P = \mathbb{P}(\mathscr{E}) \xrightarrow{\pi} X$ be a projective bundle with* $\operatorname{rank} \mathscr{E} = n + 1$*, then*

$$\bigoplus_{i=0}^{n} \mathbb{K}'(X) \to \mathbb{K}'(P),$$

$$([\mathscr{F}_i])_{i=0}^n \mapsto \sum_{i=0}^{n} [\mathscr{O}_P(i)] \cdot \pi^*[\mathscr{F}_i]$$

is an isomorphism.

Sketch of proof. (1) j^* is surjective: note that coherent sheaf on U can be extended to coherent sheaf on X by using Lemma 5.3. For the details, see first few pages of [2].

$\mathrm{Im}(i_*) \subset \ker(j^*)$: trivial.

$\ker(j^*) \subset \mathrm{Im}(i_*)$: if $\mathscr{F}$ is a coherent sheaf on X such that $j^*[\mathscr{F}] = 0$, i.e. $[\mathscr{F}|_U] = 0$, then $\mathscr{F}_U$ is a finite sum of short exact sequences of sheaves on U, each of them can be extended to coherent sheaf on X by using Lemma 5.3. More precisely, we can find coherent sheaves $\mathscr{E}_i, \mathscr{F}_i, \mathscr{G}_i$ on X with injective morphisms $\mathscr{E}_i \hookrightarrow \mathscr{F}_i$ and surjective morphisms $\mathscr{F}_i \twoheadrightarrow \mathscr{G}_i$, such that $[\mathscr{F}] = \sum_i([\mathscr{F}_i] - [\mathscr{E}_i] - [\mathscr{G}_i])$ and such that $0 \to \mathscr{E}_i|_U \to \mathscr{F}_i|_U \to \mathscr{G}_i|_U \to 0$ is exact. Then we have $[\mathscr{F}_i] - [\mathscr{E}_i] - [\mathscr{G}_i] = [\ker(\mathscr{F}_i \to \mathscr{G}_i)/\mathrm{Im}(\mathscr{E}_i \to \mathscr{F}_i)] \in \mathrm{Im}(i_*)$.

(2) We prove π^* is surjective. By (1) and diagram chasing, we may assume X is affine and $Y = \mathbb{A}^n_X$. By induction, we may assume $n = 1$. By intersect all open subsets, we may assume $X = \mathrm{Spec}(A)$ for A Artinian. Then $A = \prod_\eta \mathscr{O}_{X,\eta}$ for η runs over generic points of X. Hence $\mathbb{K}'(Y) = \prod_\eta \mathbb{K}'(\mathbb{A}^1_{k_\eta}) = \mathbb{Z}[\pi_* \mathscr{O}_{\mathrm{Spec}(k_\eta)} \mid \eta \in X]$.

The injectivity of π^* is easy when $X = \mathrm{Spec}(k)$ for a field k and $Y = \mathbb{A}^n_k$. In the general case, it is a consequence of (3).

(3) Surjectivity: similar to (2) we may assume $X = \mathrm{Spec}(k)$ for a field k and $P = \mathbb{P}^n_k$. Induction on n. Note that $j : \mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k$ is an open subscheme, with $\mathbb{P}^n_k \setminus \mathbb{A}^n_k = \mathbb{P}^{n-1}_k \overset{i}{\hookrightarrow} \mathbb{P}^n_k$, hence by (1) we have

$$\mathbb{K}'(\mathbb{P}^{n-1}_k) \xrightarrow{i_*} \mathbb{K}'(\mathbb{P}^n_k) \xrightarrow{j^*} \mathbb{K}'(\mathbb{A}^n_k) \to 0.$$

We have $\mathbb{K}'(\mathbb{A}^n_k) = \mathbb{K}'(k) \cong \mathbb{Z}$, and by the surjectivity of (2),

$$\mathbb{K}'(\mathbb{P}^n_k) = i_* \mathbb{K}'(\mathbb{P}^{n-1}_k) + \pi^* \mathbb{K}'(k).$$

