Fully Nonlinear PDEs and Related Geometric Problems

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The three well known classical PDEs are Laplace equation, heat equation, and wave equation, representing the three types of PDEs with rather distinct properties: elliptic, parabolic and hyperbolic equations, respectively.

- **Laplace equation** – elliptic
  \[ \Delta u := \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \]

- **Heat equation** – parabolic
  \[ u_t = \Delta u \]

- **Wave equation** – hyperbolic
  \[ u_{tt} = \Delta u \]
In general, an elliptic PDE of the second order can be written in the form

\[ F(\nabla^2 u, \nabla u, u, x) = 0. \]

The **ellipticity** means

\[ \{ F^{ij}[u] \} \equiv \{ F^{ij}(\nabla^2 u, \nabla u, u, x) \} > 0 \]

where, if we write \( F(A, \cdot, \cdot, \cdot) \) and \( A = \{ a_{ij} \} \in S^{n \times n} \),

\[ F^{ij} \equiv \frac{\partial F}{\partial a_{ij}} \]

\( S^{n \times n} \) is the set of \( n \) by \( n \) symmetric matrices. Equivalently, the linearized operator

\[ \mathcal{L}_u = F^{ij}[u] \nabla_i \nabla_j + \text{lower order terms} \]

is elliptic.
Equation (2) is
• **linear**, if $F$ is linear in $u$, $\nabla u$ and $\nabla^2 u$; otherwise, **nonlinear**;
• **semilinear**, if $F$ is linear in $\nabla u$ and $\nabla^2 u$;
• **quasilinear**, if $F$ is linear in $\nabla^2 u$;
• **fully nonlinear**, if $F$ is not linear in $\nabla^2 u$. 
Let’s first look at some examples of nonlinear equations from geometry.

- The minimal surface equation
  \[
  \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0.
  \]
  For \( n = 2 \),
  \[
  (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0.
  \]
- The minimal surface equation in hyperbolic space
  \[
  \left( \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \right) u_{ij} = \frac{n}{u}.
  \]
- The spacelike maximal surface equation in Minkowski space
  \[
  \text{div} \left( \frac{Du}{\sqrt{1 - |Du|^2}} \right) = 0.
  \]
  The spacelike condition
  \[
  |Du| < 1.
  \]
• The Monge-Ampère equation
\[ \det D^2 u = \psi. \]
For \( n = 2 \),
\[ u_{xx}u_{yy} - u_{xy}^2 = \psi. \]
• The complex Monge-Ampère equation
\[ \det \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = \psi. \]
• The prescribed Gauss curvature equation
\[ \det D^2 u = K \left( 1 + |Du|^2 \right)^{\frac{n+2}{2}}. \]
For spacelike hypersurfaces in Minkowski space
\[ \det D^2 u = K \left( 1 - |Du|^2 \right)^{\frac{n+2}{2}}. \]
There are important fourth order equations in geometry.

- Willmore surface equation.
- Affine maximal hypersurface equation

\[ u^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \det D^2 u = 0. \]
1. **The Isoperimetric Inequality.** Let $C$ be a simple closed curve in $\mathbb{R}^2$. Then

$$4\pi A \leq L^2$$

where $A$ is the enclosed area, $L$ denotes the length of $C$.

More generally, let $\Omega$ be a domain in $\mathbb{R}^n$. Then

$$\omega_1(n|\Omega|^{n-1} \leq |\partial \Omega|^n$$

where $\omega$ denotes the volume of the unit sphere in $\mathbb{R}^n$.

**Proof.** Consider the Neumann problem

$$\Delta u = C \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 1 \text{ on } \partial \Omega.$$ 

Let $\Gamma^+$ be the lower contact of (the graph of) $u$

$$\Gamma^+ = \{x \in \Omega : u(y) \geq u(x) + Du(x) \cdot (y - x)\}.$$
It is easy to see that
\[ B_1 \subset Du(\Gamma^+) \]
where \( B_1 \) denote the unit ball in \( \mathbb{R}^n \) centered at the origin. Consequently,
\[
|B_1| \leq |Du(\Gamma^+)| \leq \int_{\Gamma^+} \det D^2 u \leq \frac{1}{n^n} \int_{\Gamma^+} (\Delta u)^n \leq \frac{1}{n^n} C^n |\Omega|.
\]
By the Divergence Theorem,
\[
C|\Omega| = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = |\partial\Omega|.
\]
So
\[
C = \frac{|\partial\Omega|}{|\Omega|}.
\]
This completes the proof. \( \square \)
2. **Alexandrov Theorem.** An embedded closed hypersurface in $\mathbb{R}^{n+1}$ of constant mean curvature must be a sphere.

