Special Lagrangian equations

Yu YUAN

In memory of my teacher, Ding Weiyue Laoshi

Part 1 Intro: Equs

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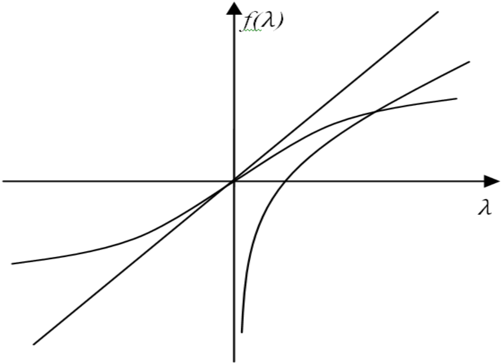
$\circ\ \sigma_k = \lambda_1 \cdots \lambda_k + \cdots = 1$

Part 1 Intro: Ellipticity & Convexity



- elliptic $\Leftrightarrow$ $f(\lambda)$ monotonic

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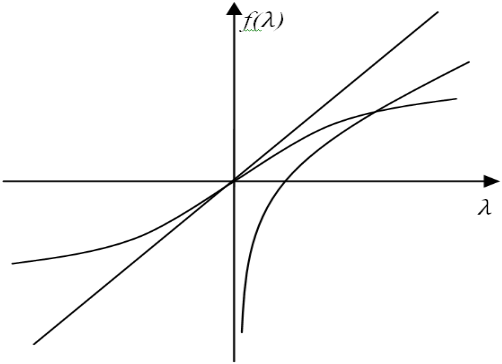
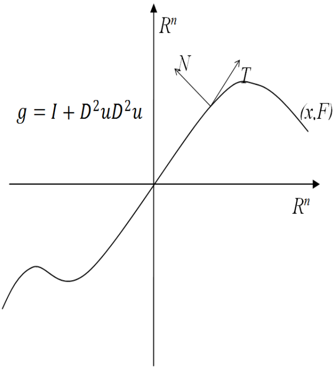


- elliptic $\Leftrightarrow$ $f(\lambda)$ monotonic
- saddle shape adds obstacles

Part I Intro: Lagrangian, and special Lagrangian



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$\Leftrightarrow JT=N$

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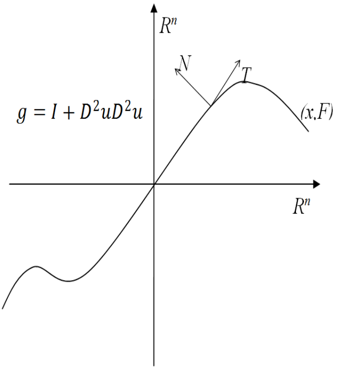

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$\Updownarrow$ Harvey-Lawson 70s

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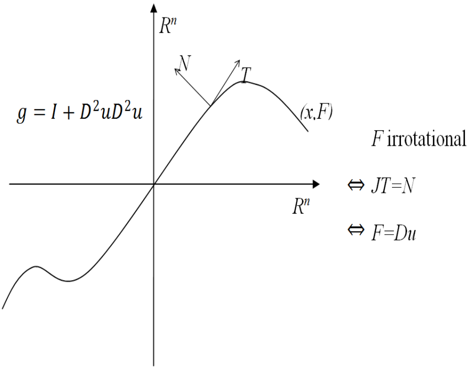

- special Lagrangian $\Leftrightarrow$ $\arctan D^2 u = \Theta$

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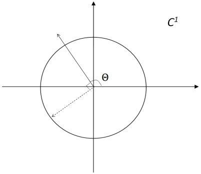
- minimal, volume minimizing compared to surfaces with same bdry, Lag or NOT (calibration argument).