By induction hypothesis,

$$\mathbb{K}'(\mathbb{P}^{n-1}_k) = \sum_{d=0}^{n-1} \mathbb{Z} \cdot \mathscr{O}_{\mathbb{P}^{n-1}_k}(d),$$

note that $\mathscr{O}_{\mathbb{P}^{n-1}_k}(d) = i_* \mathscr{O}_{\mathbb{P}^n_k}(d)$, and $0 \to \mathscr{O}_{\mathbb{P}^n_k}(-1) \to \mathscr{O}_{\mathbb{P}^n_k} \to i_* \mathscr{O}_{\mathbb{P}^{n-1}_k} \to 0$ with $i_* \mathscr{O}_{\mathbb{P}^{n-1}_k} = i_* i^* \mathscr{O}_{\mathbb{P}^n_k}$, we have

$$i_* \mathbb{K}'(\mathbb{P}^{n-1}_k) = \sum_{d=0}^{n-1} \mathbb{Z} \cdot i_* i^* \mathscr{O}_{\mathbb{P}^n_k}(d) = \sum_{d=0}^{n-1} \mathbb{Z} \cdot \mathscr{O}_{\mathbb{P}^n_k}(d+1).$$

Injectivity: if $([\mathscr{F}_i])_{i=0}^n$ are not all zero such that $\sum_{i=0}^n [\mathscr{O}_P(i)] \cdot \pi^*[\mathscr{F}_i] = 0$, let j be the maximal integer such that $[\mathscr{F}_j] \neq 0$, taking $\pi_*(- \otimes \mathscr{O}_P(-j))$ of this element, utilizing Theorem 10.4 we may obtain a contradiction. $\square$

11.2. **Chow group.** Let X be a Noetherian scheme of pure dimension n. For an integer $p \geq 0$ let $Z_p(X)$ be the free abelian group generated by all closed integral subscheme Y of X of dimension p. Let $Z_*(X) := \bigoplus_{p \geq 0} Z_p(X)$. Note that such Y is determined by its generic point, hence $Z_*(X)$ is also the free abelian group generated by all the points of X.

If Y is a closed subscheme of X, with $Y = \bigcup_i Y_i$ the irreducible components, let $Y_{i,\mathrm{red}} \subset Y_i$ be the maximal reduced part of Y_i, then $Y_{i,\mathrm{red}}$ is a closed integral subscheme of X. Let η_i be the generic point of Y_i, then $\mathscr{O}_{Y_i,\eta_i}$ is an Artinian local ring with residue field $k_{\eta_i} = k(Y_{i,\mathrm{red}})$, and that the length $\mathrm{length}_{k_{\eta_i}} \mathscr{O}_{Y_i,\eta_i}$ is finite. We define the class of Y in $Z_*(X)$ to be

$$[Y] := \sum_i \mathrm{length}_{k_{\eta_i}} \mathscr{O}_{Y_i,\eta_i} \cdot [Y_{i,\mathrm{red}}].$$

If Y is a closed integral subscheme of X, let η be the generic point of Y, then $k(Y) = \mathscr{O}_{Y,\eta} = k_\eta$. For a non-zero rational function $f \in k(Y)^\times$, define

$$\mathrm{div}(f) := \sum_{\substack{Z\subset Y \text{ closed integral} \\ \text{subscheme of codimension 1}}} \mathrm{ord}_Z(f)\cdot [Z],$$

where for $f \in \mathscr{O}_{Y,\eta_Z} \setminus \{0\}$ (note that $\mathscr{O}_{Y,\eta_Z}$ is of dimension 1), define $\mathrm{ord}_Z(f) := \mathrm{length}_{k_{\eta_Z}} \mathscr{O}_{Y,\eta_Z}/(f)$.

Exercise 11.1. The definition of ord_Z can be extended to non-zero elements of $k(Y) = \mathrm{Frac}(\mathscr{O}_{Y,\eta_Z})$, and which is well-defined.

Let $Z'_*(X) \subset Z_*(X)$ be the subgroup generated by all such $\mathrm{div}(f)$. The elements of it are called "rational equivalent to zero". Define the Chow group $\mathrm{CH}_*(X) := Z_*(X)/Z'_*(X)$.

If $f : X \to Y$ is a proper morphism, then we can define $f_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y)$ by

$$f_*[Z] := \begin{cases} 0, & \text{if } \dim f(Z) < \dim Z, \\ [k(Z) : k(f(Z))]\cdot [f(Z)], & \text{if } \dim f(Z) = \dim Z. \end{cases}$$

If f is a flat morphism, then we can define $f^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X)$ by $f^*[Z] := [f^{-1}(Z)]$. If X is regular then we have intersection theory on $\mathrm{CH}_*(X)$.