The proof: Alexandrov’s moving plane method, based on the maximum principle. It is also called *Alexandrov reflection principle*. The idea can be explained using curves of constant curvature in the plane.

Let $C$ be a simple closed curve in $\mathbb{R}^2$. Locally,

$$C = \text{graph of } f.$$ 

The curvature of $C$ is

$$\kappa = \frac{f''}{(1 + |f'|^2)^{\frac{3}{2}}}.$$ 

This is an ODE for $f$. Suppose $\kappa$ is constant. Then $C$ must be a circle, following the uniqueness of solution to the initial value problem.
3. Minkowski Type Problems. Let $\Sigma$ be a hypersurface in $\mathbb{R}^{n+1}$. We use $\kappa = (\kappa_1, \ldots, \kappa_n)$ to denote the principal curvatures of $\Sigma$ with respect to its interior normal. The $k$th Weingarten curvature $W_k$ of $\Sigma$ is defined as

$$W_k = \sigma_k(\kappa_1, \ldots, \kappa_n), \quad k = 1, \ldots, n$$

where $\sigma_k$ is the $k$th elementary symmetric function. For $k = 1, 2$ and $n$, $W_k$ corresponds to the mean, scalar and Gauss curvature, respectively.

Suppose now that $\Sigma$ is a strictly convex closed hypersurface. The Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^n$ is then a diffeomorphism. Let $\mathbf{n}^{-1}$ denote its inverse which we call the inverse Gauss map.
In 1950s, A. D. Alexandrov and S. S. Chern raised the following extended version of the Minkowski problem

Let \( 1 \leq k \leq n \) be a fixed integer, and \( \psi > 0 \) on \( S^n \). Does there exist a closed strictly convex hypersurface \( \Sigma \) in \( \mathbb{R}^{n+1} \) such that

\[
W_k(n^{-1}(x)) = \psi(x) \quad \forall \ x \in S^n.
\]

For \( k = n \) this is the classical Minkowski problem, which was studied by Minkowski, Alexandrov, Lewy, Nirenberg, Pogorelov, Cheng-Yau, etc.

**Theorem 1.** For \( k = n \), a necessary and sufficient condition is

\[
\int_{S^n} \frac{x}{\psi(x)} = 0.
\]
This turns out not to be the case for $1 \leq k < n$.

**Theorem 2 (P.-F. Guan, G. 2002).** (a) For every $1 \leq k < n$ and any nonzero real number $m$, there exists a parameter family of closed strictly convex hypersurfaces (all are small perturbations of the unit sphere) in $\mathbb{R}^{n+1}$ satisfying

$$\int_{S^n} \frac{x}{W_k(n^{-1}(x))^m} \neq 0.$$  

(b) There exists a function $f \in C^\infty(S^n)$ and a constant $\delta > 0$ such that for all $t \in (0, \delta)$, problem (3) has no solution for $\psi := (1 + tf)^{-1}$ while (4) is satisfied.
A partial existence result.

**Theorem 3.** Suppose $\psi$ is invariant under an automorphic group $G$ of $S^n$ without fixed points, i.e., $\psi(g(x)) = \psi(x)$ for all $g \in G$ and $x \in S^n$. Then there exists a closed strictly convex hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ satisfying (3).

For instance, if $\psi(-x) = \psi(x)$ for all $x \in S^n$ the problem is solvable.

The PDE:

$$\frac{\sigma_n(\lambda)}{\sigma_{n-k}(\lambda)} = \frac{1}{\psi}$$

where $\lambda = \lambda(\nabla^2u + ug) = \text{eigenvalues of } \nabla^2u + ug \text{ on } S^n$.

Subsequent work: Sheng-Trudinger-Wang.
4. **Plateau Type Problems.** Let $f$ be a smooth symmetric function of $n$ ($n \geq 2$) variables, and

$$\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$$

a disjoint collection of closed smooth embedded submanifolds of dimension $(n-1)$ in $\mathbb{R}^{n+1}$.