Part 1 Intro: algebraic form of equ



$$(1+i\lambda_1)\cdots(1+i\lambda_n)=\sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)}\,(\cos\Theta+i\sin\Theta)$$

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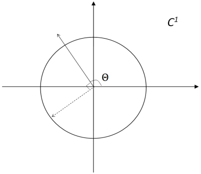

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$$Vol=\sqrt{\det g}=\sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)}=|(1+i\lambda_1)\cdots(1+i\lambda_n)|$$
$$=\cos\Theta\,(1-\sigma_2+\cdots)+\sin\Theta\,(\sigma_1-\sigma_3+\cdots)$$

"$|\;|$" $\sqrt{}$ in divergence form

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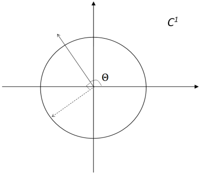

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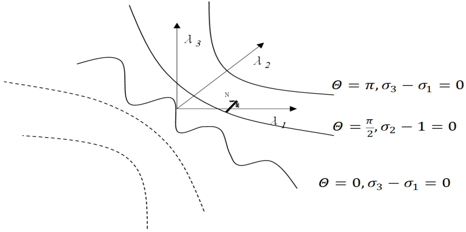
Equ $\quad \Sigma=\cos\Theta\,(\sigma_1-\sigma_3+\cdots)-\sin\Theta\,(1-\sigma_2+\cdots)=0$

Case $n=2,3$ $\Theta=\pi/2 \ \Rightarrow \ \sigma_2=1$

Part 1 Intro: level set

Case n=3



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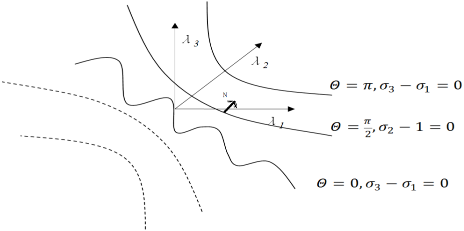

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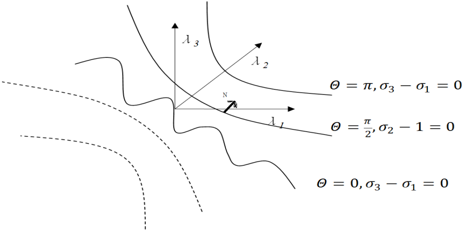

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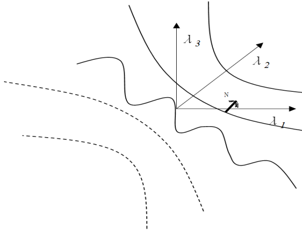

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RMK. $\left(R^{2n}, dx^2 - dy^2 \text{ or } dxdy\right)$

vol maximizing Lagrangian $\Leftrightarrow$ M-A equ

Part 2 What to do?

○ Existence
○ Properties: Liouville-Bernstein type results; regularity...
ALL depend on estimates

$$\left\| D^2 u \right\|_{L^\infty(B_1)} \leq C \left(\left\| Du \right\|_{L^\infty(B_2)} \right) \leq C \left(\left\| u \right\|_{L^\infty(B_3)} \right)$$

RMK. $\left\| D^2 u \right\|_{C^\alpha(B_{1/2})} \leq C \left(\left\| D^2 u \right\|_{L^\infty(B_1)} \right)$ can be achieved by
* PDE way w/ convexity Evans-Krylov-Safonov (Non-div);
Evan-Krylov-De Giorgi-Nash (div)
* GMT way SLag OK, M-A?
* Geometric way M-A Calabi 50s C^3 est, SLag?

Part 3 Results: i Global rigidity

- Recall Liouville, Jörgens-Calabi-Pogorelov/Cheng-Yau: global convex sol to Laplace, M-A must be quadratic

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- Caffarelli–Yuan 09 Lipschitz and $C^{1,r}$ w/ $r \in (0, 1 - 2/n]$ sol to $\det D^2 u = 1$

Theorem (Dake Wang-Y 11)

Suppose u smooth sol. to $\sum \arctan \lambda_i = \Theta$ *in* $B_1 \subset R^n$. *Then*

$$\left|D^2 u(0)\right| \leq C(n) \exp\left[C(n) \|Du\|_{L^\infty(B_1)}^{2n-2}\right] \quad \text{for } |\Theta| \geq (n-2)\pi/2$$

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·Chen-Warren-Y 07 convex sol.

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Q. Sharp exponent for Hessian estimates in dim $n \geq 3$?

