拓扑量子场论和几何不变量

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特别致谢!!!

吴可

高怡泓

过去三十年，量子场论和超弦理论深刻地影响着数学的发展，引发了多个数学领域的革命！！

Fields奖：Donaldson, Witten, Van Jones, Borchards, Drienfeld, Kontsevich, Okounkov.

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"几何物理"的兴起

专门研究几何物理的机构

Photo by C. Landgrover
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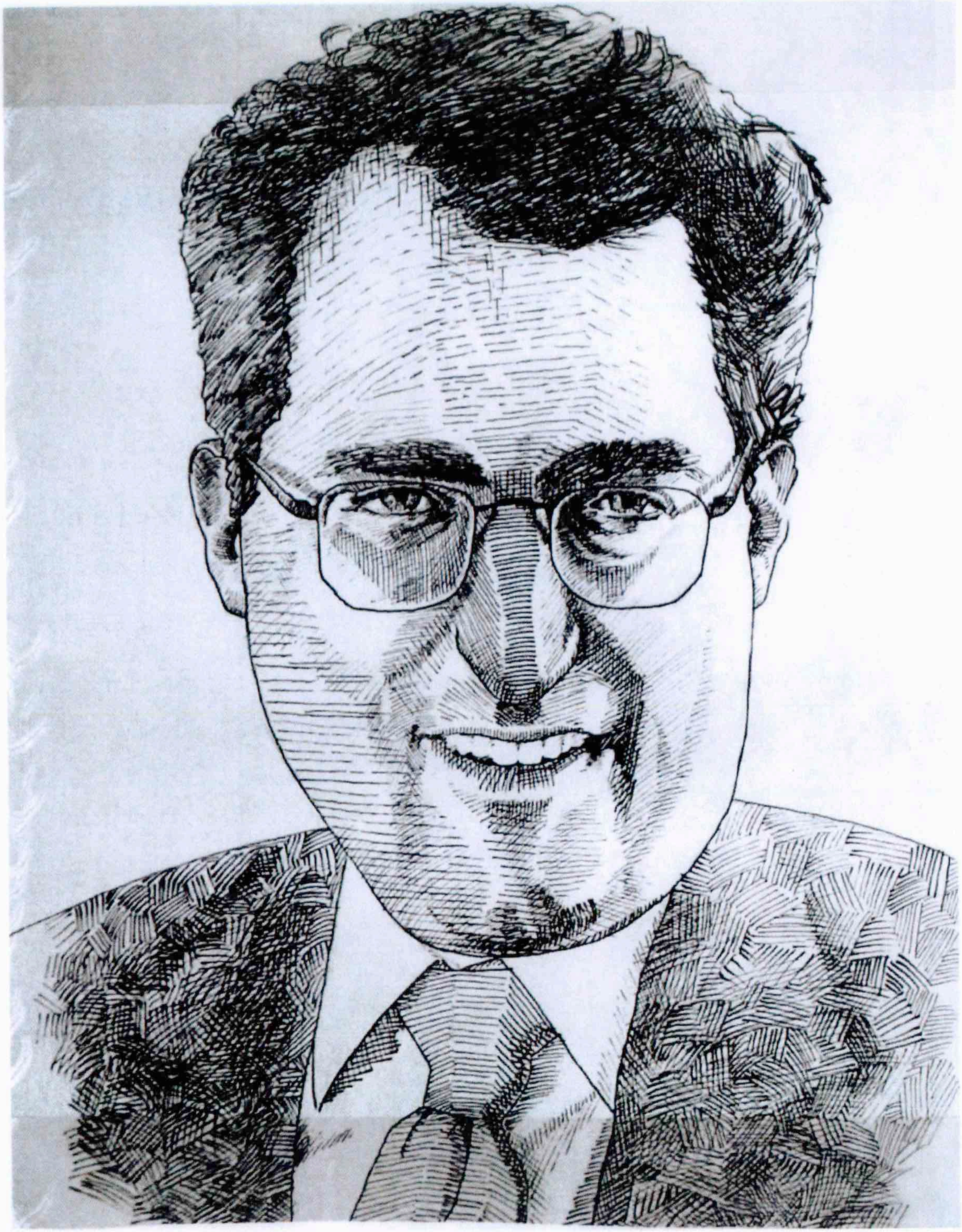
Simons 几何物理中心.

"几何物理"对数学的贡献：

- 广度是前所未有的（涉及多个数学领域）
- 深度是前所未有的（解决多个数学难题，如 $\mathbb{R}^4$ 上有无穷多不同的微分结构）
- 几个重要几何不变量
  - Donaldson 不变量，Floer 同调 } Topological Yang-Mills
  - Seiberg-Witten 不变量 } Topological Yang-Mills
  - Gromov-Witten 不变量 $\Leftarrow$ Topological σ-model
  - FJRW 不变量 $\Leftarrow$ Topological Landau-Ginzburg model



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Edward Witten

几何物理的主要

奠基人

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"拓扑量子场论"

拓扑量子场论 (TQFT):

corelation funct

$$\langle Q_1 \cdots Q_k \rangle = \int [d\Phi]\, e^{-S(\Phi)}\, Q_1(\Phi) \cdots Q_k(\Phi)$$

path integral

is independent of metric, ...