**Question.** Does there exist an immersed hypersurfaces $M$ in $\mathbb{R}^{n+1}$ of constant curvature

$$f(\kappa[M]) = K$$

with boundary

$$\partial M = \Gamma?$$

Here $\kappa[M] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of $M$ and $K$ is constant.
The Plateau problem: $f = \sigma_1$, the mean curvature of $M$, raised by Joseph-Louis Lagrange in 1760, named after Joseph Plateau who experimented with soap films, and solved independently by Jesse Douglas and Tibor Rado in 1930’s. But there were a lot of subsequent developments and research activities, especially in geometric measure theory.
For $f = \sigma_n$, the Gauss curvature, $M$ is locally determined by
\[ \det D^2 u = K(1 + |Du|^2)^{\frac{n+2}{2}}. \]
This equation is elliptic for strictly convex solutions. We require $M$ to be \textit{locally strictly convex}, i.e., the second fundamental form is $M$ is positive definite everywhere.

- The second fundamental form of each $\Gamma_k$ is nondegenerate everywhere. For $n = 1$ this means that the curvature of $\Gamma_k$ never vanishes.
- There are topological obstructions (H. Rosenberg).
Existence results.

- Caffarelli-Nirenberg-Spruck (1980’s): The Dirichlet problem is solvable over a strictly convex domain, provided that there is a strictly convex subsolution.
- Spruck-G. (1993, 1998): On any smooth bounded domain, as long as there is a strictly convex subsolution.
- Spruck-G. (2002): If $\Gamma$ bounds a locally strictly convex hypersurface, it bounds one with constant Gauss curvature.
  It was also independently proved by Trudinger-Wang (2002).
- Spruck-G. (2004): This is true for more general function $f$. 
Some of the technical issues.

- The Dirichlet problem in general domain.
- Perron’s method for locally convex hypersurfaces.
- Local gradient estimates.
- Area minimizing for locally convex hypersurfaces.
Some general questions to understand.

- Global bahavior/properties of solutions
  - Liouville type theorem
  - Bernstein theorem
  - Symmetry

- Existence/expresssion of solutions.
  - Separation of variables, eigenfunction expansions
  - Poisson representation
  - d’Alembert’s formula
  - A priori estimates
In the rest of this talk we shall mainly concerned with equations of the form

\[ f(\lambda(A[u])) = \psi \]

on a Riemannian manifold \((M^n, g)\), where

- \(f\): a smooth symmetric function of \(n\) variables defined in \(\Gamma \subset \mathbb{R}^n\)
- \(\Gamma\): a symmetric open and convex cone with vertex at the origin, \(\partial \Gamma \neq \emptyset\), and \(\Gamma_n \subseteq \Gamma\) where

\[ \Gamma_n \equiv \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \} \subseteq \Gamma. \]

- \(A[u] = \nabla^2 u + \chi\).
- \(\lambda(A) = (\lambda_1, \cdots, \lambda_n)\) denotes the eigenvalues of \(A\).

Note that \(F(A) = f(\lambda(A))\) if and only if \(F(PAP^T) = F(A)\) for any orthogonal matrix \(P\).
Examples of $f$. This covers a very broad class of equations.

- $f = \sigma_k^\frac{1}{k}$ or $(\sigma_k/\sigma_l)^\frac{1}{k-l}$ define on

$$
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \ 1 \leq j \leq k \}
$$

where $\sigma_k(\lambda)$ is the elementary symmetric function

$$
\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \ k = 1, \ldots, n.
$$

In particular, $\sigma_1(\lambda) = \Delta u$, $\sigma_n(\lambda) = \det \nabla^2 u$.
- The special Lagrange equation: $f(\lambda) = \sum \tan^{-1} \lambda_i$.
- $f = \log P_k$ where

$$
P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \ 1 \leq k \leq n
$$

defined in the cone

$$
\mathcal{P}_k := \{ \lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0 \}.
$$
• The inverse sum:

$$f(\lambda) = -\sum \frac{1}{\lambda^\alpha}, \quad \lambda \in \Gamma_n, \quad \alpha > 0.$$
The fundamental structure conditions [CNS1985]. To study the equations under the framework of elliptic PDE theory, we need some basic assumptions.