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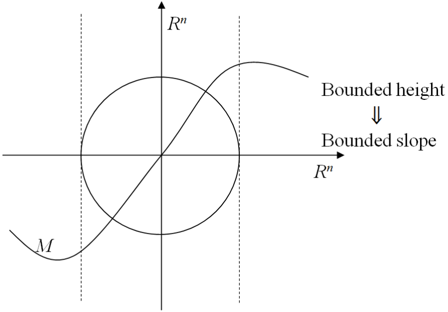
- global sol to $\sum \arctan \lambda_i = (n-2)\,\pi/2$ w/ quadratic growth is quadratic.

Idea of Warren-Y & Dake Wang-Y



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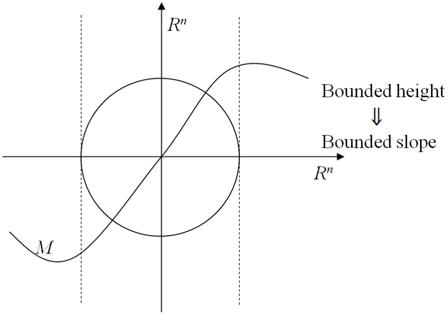

Heuristically, the reciprocal of the 2nd derivative norm/slope is super harmonic, once it is 0 somewhere inside, then it is 0 everywhere, that is, the 2nd derivative norm/slope is infinite everywhere. But we start from a 1st derivative graph $(x, Du(x))$. Impossible!

The "secret" ingredient is

$$\triangle_g \frac{1}{\sqrt{1+\lambda_{\max}^2}} \le 0 \Leftrightarrow \triangle_g \log\sqrt{1+\lambda_{\max}^2} \ge \left|\nabla_g \log\sqrt{1+\lambda_{\max}^2}\right|^2 .$$

Technically,

$$\begin{aligned} \text{slope} \;&\underset{\text{mean value}}{\overset{1}{\le}} \int \text{slope} \underset{\text{Sobolev}}{\overset{2}{\le}} \int |\nabla \text{slope}| \\ &\underset{\substack{\text{Jacobi}\\ \text{conformality}\\ \text{divergence form of } \sigma_k}}{\overset{3}{\le}} \int \text{weighted vol} \underset{\text{divergence}}{\overset{4}{\le}} \text{height} \end{aligned}$$

Step 1/2/3. * Strongly subharmonic slope

$$b = \ln \sqrt{1 + \lambda_{\max}^2}$$

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$$\int \left|\nabla_g b\right|^2 dv_g \leq C\left(n\right) \int \left|\nabla_g \varphi\right|^2 dv_g.$$

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$$\int |\nabla_g b|^2 \, dv_g \leq C(n) \int |\nabla_g \varphi|^2 \, dv_g.$$

* Twist multiplication to contain terms involving derivatives of cut-off fcns

Part 3. Results: ii A priori est. SLag w/ subcritical phases -

- Nadirashvili-Vladuct 09 $C^{1,1/3}$ solution to $\sum_{i=1}^{3} \arctan \lambda_i = 0$

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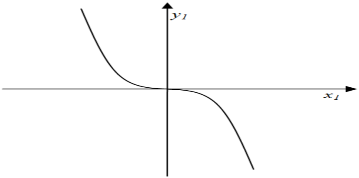
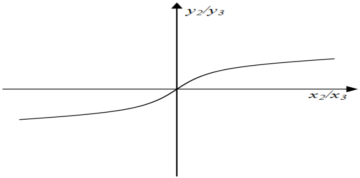
Ideas of D. Wang-Y rotation way
* "Bounded" coordinates w/ Lag angles near $(0^-, \pi/4, \pi/4)$, via Cauchy-K solve $\sigma_2\left(D^2u\right) = 1$ or $\arctan D^2u = \pi/2$
* Switch x-y coordinates, still Lag now w/ Lag angles $(-\pi/2, \pi/4, \pi/4)$, that is $\arctan D^2\tilde{u} = 0$
* Rotate down z_2, z_3 plane, keep z_1 plane to increase the phase from 0 to $\Theta < \pi/2$, then $\arctan D^2\tilde{\tilde{u}} = \Theta$ w/ $\Theta \in (0, \pi/2)$





