EXERCISE SHEET 5: ALGEBRAIC NUMBER THEORY SUMMER SCHOOL AT AMSS 2019

Exercise 1. Some simple applications of Minkowski bound:

(1) Find the class number of $\mathbb{Q}(\sqrt{m})$ for $m = 5, 6, -5, -13$.
(2) Show that the ideal class group of $\mathbb{Q}(\sqrt{-23})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and find explicitly an ideal that generates the ideal class group.

Exercise 2. Let $K = \mathbb{Q}(\sqrt[3]{m})$.

(1) Show that $\mathbb{Z}[\sqrt[3]{m}]$ is the ring of integers of K if m is square free and m is not congruent to ± 1 modulo 9.
(2) Prove that $\mathbb{Z}[\sqrt[3]{m}]$ is a principal ideal domain for $m = 3, 5, 6$, and that the class number of $\mathbb{Q}(\sqrt[3]{7})$ is 3.

Exercise 3. The aim of this exercise is to prove that the pairs $(17, \pm 70)$ are the only solutions in $\mathbb{Z}^2$ to the equation

$$y^2 + 13 = x^3. \tag{0.0.1}$$

We denote $A = \mathbb{Z}[\sqrt{-13}]$, and let $(x, y) \in \mathbb{Z}^2$ be a solution.

(1) Show that no prime ideals of A contain both $y + \sqrt{-13}$ and $y - \sqrt{-13}$.
(2) Show that there exist $(a, b) \in \mathbb{Z}^2$ such that

$$y + \sqrt{-13} = (a + b\sqrt{-13})^3.$$

Conclude that $(x, y) = (17, \pm 70)$. (You may use directly the fact that $\mathbb{Q}(\sqrt{-13})$ has class number 2).

Exercise 4. Let K be a number field, and S be a finite set of maximal ideals of $\mathcal{O}_K$. Fix an algebraic closure $\overline{K}$ of K. Let K^S/K be the maximal subextension of $\overline{K}/K$ that is unramified outside S, and put $G_{K,S} = \mathrm{Gal}(K^S/K)$.

(1) Prove that for any finite abelian group M, $\mathrm{Hom}(G_{K,S}, M)$ is a finite abelian group as well.
(2) For $K = \mathbb{Q}$ and $S = \{5, 13, 31, 101\}$, compute the dimension of $\mathrm{Hom}(G_{\mathbb{Q},S}, \mathbb{F}_5)$ over $\mathbb{F}_5$. (*Hint: use Kronecker–Weber's theorem*)