Easier case: S, Q_i are independent of metric

Example: Chern-Simons theory

$$S = \int_{M^3} \left(dA \wedge A + \frac{1}{3} A \wedge A \wedge A\right)$$

A - connection.

Cohomological TQFT (Witten):

BRST symmetry: δ — "odd" vector field (fermonic)

$\Phi \longrightarrow \delta\Phi$ – infinitesimal transformation

and $\delta^2 = 0$

Want (i) S – BRST invariant, i.e. $\delta S = 0$

(ii) Q_i – BRST closed, i.e. $\delta Q_i = 0$

$$\frac{\{\delta\text{-closed}\}}{\{\delta\text{-exact}\}} = \text{BRST cohomology}$$

Topological Invariance:

① $$\int [d\Phi]\, e^{-S(\Phi) + \delta \bar{S}(\Phi)}\, \mathcal{O}_1(\Phi) \cdots \mathcal{O}_k(\Phi) = \int [d\Phi]\, e^{-S(\Phi)}\, \mathcal{O}_1(\Phi) \cdots \mathcal{O}_k(\Phi)$$ $\Rightarrow$ independence of metric

② $$\langle \delta\mathcal{O}_1 \cdots \mathcal{O}_k \rangle = 0$$ $\Rightarrow$ path integral only depends on BRST cohomology class.

$\Uparrow$

motivated by Donaldson Theory

How to construct TQFT: "Twisting"

starting from a $N=2$ SUSY theory on $\boxed{\text{flat space}}$

want to define it on curved manifold

$\Downarrow$ (multiple $\delta_1, \delta_2, \dots$)

sacrifice some of super symmetry

TQFT (with a single δ) on curved manifold.

$N=2$ SUSY Yang-Mills $\Rightarrow$ Topological Yang-Mills

$(2,2)$ SUSY Nonlinear σ-model $\Rightarrow$ Topological σ-model

$(2,2)$ SUSY LG-model $\Rightarrow$ Topological LG-model

From TQFT $\Longrightarrow$ Instantons
(solution of Non-linear PDE)

一个神奇的现象：

on flat space, 多个 δ_i

$\Downarrow$ Twisting (把 δ_i 组合起来)

well-defined on curved space. $\delta = 0 \iff$ highly nontrivial non linear PDE

$\Uparrow$ $\boxed{\text{BRST fixed pt}}$ $\Updownarrow$ solution (instantons)

BRST Fixed pt Theorem:

$$\int [d\Phi]\, e^{-S(\Phi)}\, \mathcal{O}_1 \cdots \mathcal{O}_k$$

$$= \int_{\substack{\delta\text{-fixed pt} \\ \| \\ \{\phi : \Sigma \to X\} \\ \text{holomorphic map}}} (\qquad) \text{ — finite dim integral}$$

$\Leftarrow$ Top σ-model

mathematician "can" make sense of it

物理学家的工作已经结束！

数学家的工作刚刚开始！

Mathematical Invariants

path integral $\Leftrightarrow$ "count"

Yang-Mills instantons: $F_A + *F_A = 0 \Rightarrow$ Donaldson inv

Seiberg-Witten instantons: $\left\{\begin{array}{l} *F_A = \sigma(\phi,\phi) \\ \not{D}_A \phi = 0 \end{array}\right. \Rightarrow$ Seiberg-Witten inv

worldsheet instantons: $\partial_{\bar{z}} \phi = 0 \Rightarrow$ Gromov-Witten invariant

LG-instantons: $\partial_{\bar{z}} s_i + \overline{\partial_i W(s_1 \cdots s_n)} = 0$

$\hookrightarrow$ Fan-Jarvis-Ruan-Witt invariants

Topological σ-model $\Longrightarrow$ $\displaystyle\int_{\underset{\text{holomorphic map}}{\{\phi:\Sigma\to X\}}}$ "differential form" (observable)

$\{\phi:\Sigma\to X\}$ holomorphic map $= \mathcal{M}$ - moduli space of "instantons"

数学家的问题:

① $\mathcal{M}$ - compact ?
② $\mathcal{M}$ - smooth ?
$\Big\}$ Not true

③ what are "differential form" - more less Yes

(i) 上述三个问题数学上都很困难 (需好运!)
(ii) 需用数年的时间发展新的数学理论
(iii) 高成本 $\Rightarrow$ 高收益 (powerful new invariants in math)

<u>Topological σ-model</u>: $\int_{\mathcal{M}}$ "differential form"

$\mathcal{M}$ - moduli space of holomorphic maps

① $\mathcal{M} \subset \overline{\mathcal{M}}$ - certain geometric compactification (Gromov)

② $\overline{\mathcal{M}}$ - highly singular.