- **ellipticity**
  \[
  f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, \quad 1 \leq i \leq n,
  \]

- **concavity**
  \[
  f \text{ is a concave function in } \Gamma
  \]

- **nondegeneracy**:
  \[
  \inf_{\Omega} \psi > \sup_{\partial \Gamma} f
  \]

  where
  \[
  \sup_{\partial \Gamma} f \equiv \sup_{\lambda_0 \in \partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda).
  \]

These conditions were introduced by Caffarelli-Nirenberg-Spruck in 1985 and have become standard in the literature.
**Admissible functions.** A function $u \in C^2$ is called *admissible* if $\lambda(A[u]) \in \Gamma$.

- (11): Eq (8) is elliptic for admissible solutions.
- (12): $F(A) \equiv f(\lambda[A])$ is concave for $A$ with $\lambda[A] \in \Gamma$.
- (13): Eq (8) will not become degenerate.
- (11) & (13) & $|\nabla^2 u| \leq C$: Eq (8) becomes uniformly elliptic.

**Evans-Krylov theorem:** Suppose that (1) is uniformly elliptic, $F$ is concave w.r.t. $\nabla^2 u$ and $|u|_{C^2(\bar{\Omega})} \leq C$. Then
\[
|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C.
\]

**Schauder theory:** $C^{2,\alpha}$ estimates imply higher regularity.

**Continuity method:** $|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C$ implies the classical solvability of the Dirichlet problem.

From this point of view, conditions (11)-(13) are fundamental to the classical solvability of equation (8).
The ultimate goal is to solve equation (8). To prove the existence of classical solutions.

- The Dirichlet problem.
- On closed manifolds.

The key is to derive global $C^2$ estimates. We hope to establish this for general manifolds—without curvature restrictions, and for general domains in the case of the Dirichlet problem—without assumptions on the geometric shape of $\partial M$, the boundary of $M$. 
Question: Are assumptions (11)-(13) necessary? Sufficient?

- For the degenerate Monge-Ampère equation, the solution may fail to belong to $C^{1,1}(\Omega)$.
- The Dirichlet problem for $\text{det} D^2 u = 1$ in $\Omega \subset \mathbb{R}^n$ with $u = 0$ on $\partial \Omega$ does not have a solution unless $\Omega$ is strictly convex.
- Nadirashvili et al.: For nonconcave $F$, the solution may fail to belong to $C^{1,1}$ $(n \geq 5)$.
- CNS3: There is an equation $(n = 2)$ satisfying (11)-(13), with solution in $C^\infty(B_1) \cap C(B_1)$ but not in $C^1(B_1)$. 
Previous Work.
Caffarelli, Nirenberg and Spruck (1985)
Ivochkina
Ivochkina-Trudinger-Wang (degenerate case)
Kryov (1980’s)
Y.-Y. Li (1990)
Trudinger (1996)
Urbas (2003)
The Dirichlet problem in $\mathbb{R}^n$.

- Caffarelli, Nirenberg and Spruck (1985, CNS).

**Theorem 4** ([CNS3, Acta 1985]). Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$, $\psi \in C^{k,\alpha}(\bar{\Omega})$, $\varphi \in C^{k+2,\alpha}(\partial \Omega)$, $k \geq 2$. Assume a) (11)-(13), and in addition b), c), d) below. Then the Dirichlet problem

\[
\begin{cases}
  f(\lambda(\nabla^2 u + \chi)) = \psi & \text{in } \bar{\Omega} \\
  u = \varphi & \text{on } \partial \Omega
\end{cases}
\]

admits a unique admissible solution $u \in C^{k+2,\alpha}(\bar{\Omega})$.

Moreover, if $\psi \in C^\infty(\bar{\Omega})$, $\varphi \in C^\infty(\partial \Omega)$ then $u \in C^\infty(\bar{\Omega})$. 
The additional conditions:

b) for every $C > 0$ and $\lambda \in \Gamma$ there is a number $R = R(C, \lambda)$ such that

$$f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \geq C$$

$$f(R\lambda) \geq C$$

c) A geometric condition on $\partial\Omega$:

$$ (\kappa_1, \ldots, \kappa_{n-1}, R) \in \Gamma \text{ on } \partial\Omega \text{ for some } R > 0$$

where $(\kappa_1, \ldots, \kappa_{n-1})$ are the principal curvatures of $\partial\Omega$

d) $\chi = 0$.

- For the Monge-Ampère equation $(f = \sigma_n^{1/n})$, c) means that $\Omega$ is strictly convex.

Trudinger in 1995 removed condition (15).
• Guan (2014, for general domains).

