## Two-Dimensional Conformal Field Theory

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Institute of Mathematics, Chinese Academy of Sciences



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  - A definition and early conjectures
  - Problems and a mathematical program
- The major problems solved
  - The geometry of vertex operator algebras
  - Intertwining operators and vertex tensor categories
  - Modular invariance and Verlinde formula
  - Rigidity and modularity
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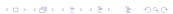
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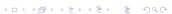
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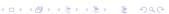
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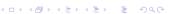
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## Part 1

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- Such a definition and the subsequent construction and study are very successful in the case of topological quantum field theories. The main reason for this success is that the Hilbert space for a topological quantum field theory is typically finite dimensional.
- For nontrivial nontopological quantum field theories, the Hilbert space must be infinite dimensional. The construction and study of these theories are much more difficult.



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- One of the Millennium Prize Problems: Establish rigorously the existence of the quantum Yang-Mills theory and prove that there is a mass gap in this theory.
- Many mathematical conjectures arising from string theory are in fact arising from two dimensional superconformal field theories or higher-dimensional supersymmetric Yang-Mills theories.
- To understand these mathematical conjectures completely, we need to construct the correspoding quantum field theories
- The simplest nontopological quantum field theories are two-dimensional conformal field theories. Many conjectures in mathematics have been derived based on the stronger conjectures that the corresponding two-dimensional conformal field theories exist.

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- A two-dimensional conformal field theory in the sense of Kontsevich-Segal is
  - a locally convex topological vector space H,
  - a nondegenerate hermitian form,
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  - satisfying additional but natural conditions.
- From now on, for simplicity, we omit the words
   "two-dimensional" so that conformal field theories mean two-domensional conformal field theories.

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- 1988, Moore and Seiberg:
  - Obtained Moore-Seiberg polynomial equations using conjectures on operator product expansion and modular invariance for intertwining operators.
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  - Conjectured that in the case of Wess-Zumino-Witten models, these invariants should be the same as those from the corresponding Chern-Simons theories.
- 1991. Witten:
  - Holomorphic factorzation of Wess-Zumino-Witten models, assuming the nondegeneracy of hermitian forms on the spaces of conformal blocks.

# Conjectures on nonlinear $\sigma$ -models associated to Calabi-Yau manifolds

- 1985, Friedan, Candelas, Horowitz, Strominger, Witten, Alvarez-Gaumé, Coleman, Ginsparg, Nemeschansky, Sen and so on: Nonlinear σ-models with Calabi-Yau manifolds as targets are N = 2 superconformal field theories.
- 1987, Gepner: A suitable orbifold and tensor product construction from an N=2 superconformal minimal model gives the nonlinear  $\sigma$ -model for the quantic Calabi-Yau manifold.
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The major problems solved

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  - A definition and early conjectures
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- 2 The major problems solved
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  - Intertwining operators and vertex tensor categories
  - Modular invariance and Verlinde formula
  - Rigidity and modularity
  - Full and open-closed conformal field theories
  - Cohomology and deformation theory
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  - Higher-genus theories and locally convex completions
  - Moduli space of conformal field theories
  - Nonrational conformal field theories



- Problem 1: Formulate precisely and prove the conjectures of Verlinde, Moore-Seiberg and Witten.
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Problems and a mathematical program

# A long term program

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# Part 2 The major problems solved

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   They are analogues of commutative associative algebras and Lie algebras, but involve additional variables. Here I omit this definition.
- Mathematicians have developed systematic methods to construct vertex operator algebras.
- Examples of vertex operator algebras were constructed in physics and mathematics from representations of Heisenberg algebras, Clifford algebras, affine Lie algebras, Virasoro algebra, superconformal algebras and W-algebras. They were also constructed from lattices (corresponding to certain types of tori). One remarkable example is the moonshine module whose automorphism group is the largest sporadic finite simple group, the Monster.

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# The first major problem in the program

- 1988, Segal: The central charge of a conformal field theory should be interpreted as twice the power of the determinant line bundle over the moduli space of Riemann surfaces with parametrized boundaries.
- The first major problem to be solved in this program: From the works above, one can easily make a conjecture on what the geometric formulation of a vertex operator algebra (including the conformal element, the Virasoro algebra and the central charge) should be. Prove that the purely algebraic formulation of a vertex operator algebra is equivalent to this infinite-dimensional analytic and geometric formulation. This turned out to be a very difficult problem.

