Equivalence Relations, Classification Problems, and Descriptive Set Theory

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Equivalence Relations

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- if $(x, y) \in E$ and $(y, z) \in E$ then $(x, z) \in E$,

for all $x, y, z \in X$.

Equivalence Relations

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2. Orbit equivalence: if $G \curvearrowright X$ is an action of a group on a set, then define

$$x_1 \sim x_2 \iff \exists g \in G \ g \cdot x_1 = x_2$$

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$$\mu \ll \nu \iff \forall A\ (\nu(A) = 0 \Rightarrow \mu(A) = 0)$$

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   - the quotient space: $X/\sim = \{ [x]_\sim : x \in X\}$
   - the quotient map: $\pi : X \to X/\sim$ by $\pi(x) = [x]_\sim$
   - the quotient topology: $A \subseteq X/\sim$ is open iff $\pi^{-1}(A) \subseteq X$ is open.

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   where T is a suitably constructed maximally consistent term-complete theory in an extended language with new constant symbols.

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Note: This classification problem is an equivalence relation, in fact an orbit equivalence relation by the conjugacy action of the general linear group.

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Note: There are only countably many finitely generated abelian groups up to isomorphism.

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A Bernoulli shift is a quadruple $(X, \mathcal{B}, \mu, T)$, where

- $X = \{1, 2, \ldots, n\}^{\mathbb{Z}}$ for some $n \geq 1$,
- $\mathcal{B}$ is the Borel σ-algebra generated by the product topology on X,
- μ is a product measure given by a probability distribution $(p_1, \ldots, p_n)$ with $\sum_{i=1}^{n} p_i = 1$,
- T is the shift: for $x = (x_n)_{n\in\mathbb{Z}}$,

$$(Tx)_n = x_{n-1}$$

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Two Bernoulli shifts $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are isomorphic if there is a measure-preserving map Φ from a μ-measure 1 subset of X onto a ν-measure 1 subset of Y such that

$$\Phi(Tx) = S\Phi(x)$$

for μ-a.e. $x \in X$.

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Ornstein, 1970: Two Bernoulli shifts are isomorphic iff they have the same entropy.

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Gromov, 1999: The classification problem for compact metric spaces up to isometry is smooth.

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Oops! E is smooth in this sense iff $|X/E| \leq |\mathbb{R}|$.
We need the map I to be somehow "computable."

Descriptive Set Theory of Equivalence Relations

Try Again:
Let X be a topological space.
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Too restrictive! In many examples, the complete invariants are not computed continuously. In fact, if E is smooth in this sense, it has to be a closed subset of $X \times X$.

Let X be a standard Borel space (a space with a σ-algebra of Borel sets that is isomorphic to the real line).
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Note: $\mathbb{R}$ itself is a standard Borel space.

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Let X be a standard Borel space and E an equivalence relation on X.
Let Y be a standard Borel space and F an equivalence relation on Y.
We say that E is Borel reducible to F, denoted $E \leq_B F$, if there is a Borel map $f : X \to Y$ such that

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Note: This notion appeared in the 1980s and was borrowed from computational complexity theory. The notion of Borel reducibility gives a sense of relative complexity between equivalence relations.

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   but
   $\mathbb{R}/\mathbb{Q} \not\leq_B =$!

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Foreman–Rudolph–Weiss, 2011: The isomorphism problem for measure-preserving transformations is not a Borel equivalence relation.

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Glimm–Effros, 1960s: Let $G \curvearrowright X$ be a Borel action of a locally compact Polish group G on a standard Borel space. Let E be the orbit equivalence relation. Then either E is smooth or else $\mathbb{R}/\mathbb{Q} \leq_B E$.

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Harrington–Kechris–Louveau, 1990: Let E be any Borel equivalence relation on a standard Borel space. Then either E is smooth or else $\mathbb{R}/\mathbb{Q} \leq_B E$.

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- E_0: the eventual agreement equivalence relation on $\{0,1\}^{\mathbb{N}}$:

$$(x,y) \in E_0 \iff \exists n \forall m \geq n\ x(m) = y(m)$$

- Consider the shift action of $\mathbb{Z}$ on $\{0,1\}^{\mathbb{Z}}$:

$$(g \cdot x)(h) = x(h-g)$$

- The Pythagorean equivalence relation on $\mathbb{R}_+$: $x \sim y \iff x/y \in \mathbb{Q}$

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G.–Jackson, 2015: Any action of a countable abelian group gives rise to a hyperfinite equivalence relation.

Weiss' Question, 1980s: Does every action of a countable amenable group give rise to a hyperfinite equivalence relation?

Orstein–Weiss, 1980: Any action of a countable amenable group gives rise to a hyperfinite equivalence relation on a conull set.

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Uspenskij, 1986: There is a universal Polish group, i.e., a Polish group which contains a copy of every other Polish group as a closed subgroup.

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Examples of turbulent equivalence relations include the measure equivalence, $\mathbb{R}^{\mathbb{N}}/\ell^p$, $\mathbb{R}^{\mathbb{N}}/c_0$, etc.

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- compact operators
- self-adjoint operators
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- general bounded linear operators is a Borel equivalence relation (Ding–G. 2014, Hjorth–Törnquist 2012).

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- (Chang–G., 2016?) the homeomorphism of all continua.

Summary

We defined four main benchmark equivalence relations (of increasing complexity):

$=$: the equality equivalence (smooth)

$\mathbb{R}/\mathbb{Q}$: the Vitali equivalence (hyperfinite)

E_{S_∞} (usually referred to as graph isomorphism)

E_G^∞: the universal orbit equivalence relation

Summary

What about

- general bounded linear operators on an infinite dimensional Hilbert space (up to unitary equivalence)?
- arbitrary countable groups up to isomorphism?
- general measure-preserving transformations up to isomorphism?
- representations of general separable C^*-algebras up to unitary equivalence?
- general separable complete metric spaces up to isometry?
- compact metric spaces up to homeomorphism?

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All have partial or complete solutions.

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Challenge to the audience:

- Develop a Spectral Theory for general bounded linear operators.
- Classify separable locally compact metric spaces up to isometry.
- Determine the exact complexity of the isomorphism of all measure-preserving transformations (von Neumann's problem)

Invariant Descriptive Set Theory, CRC Press, 2009.

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Thank you for your attention!