**Theorem 5** (Guan 2014, arXiv:1403.2133). Assume

a) (11)-(13) hold,

e) the subsolution assumption: there exists an admissible subsolution $u \in C^2(\bar{\Omega})$

\begin{equation}
\begin{aligned}
f(\lambda(\nabla^2 u + \chi) \geq \psi & \text{ in } \bar{\Omega} \\
u = \varphi & \text{ on } \partial \Omega.
\end{aligned}
\end{equation}

The Dirichlet problem (14) then has a unique admissible solution $u \in C^{k+2,\alpha}(\bar{\Omega})$.

Moreover, if $\psi \in C^\infty(\bar{\Omega})$, $\varphi \in C^\infty(\partial \Omega)$ then $u \in C^\infty(\bar{\Omega})$. 
**The proof.** To derive

\[ |u|_{C^2(\bar{\Omega})} \leq C. \]

Then $C^{2,\alpha}$ and higher order estimates follows from Evans-Krylov Theorem and Schauder theory; existence by the continuity method.
Theorem 6 (Guan, 2014). Let \( u \in C^4(M) \cap C^2(\bar{M}) \) be an admissible solution of the Dirichlet problem (14). Suppose (11)-(13) hold and that there exists an admissible subsolution \( \underline{u} \in C^2(\bar{M}) \):

\[
\begin{cases}
  f(\lambda[\nabla^2 u + \chi]) \geq \psi \text{ in } \bar{M}, \\
  u = \varphi \text{ on } \partial M.
\end{cases}
\]

Then

\[
\max_{\bar{M}} |\nabla^2 u| \leq C.
\]

The proof consists of two steps:

- a maximum principle for \( |\nabla^2 u| \)

\[
\max_{\bar{M}} |\nabla^2 u| \leq C \left( 1 + \max_{M} |\nabla u|^2 + \max_{\partial M} |\nabla^2 u| \right)
\]

- the boundary estimate

\[
\max_{\partial M} |\nabla^2 u| \leq C.
\]
The concavity and subsolution. For $\sigma > \sup_{\partial \Gamma} f$, let
\[ \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) > \sigma \} \]
and suppose $\Gamma^\sigma \neq \emptyset$. By (11) and (12) the level set
\[ \partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \} \]
i.e., boundary of $\Gamma^\sigma$, is a smooth convex hypersurface.

For $\lambda \in \partial \Gamma^\sigma$ let
\[ \nu_\lambda = \frac{Df(\lambda)}{|Df(\lambda)|} \]
be the unit normal to $\partial \Gamma^\sigma$ and $T_\lambda \partial \Gamma^\sigma$ denote the tangent plane of $\partial \Gamma^\sigma$ at $\lambda$. 
**Theorem 7.** Let $\mu \in \Gamma$ and $\beta > 0$. There exists uniform constant $\varepsilon > 0$ such that when $|\nu_\mu - \nu_\lambda| \geq \beta$,

\begin{equation}
\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon \sum f_i(\lambda) + \varepsilon.
\end{equation}

- Used in global estimates for $|\nabla^2 u|$ too.
- Apply Theorem 7 to $\mu = \lambda[u]$, $\lambda = \lambda[u]$. 
Equations on closed Riemannian manifolds. Let $(M^n, g)$ be a Riemannian manifold without boundary.

Y.-Y. Li (1990) first studied equation (14) for $\chi = g$ on closed Riemannian manifolds with nonnegative sectional curvature. John Urbas (2002) removed the curvature assumption.

Their main extra assumptions are

$$\lim_{|\lambda| \to \infty} \sum f_i = \infty$$

and

$$\lim_{|\lambda| \to \infty} \sum f_i (1 + \lambda_i^2) = \infty$$

respectively.
The notation of subsolutions on closed Riemannian manifolds. When $M$ is closed, the subsolution assumption does not make sense. Indeed, a subsolution must be a solution, or there is no solution. This is a consequence of the maximum principle.