Part 3 Parabolic side, longtime existence & estimates

Lagrangian mean curvature flow in $R^n \times R^n$

$$\partial_t U = g^{ij} \partial_{ij} U \quad \text{w/ } U = Du$$

Euclidean $\left(R^{2n}, dx^2 + dy^2\right)$, $g = I + D^2 u D^2 u \Leftrightarrow \partial_t u = \arctan D^2 u$
eg. 1-d $U(x,t) = u_x(x,t)$, potential equ of mean curv. flow and mean curv. flow in nondiv form are respectively

$$u_t = \arctan u_{xx} \quad \leftrightarrow \quad U_t = \frac{1}{1+U_x^2} U_{xx}$$

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ii) $\left\| D^l u \right\|_{L^\infty(R^n)} \leq C_l / t^{l-2}$ for $l \geq 3$ & $t > 0$
iii) $Du(x,t)$ is $C^{1/2}$ in time at $t = 0$

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Consequence: Via $U(n)$ rotation, the long time existence and ests for smooth sols also hold for locally $C^{1,1}$ initial potential u_0 being

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RMK2. C-eg by Neves-Y 09 shows all the above conds are sharp:

∘ almost convexity $-(1+\eta)I \leq D^2u_0 \leq (1+\eta)I$ not preserved (Gauss image of (x, Du_0) not a convex set in Grassmanian $LG(n,n)$)

∘ convex consequence. If just $D^2u_0 \geq -\eta I$, then graphical condition is not preserved, i.e. $D^2u = \infty$ at some point

∘ large phase consequence. If just arctan $\left[D^2u_0\right] \geq (n-1)\pi/2 - \eta$, then graphical condition is not preserved.

RMK3. But by Chau-Chen-Y, in the latter convex & large phase c-egs, the Lagrangian manifolds themselves flows smoothly & forever (didn't know in 09).

Earlier results

◦ Smoczyk-Wang 02: longtime existence in periodic setting w/ initial u_0 satisf. $0 \leq D^2 u_0 \leq C$ or equivalently $(-1+\delta) I \leq D^2 u_0 \leq (1-\delta) I$ (Direct application of Krylov Theory on convex fully nonlinear parabolic equs)

◦ Chen-Pang 09: longtime existence and UNIQUENESS of C^0 viscosity sol w/ C^0 initial potential

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$\circ$ $u_t = \arctan D^2 u$ degenerate parabolicity/diffusion $\dashrightarrow$ uniq. & longtime C^0 sol

saddle shape $\dashrightarrow$ higher derivative finite time blow up

Ideas of Chau-Chen-Y: Compactness arguments powered by Chen-Pang Uniqueness and Nguyen-Y (09) parabolic Schauder $C^{2+1,1+1/2}$ est. for Lag. heat equ

$$[u_t]_{1,1/2;Q_{1/2}} + \left[D^2 u\right]_{1,1/2;Q_{1/2}} \leq C\left(\sqrt{3}\right),$$

$$\text{provided } \left|D^2 u\right| \underset{\text{vol convex cond.}}{\leq} \sqrt{3} \text{ in } Q_1$$

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Part 4 Parabolic side, self similar sols for curvature flows w/ potential

- Lagrangian mean curvature flow in R^{2n}

$$\partial_t U = g^{ij}\partial_{ij} U \quad \text{w/ } U = Dv$$

Euclidean $\left(R^{2n}, dx^2 + dy^2\right)$, $g = I + D^2 v D^2 v \Leftrightarrow \partial_t v = \arctan D^2 v$
Pseudo-Euclidean $\left(R^{2n}, dxdy\right)$, $g = D^2 v \quad \Leftrightarrow \quad \partial_t v = \ln\det D^2 v$

- Kahler Ricci flow

$$\left.\begin{array}{r} \partial_t g_{i\bar{k}} = -R_{i\bar{k}} \\ g_{i\bar{k}} = v_{i\bar{k}} \\ Ric = -\partial\bar{\partial}\ln\det\partial\bar{\partial}v \end{array}\right\} \Leftrightarrow \quad \partial_t v = \ln\det\partial\bar{\partial}v$$

Consider self similar shrinking sols in $R^{2n} \times (-\infty, 0)$:
$v(x,t) = -tu\left(x/\sqrt{-t}\right)$

$$\arctan D^2 / \ln\det D^2 / \ln\det\partial\bar{\partial} \ \ u = \frac{1}{2}x \cdot Du(x) - u(x).$$

