Introduction to local index theory

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Elliptic operators and heat kernel

- $E, \ F$ complex vector bundles over an oriented compact manifold M, carrying Hermitian metrics g^E, g^F.
- $D : \Gamma(E) \to \Gamma(F)$ first order elliptic differential operator
- Volume form dv_M
- $D^* : \Gamma(F) \to \Gamma(E)$ formal adjoint of D :

$$\int_M \langle D\sigma, \sigma' \rangle_{g^F} dv_M = \int_M \langle \sigma, D^* \sigma' \rangle_{g^E} dv_M .$$

Mckean-Singer formula

- $\mathrm{ind}D = \dim(\ker D) - \dim(\ker D^*)$
- $\langle D^* D\sigma, \sigma\rangle = \langle D\sigma, D\sigma\rangle = |D\sigma|^2$ implies
$$\ker D = \ker D^* D,$$
$$\ker D^* = \ker DD^*$$
- $D^* D : \Gamma(E) \to \Gamma(E), \quad DD^* : \Gamma(F) \to \Gamma(F)$
nonnegative elliptic operators
- $\exp(-tD^* D) : \Gamma(E) \to \Gamma(E)$
$\exp(-tDD^*) : \Gamma(F) \to \Gamma(F)$ compact operators of traceclass for any $t > 0$

Mckean-Singer formula.

- **Mckean-Singer formula**

$$\underline{\operatorname{ind}(D) = \operatorname{Tr}\left[\exp\left(-tD^*D\right)\right] - \operatorname{Tr}\left[\exp\left(-tDD^*\right)\right]}$$

- *Proof.* If $\lambda \in \operatorname{Spec}(D^*D)$ and $\lambda \neq 0$,
  Then by $D^*Ds = \lambda s$ with $\underline{s \neq 0}$
  One gets $DD^*(Ds) = \lambda(Ds)$ with $\underline{Ds \neq 0}$.
  Thus $\lambda \in \operatorname{Spec}(DD^*)$.
- $$\underline{\sum_{\lambda \in \operatorname{Spec}(D^*D)} \exp(-t\lambda) - \sum_{\mu \in \operatorname{Spec}(DD^*)} \exp(-t\mu)}$$
  $$= \dim(\ker D) - \dim(\ker D^*) = \underline{\operatorname{ind}(D)}. \quad \text{Q.E.D.}$$

Local Gauss-Bonnet-Chern formula

- M even dimensional closed oriented manifold.
- g^{TM} Riemannian metric on TM. ∇^{TM} Levi-Civita connection. $R^{TM} = (\nabla^{TM})^2$ the curvature
- $D = d + d^* : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)$ the de Rham-Hodge operator
- **Local Gauss-Bonnet-Chern theorem** (Patodi, 1970)

$$\lim_{t\to 0^+} \left(\mathrm{Tr}\left[\exp\left(-tD^*D\right)(x,x)\right] - \mathrm{Tr}\left[\exp\left(-tDD^*\right)(x,x)\right]\right) dv(x)$$

$$= \left\{ \mathrm{Pf}\left(-\frac{R^{TM}(x)}{2\pi}\right) \right\}^{\mathrm{top}}$$

- **Gauss-Bonnet-Chern theorem :**

$$\chi(M) = \int_M \mathrm{Pf}\left(-\frac{R^{TM}(x)}{2\pi}\right)$$

The Dirac operator

- M an even dimensional compact spin manifold
- g^{TM} a Riemannian metric on TM, ∇^{TM} the Levi-Civita connection, $R^{TM} = (\nabla^{TM})^2$ the curvature
- $S(TM) = S_+(TM) \oplus S_-(TM)$
  the Hermitian bundle of spinors associated to (TM, g^{TM})
- $\nabla^{S(TM)} = \nabla^{S_+(TM)} \oplus \nabla^{S_-(TM)}$
- For any $X \in TM$, $c(X)$ denotes the Clifford action of X on $S(TM)$. It exchanges $S_\pm(TM)$.

The Dirac operator

- (E, g^E) a Hermitian vector bundle over M carrying a Hermitian connection ∇^E
- $\nabla^{S(TM)\otimes E} = \nabla^{S(TM)} \otimes \mathrm{Id}_E + \mathrm{Id}_{S(TM)} \otimes \nabla^E$
- $e_1, \cdots, e_{\dim M}$ oriented orthonormal basis of TM
- The Dirac operator :

$$D^E = \sum_{i=1}^{\dim M} c(e_i)\nabla_{e_i}^{S(TM)\otimes E} : \Gamma(S(TM)\otimes E) \to \Gamma(S(TM)\otimes E)$$

$$D_\pm^E := D^E\big|_{\Gamma(S_\pm(TM)\otimes E)} : \Gamma(S_\pm(TM)\otimes E) \to \Gamma(S_\mp(TM)\otimes E)$$

- D^E elliptic and self-adjoint : $\left(D_+^E\right)^* = D_-^E$

The Atiyah-Singer index theorem for Dirac operators

- $\mathrm{ind}(D_+^E) = \dim(\ker D_+^E) - \dim(\ker D_-^E)$
- **The Atiyah-Singer index theorem (1963)**

$$\underline{\mathrm{ind}\left(D_+^E\right) = \left\langle \widehat{A}(TM)\mathrm{ch}(E), [M] \right\rangle}$$

$$= \int_M \det{}^{1/2}\left(\frac{R^{TM}/4\pi}{\sinh(R^{TM}/4\pi)}\right) \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}\left(\nabla^E\right)^2\right)\right]$$