The tangent cone at infinity (Guan, DJM 2014). Define

$$S_{\mu}^\sigma = \{ \lambda \in \partial \Gamma^\sigma : \nu_\lambda \cdot (\mu - \lambda) \leq 0 \}$$

and

$$\mathcal{C}^+_\sigma = \{ \mu \in \mathbb{R}^n : S_{\mu}^\sigma \text{ is compact} \}.$$

which we call the tangent cone at infinity to $\Gamma^\sigma$. 
Theorem 8 (Guan, DJM 2014). a) $C_{o}^+$ is open.

b) Assume (11)-(13) and that there exists $u \in C^{2}(\overline{M})$ with

\begin{equation}
\lambda(\nabla^2 u + \chi)(x) \in C_{\psi(x)}^+; \quad \forall \ x \in \overline{M}.
\end{equation}

Then

\begin{equation}
\max_{\overline{M}} |\nabla^2 u| \leq C \left( 1 + \max_{\partial M} |\nabla^2 u| \right)
\end{equation}
The enlarged cone $\tilde{C}_\sigma^+$. We now construct a larger cone from $C^+\sigma$. Note that the unit normal vector of any supporting hyperplan to $\Gamma^\sigma$ belongs to $\Gamma_n$. We define $\tilde{C}_\sigma^+$ to be the region in $\mathbb{R}^n$ bounded by those supporting hyperplans to $C^+\sigma$ with unit normal vector in $\partial \Gamma_n$; so $\tilde{C}_\sigma^+ = \mathbb{R}^n$ if there are no such supporting planes. Clearly, if $\tilde{C}_\sigma^+ \neq \mathbb{R}^n$ it is an open symmetric convex cone with vertex at $a1$ for some $a \in \mathbb{R}^n$ where $1 = (1, \ldots, 1) \in \mathbb{R}^n$. Moreover, $\mu + \Gamma_n \subset \tilde{C}_\sigma^+$ for $\mu \in \tilde{C}_\sigma^+$ and $\tilde{C}_\sigma^+ \subset \tilde{C}_\rho^+$ if $\sigma \geq \rho$.

**Theorem 9 (Guan, 2015).** Let $\mu \in \tilde{C}_\sigma^+$ and $d(\mu)$ denote the distance from $\mu$ to $\partial \tilde{C}_\sigma^+$. Then there exist $\delta, \varepsilon > 0$ such that for any $\lambda \in \partial \tilde{C}_\sigma^+$, either

\begin{equation}
(26) \quad f_i(\lambda) \geq \delta \sum f_k(\lambda)
\end{equation}

or

\begin{equation}
(27) \quad f_i(\lambda)(\mu_i - \lambda_i) \geq \varepsilon \sum f_i(\lambda) + \varepsilon.
\end{equation}
**Theorem 10 (Guan, 2015).** Assume (11)-(13) and that there exists \( u \in C^2(M) \) with

\[
\lambda(\nabla^2 u + \chi)(x) \in C^\psi(x), \quad \forall \ x \in \bar{M}.
\]

Then

\[
\max_{\bar{M}} |\nabla^2 u| \leq C \left( 1 + \max_{\partial M} |\nabla^2 u| \right).
\]
Szekelyhidi (2015) introduced the another notion of generalized subsolutions:

$$(30) \quad (\lambda(\nabla^2 u + \chi)(x) + \Gamma_n) \cap \partial \Gamma^\psi(x) \text{ is compact } \forall \ x \in \bar{M}. $$

and proved the same estimates under a little stronger assumptions. The following result clarify the relations.

**Theorem 11** (Guan, 2016). a) *For Type I cone, $C^+_\sigma = \tilde{C}^+_\sigma$. b) Assumptions (28) and s (30) are equivalent.*

According to CNS, a cone $\Gamma$ is Type 1 if each $\lambda_i$-axis belongs to $\partial \Gamma$. For instance, $\Gamma_k \ (k \geq 2)$ are Type 1.
• Canonical $\chi$: $\chi = 0$ or $\chi = g$. For instance,

$$\text{(31)} \quad \det(\nabla^2 u + g) = K(x)(-2u - |\nabla u|^2) \det g$$

is the Darboux equation (isometric embedding).

• $\nabla^2 u + \text{Ric}_g$: the Bakry-Emery Ricci tensor of the Riemannian measure space $(M^n, g, e^{-u} dg)$.

• Te Ricci soliton equation: $\nabla^2 u + \text{Ric}_g = \lambda g$

• $\nabla^2 u + ug$ on $S^n$. (In classical geometry. Minkowski problem, extensions proposed by Alexandrov, Chern; Christoffel-Minkowski problem.)