Th'm (Chau-Chen-Y. 10)

- Let u be an entire smooth sol to $\arctan\lambda_1 + \cdots + \arctan\lambda_n = \frac{1}{2}x \cdot Du(x) - u$ in R^n. Then $u = u(0) + \frac{1}{2}\left\langle D^2u(0)\,x, x\right\rangle$.
- Let u be an entire smooth convex sol to $\ln\det D^2u = \frac{1}{2}x \cdot Du(x) - u$ in R^n satisfying $D^2u(x) \geq \frac{2(n-1)}{|x|^2}$ for large $|x|$. Then u is quadratic.
- Let u be an entire smooth pluri-subharmic sol to $\ln\det\partial\bar{\partial}u = \frac{1}{2}x \cdot Du(x) - u$ in C^m satisfying $\partial\bar{\partial}u \geq \frac{2m-1}{2|x|^2}$ for large $|x|$. Then u is quadratic.

Th'm (Drugan-Lu-Y. 13)

Let u be an entire smooth pluri-subharmonic sol to $\ln\det\partial\bar{\partial}u = \frac{1}{2}x \cdot Du(x) - u$ in $\Omega \subseteq C^m$ s.t. the metric $g = \partial\bar{\partial}u$ is complete. Then $u(x)$ is quadratic.

RMK. For arctan case: Y. 09 bounded Hessian, Chau-Chen-He 09 $\left|D^2 u\right| \leq 1-\delta$, R. Huang-Z. Wang 10, $\left|D^2 u\right| \leq 1$, rigidity was derived. For $\ln\det D^2$ case w/ similar lower bound, R. Huang-Z. Wang 10 derived the rigidity.
Y. 15 Every Euclid complete graphical shrinker (x, Du) in $\Omega \times R^n$ in the sense $|x|^2 + |Du|^2 = \infty$ on $\partial\Omega$, is a (Lagrangian) plane.

RMK. For $\ln\det D^2$ case: Q. Ding-Xin 12, rigidity of entire sol, without any lower; Drugan-Lu-Y works for real M-A case w/ complete metric

RMK. For $\ln\det \partial\bar\partial$ case: $m = 1$ W.L. Wang 15 every entire self similar sol on complex plane is quadratic; general dim m W.L. Wang 15 every entire self similar sol w/ real convex potential is quadratic.

Ideas of Chau-Chen-Y & Drugan-LU-Y

*

$$g^{ij}\partial_{ij}\Theta\left(x\right)=\frac{1}{2}x\cdot D\Theta\left(x\right)$$

The amplifying force in the right forces bounded $\Theta\left(x\right)=\arctan D^2u\left(x\right)$ to be constant.

* Self similarity and Euler's formula show smooth potential u is homog. order two, then quadratic

$$\Theta\left(0\right)=\frac{1}{2}x\cdot Du\left(x\right)-u\left(x\right)$$

Part 5 Questions

Q1. Pointwise argument toward slope (D^2u) est for SLag?

Recall for minimal surface equ $\operatorname{div}\left(Df/\sqrt{1+|Df|^2}\right)=0$,

Bombieri-De Giorgi-Miranda (60s), Trudinger (70s) integral way

$$|Df(0)| \leq C(n)\exp\left[C(n)\|f\|_{L^\infty(B_1)}\right]$$

Korevaar 80s Pointwise way, slick, relying on pointwise Jacobi.

Q2. Construction of nontrivial sol to $\arctan D^2u=(n-2)\frac{\pi}{2}$ in $R^{n>3}$. $\exists$? homog order 2 sol to $\arctan D^2u=0$ in dim $n\geq 5$?

Q3. Any further regularity beyond C^0 for C^0 viscosity sol to $\arctan D^2u=\Theta$ w/ $|\Theta|<(n-2)\pi/2$?

Q4. Every entire sol to $\ln\det\partial\bar{\partial}u=\frac{1}{2}x\cdot Du(x)-u$ in C^m is quadratic?

RMK. Self-similarity makes sol to the eigenvalue equ more rigid. In contrast, $\exists$ nontrivial (non flat) entire and complete sol to equ $\ln\det\partial\bar{\partial}u=0$ in C^m by LeBrun, Hitchin ... 80s.