11.3. **Chern class for line bundle.** Let $\mathscr{L}$ be a line bundle on X. Then for each p, we can define the first Chern chass to be a map $c_1(\mathscr{L}) : \mathrm{CH}_p(X) \to \mathrm{CH}_{p-1}(X)$, $[Z] \mapsto [\mathrm{div}(\ell)]$, where $Z \xhookrightarrow{i} X$ is a closed integral subscheme of X and ℓ is any non-zero rational section of $i^*\mathscr{L}_{\eta_Z}$.

The Chern class has the following properties:

- Projection formula: if $f : X \to Y$ is proper, $\mathscr{L}$ is a line bundle on Y, then $f_*(c_1(f^*\mathscr{L})(\alpha)) = c_1(\mathscr{L})(f_*(\alpha))$.
- If $\mathscr{L}$ and $\mathscr{L}'$ are line bundles on X, then $c_1(\mathscr{L}) \circ c_1(\mathscr{L}') = c_1(\mathscr{L}') \circ c_1(\mathscr{L})$.

The Chow group has the properties similar to $\mathbb{K}'$-group (Theorem 11.1):

Theorem 11.2. *Let X be a Noetherian scheme.*

(1) *Let $i : Y \hookrightarrow X$ be a closed subscheme, $U := X \setminus Y$, $j : U \hookrightarrow X$ be an open subscheme, then*

$$\mathrm{CH}_*(Y) \xrightarrow{i_*} \mathrm{CH}_*(X) \xrightarrow{j^*} \mathrm{CH}_*(U) \to 0$$

is exact.

(2) *Let $\pi : Y \to X$ be an affine bundle of relative dimension n, then for each p, $\pi^* : \mathrm{CH}_p(X) \to \mathrm{CH}_{p+n}(Y)$ is an isomorphism.*

(3) *Let $P = \mathbb{P}(\mathscr{E}) \xrightarrow{\pi} X$ be a projective bundle with $\mathrm{rank}\,\mathscr{E} = n+1$, then*

$$\bigoplus_{i=0}^n \mathrm{CH}_*(X) \to \mathrm{CH}_*(P),$$

$$(\alpha_i)_{i=0}^n \mapsto \sum_{i=0}^n c_1(\mathscr{O}_P(1))^i(\pi^*(\alpha_i))$$

is an isomorphism.

12. Lecture 12, August 22

The goal of this talk is

- Chern classes for vector bundles
- $\mathrm{CH}_*(X)$ as a $\mathbb{K}(X)$-module
- define $\mathrm{CH}^*(X) := \mathbb{K}(X)[X]$ as a ring, if X is regular then $\mathrm{CH}_*(X)$ is a $\mathrm{CH}^*(X)$-module
- $\mathrm{CH}^*(X) \otimes \mathbb{Q} \cong \mathbb{K}(X) \otimes \mathbb{Q}$
- Grothendieck-Riemann-Roch

12.1. **Chern classes for a vector bundle.** Let $\mathscr{E}$ be a vector bundle on X of rank n, and $\mathbb{P}(\mathscr{E}) \xrightarrow{\pi} X$ be the corresponding projective bundle of relative dimension $n-1$, endowed with $\mathscr{O}(1) := \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

Recall that $\mathrm{CH}_*(\mathbb{P}(\mathscr{E})) = \bigoplus_{i=0}^{n-1} c_1(\mathscr{O}(1))^i(\pi^* \mathrm{CH}_*(X))$. Hence for an element $\alpha \in \mathrm{CH}_*(X)$, we have $c_1(\mathscr{O}(1))^n(\pi^*(\alpha)) = \sum_{i=0}^{n-1}(-1)^i c_1(\mathscr{O}(1))^{n-1-i}(\pi^*(\alpha_{i+1}))$ for unique $\alpha_1, \cdots, \alpha_n \in \mathrm{CH}_*(X)$. For each p and $1 \leq i \leq n$, we define the map $c_i(\mathscr{E}) : \mathrm{CH}_p(X) \to \mathrm{CH}_{p-i}(X)$ by $\alpha \mapsto \alpha_i$, and define $c(\mathscr{E})[t] := t^n + \sum_{i=1}^n (-1)^i c_i(\mathscr{E}) t^{n-i} \in \mathrm{End}(\mathrm{CH}_*(X))[t]$. We have $c(\mathscr{E})[c_1(\mathscr{O}(1))] = 0$, namely, $c(\mathscr{E})[t]$ is the characteristic polynomial of $c_1(\mathscr{O}(1))$ acting on $\mathrm{CH}_*(\mathbb{P}(\mathscr{E}))$.

