EXERCISE SHEET 6: ALGEBRAIC NUMBER THEORY SUMMER SCHOOL AT AMSS 2019

Exercise 1. The aim of this exercise is to give a proof of the finiteness of the ideal class group of a number field K following Dedekind.

As explained in class, we have to show that there exists a constant C_K, which depends only on K, such that for all ideal $I \subseteq \mathcal{O}_K$, there exists $\alpha \in I \backslash \{0\}$ such that $|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| \leqslant C_K \mathrm{N}(I)$. Let $n = [K : \mathbb{Q}]$. We choose an integral basis $(\alpha_1, \cdots, \alpha_n)$ for $\mathcal{O}_K$.

(1) Prove that there exist integers $x_i \in [-\sqrt[n]{\mathrm{N}(I)}, \sqrt[n]{\mathrm{N}(I)}]$ with $1 \leqslant i \leqslant n$ such that not all x_i are zero and $\alpha := \sum_i x_i \alpha_i \in I$.
(2) Prove that there exists a constant $C_K > 0$ such that $|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| \leqslant C_K \mathrm{N}(I)$.

Exercise 2. Let $K = \mathbb{Q}(\zeta_p)$ be the p-th cyclotomic field, where p is an *odd* prime, and $K^+ = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$. Denote respectively by U_K and U_{K^+} the group of units in K and K^+.

(1) Let u be a unit of K. Show that $u/\bar{u}$ is a root of unity.
(2) Let u be as in (1), write $u/\bar{u} = \pm\zeta_p^k$ for some $k \in \mathbb{Z}$. Show that it is impossible to have $-$ sign.
(3) Show that $U_K = U_{K^+} \times \langle \zeta_p \rangle$, i.e. every unit u in K writes uniquely as $u = \varepsilon \cdot \zeta_p^k$ with $\varepsilon \in U_{K^+}$. (Hint: Consider the map $\phi : U_K \to U_{K^+}$ given by $u \mapsto u/\bar{u}$.)

Exercise 3. Let K be a number field. We say an element $\alpha \in K$ is *totally positive* if $\sigma(\alpha) > 0$ for every real embedding $\sigma : K \hookrightarrow \mathbb{R}$. Denote by $\mathcal{I}_K$ the group of fractional ideals of K, and by $\mathcal{P}_K^+$ the subgroup of principal ideals generated by a totally positive element. Define the strict ideal class group of K as

$$\mathcal{C}l_K^+ = \mathcal{I}_K / \mathcal{P}_K^+.$$

(1) Show that the kernel of the natural surjection $f : \mathcal{C}l_K^+ \to \mathcal{C}l_K$ has at most $2^{r_1 - 1}$ elements, where r_1 denotes the number of real embeddings of K. Conclude that $\mathcal{C}l_K^+$ is a finite abelian group.
(2) Assume that K is real quadratic. Let u denote the fundamental unit of K. Prove that $\mathrm{Ker}(f)$ has order 2 if $\mathrm{N}_{K/\mathbb{Q}}(u) = 1$, and $\mathrm{Ker}(f) = \{1\}$ if $\mathrm{N}_{K/\mathbb{Q}}(u) = -1$.

Exercise 4. Let K be a real quadratic field with discriminant d_K. Then fundamental unit of K is defined to be the unique unit ε of K such that $\varepsilon > 1$ and $U_K = \{\pm 1\} \times \varepsilon^{\mathbb{Z}}$.

(1) Let $u > 1$ be a unit of K. Show that $u \geqslant (\sqrt{d_K} + \sqrt{d_K - 4})/2$ if $\mathrm{N}_{K/\mathbb{Q}}(u) = -1$, and $u \geqslant (\sqrt{d_K} + \sqrt{d_K + 4})/2$ if $\mathrm{N}_{K/\mathbb{Q}}(u) = 1$. (Hint: consider $\mathrm{Disc}_{K/\mathbb{Q}}(1, u)$ and use the equality that $\mathrm{Disc}_{K/\mathbb{Q}}(1, u) \geqslant d_K$.)
(2) Show that if d_K is divisible by a prime number p with $p \equiv 3 \mod 4$, then K does not contain any units u with $\mathrm{N}_{K/\mathbb{Q}}(u) = -1$.
(3) Find the fundamental unit of $K = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5})$.