$\int_{[\overline{\mathcal{M}}]^{vir}}$ " " — virtual integration

③ "diff $\overrightarrow{\text{form}}$" (Donaldson)

$\Downarrow$

$\int_{[\overline{\mathcal{M}}]^{vir}}$ differential form $\Rightarrow$ Gromov-Witten invariant

"rigorously"

(Ruan: X semi positive
. . . .
- - - - - .)

- A powerful new inv in symplectic geometry

Gromov-Witten Invariants in Mathematics

- Connection to many other area of mathematics
- Revolutionize Enumerative Algebraic Geometry

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代数几何一个有一百多年历史
的古老分支

such as integrable system, combinatorics, representation theory

- One of most popular area in math

等价的物理模型 $\Longrightarrow$ 数学中一些神秘的 "duality"

A-twist $\Rightarrow$ A-model (Gromov-Witten Theory)

$\Updownarrow$ "trivially"　　　　$\Updownarrow$ mirror sysmetry

B-twist $\Rightarrow$ B-model (classical 代数几何)

Mirror symmetry $\Rightarrow$ amazing predications

Example: $X = \{ x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0 \} \subset \mathbb{CP}^4$

Gromov-Witten inv　$n_d \in \mathbb{Q}$　　$d = 1, 2, \cdots$.

① $n_1 = \#$ of lines
② $n_2 = \#$ of conic

$\Big\}$ classical

③ n_3: 数学家先算错了, 物理学家用 Mirror Symmetry $\Rightarrow$ 正确答案!

Gromov-Witten Theory：一个物理学家和数学家密切合作的领域

Gromov-Witten 不变量

- 有广泛应用的数学不变量
- Topological string

目前感兴趣的问题：

① 计算高亏格不变量

② 研究它的结构 ⟹ modular form

任何一个会议，都是物理学家和数学家"欢聚一堂"！

Topological Landau-Ginzburg (LG) model

Super potential $W: \mathbb{C}^n \to \mathbb{C}$ - holomorphic funct

Witten equation (1993) : $\bar{\partial} s_i + \overline{\partial_i W(s_1 \cdots s_n)} = 0$

$\Uparrow$

instanton equation

$s_i \in \Omega^0(L_i)$ $\quad \begin{matrix} L_i \\ \downarrow \\ \Sigma \end{matrix}$ - line bundle

BRST-formulation (Vafa-Hori, Sharp)

$\overline{\mathcal{M}}$ = space of solutions of Witten equations

Fan-Jarvis-Ruan spent 7 years (2002-2009) to develop a full mathematical package to treat $\overline{\mathcal{M}} \Rightarrow$ FJRW-theory

Landau-Ginzburg / Calabi-Yau correspondence

(Another physical equivalent model $\Longrightarrow$ math conjecture)

Motivating example: $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$

LG-model

W - super potential

$\Downarrow$

FJRW - invariant

$N_{g,d} \in \mathbb{Q}$

$\Downarrow$

generating function

$\mathcal{F}_g^{LG} = \sum_{d \geq 0} N_{g,d}\, t^d$

$X_W = \{W = 0\} \subset \mathbb{CP}^4$ defines the best known examples of Calabi-Yau 3-fold ($\Rightarrow$ string theory model)

$\Downarrow$

Gromov-Witten invariant

$n_{g,\beta} \in \mathbb{Q}$

$\Downarrow$

$\mathcal{F}_g^{GW} = \sum_{\beta \geq 0} n_{g,\beta}\, q^\beta$

$$\mathcal{F}_g^{LG} \xleftrightarrow[\text{continuation}]{\text{analytic}} \mathcal{F}_g^{GW}$$

LG/CY correspondence.

physical

mathematical

<u>Other TQFT</u> ? coupled with gauge theory

G - action

$\circlearrowleft$

$X \xrightarrow[\text{moment map}]{\mu} \mathfrak{g}^*$

<u>Topological LG-model coupled with gauge theory</u>

- Gauged Witten equation: $\left\{ \begin{array}{l} \bar\partial_A s_i + \overline{\partial_i W(s_1 \cdots s_n)} = 0 \\ * F_A = \mu(s_1 \cdots s_n) \end{array} \right.$

  A - gauge field on Σ

Fan - Jarvis - Ruan: "new theory"

结束语:

- 物理学家通过 TQFT 提出诸多数学新理论的框架
- 严格的建立这些数学新理论需数学家长期持久的努力
- 物理学家通过各种物理模型的等价性提出诸多深刻的数学猜测 (如 Mirror Symmetry, LG/CY correspondence)
- 而证明这些深刻的数学猜测尚需物理学家和数学家的共同努力.

共建数学和物理的“和谐社会”

谢谢大家！

THANK YOU!