Equivalence Relations, Classification Problems, and Descriptive Set Theory

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September 9, 2015
Equivalence Relations

An equivalence relation on a set $X$ is a binary relation $E \subseteq X \times X$ such that:

1. $\left( x, x \right) \in E$, for all $x \in X$.
2. If $\left( x, y \right) \in E$, then $\left( y, x \right) \in E$.
3. If $\left( x, y \right) \in E$ and $\left( y, z \right) \in E$, then $\left( x, z \right) \in E$, for all $x, y, z \in X$. 
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Equivalence Relations

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   $$g_1 \sim g_2 \iff g_1^{-1}g_2 \in H \iff g_1H = g_2H$$

2. **Orbit equivalence**: if $G \curvearrowright X$ is an action of a group on a set, then define

   $$x_1 \sim x_2 \iff \exists g \in G \; g \cdot x_1 = x_2$$
Equivalence Relations

Examples in measure theory:

3. Vitali set: Consider the cosets of \( \mathbb{Q} \) in \( \mathbb{R} \). Using AC, find a set \( V \) that meets each coset at exactly one point. \( V \) is not Lebesgue measurable.

4. Measure equivalence: two measures are equivalent iff they are absolutely continuous to each other. \( \mu \ll \nu \iff \forall A (\nu(A) = 0 \Rightarrow \mu(A) = 0) \)
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Equivalence Relations

Examples in topology:

5. Quotient space: If $X$ is a topological space and $\sim$ an equivalence relation on $X$, then define

\[ X/\sim = \{ [x] \sim : x \in X \} \]

the quotient map:

\[ \pi : X \to X/\sim \text{ by } \pi(x) = [x] \sim \]

the quotient topology:

\[ A \subseteq X/\sim \text{ is open iff } \pi^{-1}(A) \subseteq X \text{ is open.} \]
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Equivalence Relations

Examples in logic:

Gödel's Completeness Theorem: Every consistent set of first-order sentences has a model.
Henkin constructed a model using all first-order terms and defining
\[ t \sim s \iff T \vdash t = s \]
where \( T \) is a suitably constructed maximally consistent term-complete theory in an extended language with new constant symbols.
Equivalence Relations

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Classification Problems: Examples

Example: Classify square matrices up to similarity: $A$ and $B$ are similar iff there is a nonsingular matrix $S$ such that $A = S^{-1}BS$.

Two square matrices are similar iff they have the same Jordan normal form.

Note: This classification problem is an equivalence relation, in fact an orbit equivalence relation by the conjugacy action of the general linear group.
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Example Classify finitely generated abelian groups up to isomorphism. Every finitely generated abelian group is isomorphic to a direct sum $\mathbb{Z}/p^r \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^r \mathbb{Z} \oplus \mathbb{Z}^m$. Note: There are only countably many finitely generated abelian groups up to isomorphism.
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Example: Classify all Bernoulli shifts up to isomorphism.

A Bernoulli shift is a quadruple \((X, B, \mu, T)\), where

- \(X = \{1, 2, \ldots, n\}\) for some \(n \geq 1\),
- \(B\) is the Borel \(\sigma\)-algebra generated by the product topology on \(X\),
- \(\mu\) is a product measure given by a probability distribution \((p_1, \ldots, p_n)\) with \(\sum_{i=1}^{n} p_i = 1\),
- \(T\) is the shift: for \(x = (x_n)_{n \in \mathbb{Z}} \in X\), \((Tx)_n = x_{n-1}\).
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Example Classify all Bernoulli shifts up to isomorphism. Two Bernoulli shifts \((X, \mathcal{B}, \mu, T)\) and \((Y, \mathcal{C}, \nu, S)\) are isomorphic if there is a measure-preserving map \(\Phi\) from a \(\mu\)-measure 1 subset of \(X\) onto a \(\nu\)-measure 1 subset of \(Y\) such that

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Ornstein, 1970: Two Bernoulli shifts are isomorphic iff they have the same entropy.
A classification problem is smooth if one can assign a real number as a complete invariant of the object under classification.