Theorem 12.1. *If $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$ is a short exact sequence of vector bundles on X, then $c(\mathscr{E}_2)[t] = c(\mathscr{E}_1)[t] \cdot c(\mathscr{E}_3)[t]$.*

Corollary 12.2. *If $0 = \mathscr{E}_n \subset \mathscr{E}_{n-1} \subset \cdots \subset \mathscr{E}_0 = \mathscr{E}$ is a filtration of $\mathscr{E}$ such that $\mathscr{E}_i/\mathscr{E}_{i+1}$ is line bundle for every i, then $c(\mathscr{E})[t] = \prod_{i=0}^{n-1}(t - c_1(\mathscr{E}_i/\mathscr{E}_{i+1}))$.*

Remark 12.3. In fact these two results are equivalent. Firstly, for a vector bundle $\mathscr{E}$ on X of rank n, we can define the *flag variety* $\mathrm{Flag}(\mathscr{E})$ which classifies the filtrations of $\mathscr{E}$. It is constructed by the following way: construct the projective bundle $\mathbb{P}(\mathscr{E}) \xrightarrow{\pi} X$, then there is a surjective morphism of sheaves $\pi^*\mathscr{E} \twoheadrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ whose kernel is a vector bundle of rank $n-1$; repeat this process on its kernel, then the rank of the bundle decreases, finally we obtain $\mathrm{Flag}(\mathscr{E}) \to X$ which is a flat morphism. The pullback of $\mathscr{E}$ to $\mathrm{Flag}(\mathscr{E})$ has universal filtration. Therefore we consider the flat morphism $f : \mathrm{Flag}(\mathscr{E}_1) \times \mathrm{Flag}(\mathscr{E}_3) \to X$, which induces $f^* : \mathrm{CH}_*(X) \hookrightarrow \mathrm{CH}_*(\mathrm{Flag}(\mathscr{E}_1) \times \mathrm{Flag}(\mathscr{E}_3))$, then the pullback of $\mathscr{E}_1$ and $\mathscr{E}_3$ have filtrations, hence the pullback of $\mathscr{E}_2$ also have filtration. This technique is called "splitting principle".

In particular, if $\mathscr{E} = \bigoplus_{i=1}^n \mathscr{L}_i$ is a direct sum of line bundles, then $c(\mathscr{E})[t] = \prod_{i=1}^n (t - c_1(\mathscr{L}_i))$, hence $c_d(\mathscr{E}) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} \prod_{j=1}^d c_1(\mathscr{L}_{i_j})$.

12.2. **Chern character.** We define a ring homomorphism

$$\mathrm{ch} : \mathbb{K}(X) \to \mathrm{End}(\mathrm{CH}_*(X)) \otimes \mathbb{Q},$$

which is contravariant in X, such that if $\mathscr{E}$ has a filtration $0 = \mathscr{E}_n \subset \mathscr{E}_{n-1} \subset \cdots \subset \mathscr{E}_0 = \mathscr{E}$ of line bundles, then

$$\mathrm{ch}(\mathscr{E}) := \sum_{i=0}^{n-1} \exp(c_1(\mathscr{E}_i/\mathscr{E}_{i+1})).$$

Note that the right hand side is symmetric in i, hence the $\mathrm{ch}(\mathscr{E})$ can be expressed by $c_d(\mathscr{E})$ for $d \geq 1$.

Theorem 12.4. *If X is regular and is of finite type over a field or $\mathbb{Z}$, then there is an isomorphism*

$$\begin{aligned} \mathbb{K}(X) \otimes \mathbb{Q} &\xrightarrow[\cong]{\mathrm{ch}} \mathrm{CH}_*(X) \otimes \mathbb{Q}, \\ [\mathscr{F}] &\mapsto \mathrm{ch}([\mathscr{F}])([X]). \end{aligned}$$

12.3. **Todd class.** We define the Todd class

$$\begin{aligned} \mathrm{Td} : \mathbb{K}(X) &\to \mathrm{End}(\mathrm{CH}_*(X)) \otimes \mathbb{Q}, \\ [\mathscr{E}] &\mapsto \prod_{i=1}^n \frac{c_1(\mathscr{L}_i)}{1 - \exp(-c_1(\mathscr{L}_i))}, \qquad \text{if } [\mathscr{E}] = \sum_{i=1}^n [\mathscr{L}_i]. \end{aligned}$$

12.4. **Grothendieck-Riemann-Roch theorem.**

Theorem 12.5 (Grothendieck-Riemann-Roch)**.** *Let $f : X \to Y$ be a smooth projective morphism of regular schemes. Then for any $\alpha \in \mathbb{K}(X)$, $f_*(\mathrm{ch}(\alpha)\,\mathrm{Td}(T_{X/Y})) = \mathrm{ch}(f_*(\alpha))$.*

Here $T_{X/Y}$ is the relative tangent bundle, namely, $T_{X/Y} = (\Omega^1_{X/Y})^\vee$. If both of X and Y are varieties over a field k, then $0 \to T_{X/Y} \to T_X \to f^*T_Y \to 0$ exact.

