EXERCISE SHEET 3: ALGEBRAIC NUMBER THEORY SUMMER SCHOOL AT AMSS 2019

Exercise 1. Let $f(x) \in \mathbb{C}[x]$ be a nonconstant polynomial not of the form $g(x)^2$ for any $g(x) \in \mathbb{C}[x]$. Let $A = k[x,y]/(y^2 - f(x))$.

1. Prove that A is a domain.
2. Prove that A is a Dedekind domain if $f(x)$ has only simple roots. (*Hint: mimic the case of* $\mathbb{Q}(\sqrt{d})$ *to prove that* A *is integrally closed*)
3. If we allow the roots of $f(x)$ to have multiplicities, what is the integral closure of A in its fraction field?

Exercise 2. Let $K = \mathbb{Q}(\alpha)$ with $\alpha^3 = \alpha + 1$.

1. Show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.
2. Find the explicit decomposition of primes $p = 3, 5, 23$ in $\mathcal{O}_K$.
3. Prove that $\sqrt{\alpha}, \sqrt[3]{\alpha} \notin K$. (Hint: try to find prime p such that there exists a surjective map $\mathcal{O}_K \twoheadrightarrow \mathbb{F}_p$ such that the image of α can not has square or cubic root.)

Exercise 3. Let $K = \mathbb{Q}(\alpha)$ with $\alpha^5 = 2$.

1. Determine all the primes p that are ramified in K.
2. Prove that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.
3. Prove that if p is a prime unramified in K and $5 \nmid (p^2 - 1)$, then p decomposes in $\mathcal{O}_K$ as $(p) = \mathfrak{p}\mathfrak{p}'$ with $f(\mathfrak{p}|p) = 1$ and $f(\mathfrak{p}'|p) = 4$.

Exercise 4. Let A be a Dedekind domain. Let $I \subset A$ be a nonzero ideal with prime decomposition $I = \prod_{i=1}^r \mathfrak{p}_i^{e_i}$. Prove that $I^{-1} = \prod_{i=1}^r (\mathfrak{p}_i^{-1})^{e_i}$ and $I^{-1}I = A$

(*Remark: This is one of intermediate steps to deduce the UFL for fractional ideals from UFL for ideals. So you can only use UFL for ideals and the fact that* $\mathfrak{p}^{-1}\mathfrak{p} = A$ *for any nonzero prime* $\mathfrak{p}$)