All above examples are smooth.

Effros, 1965: The classification problem for representations of Type I separable $\mathcal{C}^*$-algebras up to unitary equivalence is smooth.

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Classification Problems: Generalizations

Questions:
- What about general bounded linear operators on an infinite-dimensional Hilbert space (up to unitary equivalence)?
- What about arbitrary countable groups up to isomorphism?
- What about general measure-preserving transformations up to isomorphism?
- What about representations of general separable C*-algebras up to unitary equivalence?
- What about general separable complete metric spaces up to isometry?
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Descriptive Set Theory of Equivalence Relations

We develop a framework to study equivalence relations and classification problems.

First Try: Let $X$ be a set (of mathematical objects). Let $E$ be an equivalence relation on $X$ ($E$ is a notion of equivalence). We say that $E$ is smooth if there is a map $I: X \to \mathbb{R}$ such that $(x_1, x_2) \in E \iff I(x_1) = I(x_2)$.

Oops! $E$ is smooth in this sense iff $|X/E| \leq |\mathbb{R}|$. We need the map $I$ to be somehow “computable.”
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Try Again:
Let $X$ be a topological space.
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Too restrictive! In many examples, the complete invariants are not computed continuously. In fact, if $E$ is smooth in this sense, it has to be a closed subset of $X \times X$. 
Let $X$ be a standard Borel space (a space with a $\sigma$-algebra of Borel sets that is isomorphic to the real line). Let $E$ be an equivalence relation on $X$. We say that $E$ is smooth if there is a Borel map $I : X \to \mathbb{R}$ such that 

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Note: $\mathbb{R}$ itself is a standard Borel space.
Let $X$ be a standard Borel space and $E$ an equivalence relation on $X$.
Let $Y$ be a standard Borel space and $F$ an equivalence relation on $Y$.
We say that $E$ is Borel reducible to $F$, denoted $E \leq_B F$, if there is a Borel map $f : X \to Y$ such that
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(x_1, x_2) \in E \iff (f(x_1), f(x_2)) \in F.
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**Note:** This notion appeared in the 1980s and was borrowed from computational complexity theory. The notion of Borel reducibility gives a sense of relative complexity between equivalence relations.
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2. The Vitali equivalence relation $\mathbb{R}/\mathbb{Q}$: $x \sim y$ iff $x - y \in \mathbb{Q}$. 

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**Feldman, 1957:** The isomorphism problem for measure-preserving transformations is not smooth. Therefore, there is no notion of generalized entropy which can serve as the complete invariant of a measure-preserving system.

**Foreman–Rudolph–Weiss, 2011:** The isomorphism problem for measure-preserving transformations is not a Borel equivalence relation.
The Vitali Equivalence Relation

Glimm–Effros, 1960s: Let $G \curvearrowright X$ be a Borel action of a locally compact Polish group $G$ on a standard Borel space. Let $E$ be the orbit equivalence relation. Then either $E$ is smooth or else $\mathbb{R}/\mathbb{Q} \leq B_E$.

Harrington–Kechris–Louveau, 1990: Let $E$ be any Borel equivalence relation on a standard Borel space. Then either $E$ is smooth or else $\mathbb{R}/\mathbb{Q} \leq B_E$. 

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- $E_0$: the eventual agreement equivalence relation on $\{0, 1\}^\mathbb{N}$:
  
  $$(x, y) \in E_0 \iff \exists n \forall m \geq n \ x(m) = y(m)$$

- Consider the shift action of $\mathbb{Z}$ on $\{0, 1\}^\mathbb{Z}$:
  
  $$(g \cdot x)(h) = x(h - g)$$

- The **Pythagorean equivalence relation** on $\mathbb{R}_+$:
  
  $x \sim y \iff x/y \in \mathbb{Q}$
The Vitali Equivalence Relation


G.–Jackson, 2015: Any action of a countable abelian group gives rise to a hyperfinite equivalence relation.