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- Main hard part: Prove that certain formal series obtained from vertex operators and the Virasoro operators are expansions of certain analytic functions coming from genus-zero Riemann surfaces and the determinant line bundle. This was done by using a theorem of Fischer and Grauert in the deformation theory of complex manifolds and the holomorphicity of the sewing isomorphisms for the determinant lines.
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- The graded dimension (vacuum character) of a vertex operator algebra in general is not modular invariant and thus is not enough to construct a genus-one conformal field theory. For affine Lie algebras and the Virasoro algebra, one needs to use all modules to obtain a modular invariant vector space. The modular invariance requirement forces us to consider modules for the vertex operator algebra, not just the algebra itself.
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# The operator product expansion conjecture of Moore and Seiberg

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$$\Phi_{\alpha,a}(z_1)\Phi_{\beta,b}(z_2) = \sum_{k} \sum_{c \in V_{kr}^l; c \in V_{kf}^l} F_{pk} \begin{bmatrix} i & j \\ l & r \end{bmatrix}_{ab}^{cc} \times \sum_{K \in \mathcal{H}_k} \xi_{p,K,d}^{\alpha,\beta} \begin{bmatrix} i & j \\ l & r \end{bmatrix} (z_1, z_2, z_3)\Phi_{K,c}(z_3)$$
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- This operator product expansion is in fact equivalent to the associativity for intertwining operators:

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# The modular invariance conjecture of Moore and Seiberg

1988, Moore and Seiberg: "The final equation is obtained from the two-point function on the torus. The conformal blocks for the two-point function of  $\beta_1 \in \mathcal{H}_{j_1}, \beta_2 \in \mathcal{H}_{j_2}$  are given by

$$\operatorname{Tr}_{i}\left[q^{L_{0}-\frac{c}{24}}\binom{i}{j_{1}p}_{z_{1}}(\beta_{1}\otimes\cdot)\binom{p}{j_{2}i}_{z_{2}}(\beta_{2}\otimes\cdot)\right]\cdot (dz_{1})^{\Delta_{\beta_{1}}}(dz_{2})^{\Delta_{\beta_{2}}}. \tag{4.13}$$

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- By stating that the conformal blocks for the two-point function are given by the traces above, Moore and Seiberg in fact assumed the modular invariance of the space spanned by these traces. This modular invariance was used as an additional hypothesis, not a result.
- Mathematically, it was clearly a powerful conjecture. Many
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- This partial result stated that when the vertex operator algebra satisfies a positive energy condition, a complete reducibility condition and a condition now called  $C_2$ -cofiniteness condition, the q-traces of products of n suitable modified vertex operators for irreducible modules can be analytically extended to meromorphic doubly-periodic functions on the plane with periods 1 and  $\tau = (\log q)/2\pi i$  and span a vector space that is invariant under the action of the full modular group  $SL(2,\mathbb{Z})$ .

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## A surprise

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- The nondegeneracy of the bilinear form is in fact equivalent to the rigidity of the corresponding braided tensor category. This is another indication why the proof of the rigidity needed the (full) modular invariance theorem and was so difficult, even in the case of Wess-Zumino-Witten models.

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- The problems and solutions discussed above are all for closed conformal field theories. We also need to construct open-closed conformal field theories.
- One needs to construct open-string vertex operator agebras. Modules for open-string vertex algebras in fact correspond exactly to the important D-branes introduced and studied by string theorists.
- The connection between the open part and the closed part of an open-closed conformal field theory is given by what is called Cardy condition, studied first by Cardy.
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- To study the moduli space of conformal field theories, one needs a deformation theory of conformal field theories. In particular, one needs a deformation theory of vertex operator algebras.
- Just as for associative algebras, Lie algebras and other algebras, one needs a correct cohomology theory for vertex operator algebras to develop a deformation theory and to study the structure and representation theory of vertex operator algebras. There had been proposals for such a theory before 2010, but unfortunately these proposals are not the correct cohomology theory.
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# Part 3 Open problems

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- The major problem to be solved in the higher-genus case is a convergence problem similar to the convergence problem for products and iterates of intertwining operators in the genus-zero case and the convergence problem for traces of products and iterates of intertwining operators.
- To prove this convergence, one needs to prove some conjectures on certain types of functions on the infinite-dimensional Teichmüller spaces and moduli spaces of Riemann surfaces with parametrized boundaries.
- Recently Radnell, Schipper and Staubach have been making good progress in the study of these Teichmüller spaces and moduli spaces. I hope that they will soon be able to establish those conjectures as theorems on functions on these spaces.



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