Example 12.6. Let X be a smooth projective curve over k and let $Y = \mathrm{Spec}(k)$. then we have $\mathrm{ch}(\mathscr{E}) = \mathrm{rank}\,\mathscr{E} + c_1(\mathscr{E})$ and $\mathrm{Td}(T_{X/Y}) = 1 + c_1(T_{X/Y})/2 = 1 - c_1(\Omega^1_{X/k})/2$. Hence

$$\begin{aligned} f_*(\mathrm{ch}(\mathscr{E})\,\mathrm{Td}(T_{X/Y})) &= f_*([\mathrm{rank}\,\mathscr{E} + c_1(\mathscr{E})] \cdot [1 - c_1(\Omega^1_{X/k})/2]) = -\frac{\mathrm{rank}\,\mathscr{E}}{2} \deg \Omega^1_{X/k} + \deg \mathscr{E}, \\ \mathrm{ch}(f_*\mathscr{E}) &= \dim_k H^0(X, \mathscr{E}) - \dim_k H^1(X, \mathscr{E}), \end{aligned}$$

which recovers Riemann-Roch theorem on curves.

Remark 12.7. We have $\mathrm{Td}(T_X) = f^* \mathrm{Td}(T_Y) \cdot \mathrm{Td}(T_{X/Y})$, therefore the Grothendieck-Riemann-Roch theorem can be rewritten as $f_*(\mathrm{ch}(\alpha)\,\mathrm{Td}(T_X)) = \mathrm{ch}(f_*\alpha) \cdot \mathrm{Td}(T_Y)$, which is more symmetric and is Grothendieck's original formulation. This also means the following diagram commutes:

$$\begin{tikzcd}
\mathbb{K}(X) \otimes \mathbb{Q} \arrow[r, "\mathrm{ch}", "\cong"'] \arrow[d, "f_*"] & \mathrm{CH}_*(X) \otimes \mathbb{Q} \arrow[r, "\mathrm{Td}(T_X)"] & \mathrm{CH}_*(X) \otimes \mathbb{Q} \arrow[d, "f_*"] \\
\mathbb{K}(Y) \otimes \mathbb{Q} \arrow[r, "\mathrm{ch}", "\cong"'] & \mathrm{CH}_*(Y) \otimes \mathbb{Q} \arrow[r, "\mathrm{Td}(T_Y)"] & \mathrm{CH}_*(Y) \otimes \mathbb{Q}.
\end{tikzcd}$$

Remark 12.8. When X is a smooth projective variety over k and $Y = \mathrm{Spec}(k)$, this is Hirzebruch-Riemann-Roch theorem.

Idea of proof of Theorem 12.5. Note that if Grothendieck-Riemann-Roch theorem holds for f and g, then it also holds for $g \circ f$. Utilizing normal cone construction, we can reduce it to the case that $X = \mathbb{P}(\mathscr{E})$ with $\mathscr{E}$ a vector bundle on Y of rank 2, hence $f : X \to Y$ is of relative dimension 1. In this case we have

$$\mathbb{K}(X) = \mathbb{K}(Y) + \mathbb{K}(Y)[\mathscr{O}(-1)],$$
$$\mathrm{CH}_*(X) = \mathrm{CH}_*(Y) + c_1(\mathscr{O}(-1))\,\mathrm{CH}_*(Y).$$

The X is the moduli space of quotient bundles of $\mathscr{E}$ of rank 1, and the tangent space $T_{X/Y}$ classifies the deformation of $0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathscr{E}/\mathscr{E}_1 \to 0$. We have $0 \to \mathscr{O}_X \to f^*\mathscr{E}^\vee(1) \to T_{X/Y} \to 0$, hence $T_{X/Y} = (f^* \det \mathscr{E})^\vee(2)$. Now the Grothendieck-Riemann-Roch theorem can be checked in this case by explicit computation. ☐

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