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Other Benchmark Equivalence Relations

Polish group: a topological group with a Polish topology, i.e., separable completely metrizable topology

Becker–Kechris, 1993: For any Polish group $G$ there is a universal action of $G$, i.e., a Borel action of $G$ on some standard Borel space $X$ such that $E_X^G$ is the most complexity among all orbit equivalence relations by $G$.

We denote this universal $G$-orbit equivalence relation by $E^G$.

Mackey, 1963: If $G$ is a Polish group and $H \leq G$ is a closed subgroup or a topological quotient of $G$, then $E_H^G \leq B_{E_X^G}$.

Uspenskij, 1986: There is a universal Polish group, i.e., a Polish group which contains a copy of every other Polish group as a closed subgroup.
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Among many equivalence relations of the form $E_G$, I will mention a selected few that were studied intensively.

- $S^\infty$: the permutation group of $\mathbb{N}$ over $\mathbb{Q}$.
- $\mathbb{B}$: the Borel bireducible to:
  - (Friedman–Stanley, 1989) the isomorphism relation for all countable groups/graphs/trees/fields;
  - (Camerlo–G., 2001) the isomorphism relation for all countable Boolean algebras;
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Examples of turbulent equivalence relations include the measure equivalence, $\mathbb{R}^N/\ell^p$, $\mathbb{R}^N/c_0$, etc.
Other Benchmark Equivalence Relations

\( U(H) \) or \( U_\infty \): the unitary group of the infinite dimensional separable complex Hilbert space
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The unitary equivalence of
- compact operators
- self-adjoint operators
- unitary operators
- general bounded linear operators

are all important problems in functional analysis.
Other Benchmark Equivalence Relations

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- general bounded linear operators is a Borel equivalence relation (Ding–G. 2014, Hjorth–Törnquist 2012).
Other Benchmark Equivalence Relations

Uspenskij’s universal Polish groups

- the isometry group of the universal Urysohn space $\text{Iso}(\mathbb{U})$
  1990;

- the homeomorphism group of the Hilbert cube $H([0, 1]^\mathbb{N})$
  1986

These give rise to universal orbit equivalence relations.

Kechris’ Question, 1980s: Is there a surjectively universal Polish group, i.e., one that has all other Polish groups as a quotient group?

Ding, 2012: Yes! (by a complicated construction)
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We defined four main benchmark equivalence relations (of increasing complexity):

\[ = \]: the equality equivalence (smooth)

\( \mathbb{R}/\mathbb{Q} \): the Vitali equivalence (hyperfinite)

\( E_{S_\infty} \) (usually referred to as graph isomorphism)

\( E_{G^\infty} \): the universal orbit equivalence relation
What about

- general bounded linear operators on an infinite dimensional Hilbert space (up to unitary equivalence)?
- arbitrary countable groups up to isomorphism?
- general measure-preserving transformations up to isomorphism?
- representations of general separable C*-algebras up to unitary equivalence?
- general separable complete metric spaces up to isometry?
- compact metric spaces up to homeomorphism?
All have partial or complete solutions.
Summary

All have partial or complete solutions.

There is much to be done...
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Challenge to the audience:

- Develop a Spectral Theory for general bounded linear operators.
Summary

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Challenge to the audience:

▶ Develop a Spectral Theory for general bounded linear operators.

▶ Classify separable locally compact metric spaces up to isometry.
All have partial or complete solutions.

There is much to be done...

Challenge to the audience:

▶ Develop a Spectral Theory for general bounded linear operators.
▶ Classify separable locally compact metric spaces up to isometry.
▶ Determine the exact complexity of the isomorphism of all measure-preserving transformations (von Neumann’s problem)

Thank you for your attention!