- If $E = \mathbf{C}$ trivial bundle, denote D^E simply by D
- $\widehat{A}(M) = \langle \widehat{A}(TM), [M] \rangle \in \mathbf{Z}$ (Borel-Hirzebruch (1960))
- Spin condition essential : $\widehat{A}(\mathbf{C}P^2) = -\frac{1}{8}$

The Lichnerowicz formula

- $k^{g^{TM}}$ – the scalar curvature of g^{TM},

$$\frac{\mathrm{vol}\left(B_x^{g^{TM}}(r)\right)}{\mathrm{vol}\left(B_0^{\mathbf{R}^n}(r)\right)} = 1 - \frac{k^{g^{TM}}(x)}{6(n+2)} r^2 + o\left(r^2\right)$$

- **Lichnerowicz formula (1963) :** $D^2 = -\Delta + \dfrac{k^{g^{TM}}}{4}$
- The Bochner-Laplacian $-\Delta \geq 0$

The Lichnerowicz theorem

- **Lichnerowicz Theorem (1963) :** *If* $k^{g^{TM}} > 0$ *over* M*, then* $\widehat{A}(M) = 0$.
- Proof. As $k^{g^{TM}} > 0$, one gets $D^2 = -\Delta + \frac{k^{g^{TM}}}{4} > 0$ which implies $\widehat{A}(M) = \mathrm{ind}(D_+) = 0$.
- Geometric condition leads to topological conclusion through analytic arguments
- Spin condition essential : $\widehat{A}(\mathbf{C}P^2) = -\frac{1}{8}$

Local index theorem for Dirac operators

- The following identity holds at any $x \in M$,

$$\lim_{t \to 0^+} \left(\mathrm{Tr}\left[\exp\left(-tD_-D_+\right)(x,x)\right] - \mathrm{Tr}\left[\exp\left(-tD_+D_-\right)(x,x)\right]\right) dv(x)$$

$$= \left(\det{}^{1/2}\left(\frac{R^{TM}/4\pi}{\sinh(R^{TM}/4\pi)}\right) \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}\left(\nabla^E\right)^2\right)\right]\right)^{\mathrm{top}}$$

- First (indirect) proof : Atiyah-Bott-Patodi, Gilkey
- Direct proofs : Bismut, Getzler, Berline-Vergne (all inspired by physicists)
- Independent direct proof à la Patodi : Yanlin Yu

The Atiyah-Patodi-Singer index theorem

- Now assume M has a boundary ∂M, also assume that all geometric data are of "product structure" near ∂M
- D_+ not elliptic - need to impose boundary conditions : the Atiyah-Patodi-Singer global condition
- Atiyah-Patodi-Singer theorem (1974).
  $\mathrm{ind}\left(D^E_{+,\mathrm{APS}}\right)$

$$
= \underline{\int_M \left(\det{}^{1/2}\left(\frac{R^{TM}/4\pi}{\sinh(R^{TM}/4\pi)}\right)\mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}\left(\nabla^E\right)^2\right)\right]\right)}
$$

$$
\underline{-\overline{\eta}\left(D_+^{E|\partial M}\right)} \qquad \in \ \mathbf{Z}
$$

The Atiyah-Patodi-Singer η-invariant

- $D_+^{E|\partial M} : \Gamma(S_+(TM)|_{\partial M}) \to \Gamma(S_+(TM)|_{\partial M})$ induced Dirac operator on ∂M : elliptic and formally self-adjoint

- For any $\mathrm{Re}(s) >> 0$, set

$$\eta\left(D_+^{E|\partial M}, s\right) = \sum_{\lambda \in \mathrm{Spec}(D_+^{E|\partial M}) \setminus \{0\}} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}$$

- $\eta(D_+^{E|\partial M})$ meromorphic on $\mathbf{C}$, holomorphic at $s = 0$. Set $\eta(D_+^{E|\partial M}) = \eta(D_+^{E|\partial M}, 0)$,

$$\overline{\eta}\left(D_+^{E|\partial M}\right) = \frac{\dim\left(\ker D_+^{E|\partial M}\right) + \eta\left(D_+^{E|\partial M}\right)}{2}$$

The APS index theorem and η-invariant

- $\overline{\eta}(D_+^{E|\partial M})$ is a spectral invariant, hard to compute.
- Has implications with many parts of mathematics, including geometry, topology, number theory, as well as mathematical physics (like Chern-Simons gauge theory)
- Atiyah : "In many ways the papers on spectral asymmetry were perhaps the most satisfying ones I was involved with."
- Indeed, when asked to cite a single most representative result in whole life, Atiyah choosed to cite this APS index theorem !

The APS index theorem and η-invariant

- As a recent simple application of the computation of η invariant, Zizhou Tang and I proved the following purely Riemannian geometric result

- **Theorem.** (Zizhou Tang - Z, Adv. in Math., 2014) *Let p be any point on a fake $\mathbf{H}P^2$ (called an Eells-Kuiper quaternionic projective plane), there exists a Riemannian metric on this fake $\mathbf{H}P^2$ such that all geodesics passing through p are simply closed and of the same length.*

- This anwsers a longstanding open question in Riemannian geometry (due to Bérard-Bergery and Besse)

1986 : a new era

- 1986 : Bismut published his heat kernel proof of the Atiyah-Singer families index theorem for Dirac oerators (Inventiones Math.)
  by using Quillen's concept of superconnection

- ICM1986 invited talk : Bismut
  (title : Index theorem and the heat equation)

- Opened a new era in local index theory, hope can report later ...

- ICM1998 Plenary lecture : Bismut

Thanks!