• In conformal geometry. The Schouten tensor of $(M^n, e^{2u} g)$

$$\text{(32)} \quad \chi = du \otimes du - \frac{1}{2}|\nabla u|^2 g + S_g$$

where $S_g$ is the Schouten tensor of $(M^n, g)$; the Ricci tensor

$$A[u] = \nabla^2 u + \gamma \Delta u g + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \text{Ric}_g.$$
More general equations. For the more general equation

\[ F(D^2u + \chi) = \psi, \]

we assume the function \( F \) to be defined in an open convex cone \( \Gamma \) in \( S^{n \times n} \), the (inner product) space of \( n \times n \) symmetric matrices, with vertex at 0, \( \Gamma^+ \subseteq \Gamma \neq S^{n \times n} \) where \( \Gamma^+ \) denotes the cone of positive matrices, and to satisfy the fundamental structure conditions:

(a) the ellipticity condition.
(b) the concavity condition.
Equations on complex manifolds.

Let \((M^n, \omega)\) be a compact Hermitian manifold of complex dimension \(n \geq 2\) with smooth boundary \(\partial M\) which may be empty (\(M\) is closed) and let, for a function \(u \in C^2(M)\), \(\chi[u] := \chi(\cdot, du(\cdot), u(\cdot))\) be a real \((1,1)\) form on \(M\), and define
\[
\chi_u := \chi[u] + \sqrt{-1}\partial \bar{\partial} u.
\]
The equation:
\[
(33) \quad f(\lambda(\chi_u)) = \psi(z, du, u) \quad \text{in} \quad M
\]
where \(\lambda(\chi_u) = (\lambda_1, \cdots, \lambda_n)\) denote the eigenvalues of \(\chi_u\) with respect to the metric \(\omega\).

This covers most of the important equations in complex geometry.

- In local coordinates:
  \[
  \chi_u = \sqrt{-1}(u_{ij} + \chi_{ij})dz_i \wedge d\bar{z}_j.
  \]
• **Guan-Nie:** $\chi_u := \chi[u] + \sqrt{-1} \partial \bar{\partial} u$ where $\chi$ depends linearly on $du$.

• **Guan-Qiu-Yuan:** $A[u] = \Delta u \omega - \gamma \sqrt{-1} \partial \bar{\partial} u + \chi$ where $\chi = \chi(du)$. 
• The complex Monge-Ampère equation:

\[ \chi_u^n = \psi^n \omega^n \]

corresponds to equation (8) for \( f = \sigma_1^{1/n} \). It plays a central role in Kähler geometry.

• Calabi-Yau Theorem. Yau’s proof of Calabi conjecture; Aubin independently for \( c_1(M) < 0 \).

• Extension to Hermitian case.
  – Cherrier: \( n = 2 \);
  – Tosatti-Weinkove.
• The Dirichlet problem.
  – Caffarelli-Kohn-Nirenberg-Spruck: strongly pseudocon-
    vex domain in $\mathbb{R}^n$;
  – Guan (1998): general domain in $\mathbb{C}^n$;
  – Guan adn Qun Li (2010): on Hermitian manifolds.

• Some applications.
  – Chern-Levine-Nirenberg conjecture: Pengfei Guan
  – Donaldson’s conjecture on geodesics in the space of
    Kähler metrics: Mabuchi, Donaldson, Xiuxiong Chen, many
    others ..... 
    – Pluricomplex Green functions.
    – Totally real submanifolds.
There have also been increasing interests in other fully nonlinear equations from Kähler geometry.

- **Donaldson:**

\[
\chi_u^n = \psi \chi_u^{n-1} \wedge \omega, \quad \chi_u > 0,
\]

proposed in the setting of moment maps, where he assumes \( \chi \) is also Kähler and \( \psi \) is the Kähler class invariant

\[
\psi = c_1 = \frac{\int_M \chi^n}{\int_M \chi^{n-1} \wedge \omega}.
\]

This corresponds to \( f = \sigma_n/\sigma_{n-1} \). The equation is also closely related to the lower bound and properness of the Mobuchi energy.
Conformal metrics on Hermitian manifolds. Let $\alpha = e^{\pm u}\omega$ be a conformal metric on $M$. The Chern-Ricci form is given by
\[ \pm \text{Ric}_\alpha = \pm \sqrt{-1} \partial \bar{\partial} \log \alpha^n = \sqrt{-1} \partial \bar{\partial} u \pm \text{Ric}_\omega \]
Consequently, the problem of determining a metric in the conformal class of $\omega$ with special properties of the Chern-Ricci form leads to the following equation for $\chi = \pm \text{Ric}_\omega$
\[ F \left( \frac{\chi}{e^{\pm u}\omega} \right) = \psi \quad \text{on} \quad M. \]
(36)
The sign in front of $\text{Ric}_\alpha$ is determined by requiring $\chi \in \Gamma$. The negative sign case will be much more difficult to study.

For $F(A) = (\det A)^{1/n}$, this equation is related to a conjecture of Yau on the holomorphic sectional curvature of a Kähler manifolds which was recently solved by Wu-Yau and Tosatti-Yang.
Balanced metrics with prescribed volume form Recall that a Hermitian metric \( \omega \) is balanced if \( d(\omega^{n-1}) = 0 \) and Gauduchon if \( \partial \bar{\partial}(\omega^{n-1}) = 0 \). A well know result due to Gauduchon asserts that any conformal class of Hermitian metrics on a compact (closed) complex manifold contains a Gauduchon metric. Fu-Wang-Wu introduced an equation of prescribed volume for balanced metrics, which can be described as below following Tosatti-Weinkove.

Let \( \omega_0 \) be a balanced metric on a closed Hermitian manifold \((M, \omega)\). We seek a balanced metric \( \eta \) such that

\[
\eta^{n-1} = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial}(u \omega^{n-2})
\]

for some \( u \in C^\infty(M) \) and prescribed volume

\[
\frac{\eta^n}{\omega^n} = \psi \quad \text{on } M. \tag{37}
\]

This is a Monge-Ampère type equation of \((n-1, n-1)\) forms. Nevertheless, by the Hodge star duality approach of Tosatti-Weinkove, equation (37) can be converted to an equation of form
(38) for \( F(A) = P_{n-1}(A) \) where \( \chi \) depends linearly on \( u \) and the gradient of \( u \); More precisely,
\[
\chi = (\text{tr} \tilde{\chi} - (n - 1) \tilde{\chi}
\]
where
\[
\tilde{\chi} = \frac{1}{(n-1)!} \ast (\omega_0^{n-1} - \sqrt{-1} \bar{\partial} u \bar{\partial} (\omega^{n-2}) + \sqrt{-1} \bar{\partial} u \partial (\omega^{n-2}) + \sqrt{-1} u \partial \bar{\partial} (\omega^{n-2}))
\]
and \( \ast \) denotes the Hodge star-operator.
• **Gauduchon conjecture.**

Let complex dimension \( n \geq 2 \) and let \( \Omega \) be a closed real \((1,1)\) form on \( M \) with \([\Omega] = c_1^{BC}(M)\) in the Bott-Chern cohomology group \( H_{BC}^{1,1}(M, \mathbb{R})\). In 1984, Gauduchon conjectured that there exists a Gauduchon metric \( \tilde{\omega} \) on \( M \) with Chern-Ricci curvature \( Ric_{\tilde{\omega}} = \Omega \).

This is a natural extension of the Calabi conjecture for Kähler manifolds solved by Yau. It was discovered by Tosatti and Weinkove and independently by Popovici that the Gauduchon conjecture reduces to solving a Monge-Ampère type equation of the form

\begin{equation}
\det(\Phi_u) = e^{F+b} \det(\omega^{n-1}) \text{ in } M
\end{equation}

with

\[\Phi_u = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + \Re(\sqrt{-1} \partial u \wedge \bar{\partial} \omega^{n-2}) > 0\]

and \( \sup_M u = 0 \), where \( \omega_0 \) is any Hermitian metric (positive definite \((1,1)\) form) and \( \omega \) a Gauduchon metric. This equation is equivalent to one for \( f = P_{n-1} \) with linear dependences of \( \chi \) on \( du \).
Results of Guan-Xiaolan Nie.

We study equation (38) on closed Hermitain manifold with $\chi$ depending linearly on $du$. Our result applies to $P_{n-1}$ giving the estimates need in proving Gaudochun’s conjecture.

**Theorem 12.** Let $\psi \in C^2(M)$ and $u \in C^4(M)$ be an admissible solution of (8). Suppose that there exists a function $u \in C^2(M)$ satisfying

\[
\lambda(\chi_u(z)) \in \tilde{C}^+(\psi(z)) \quad \forall \ z \in M,
\]

and that at any fixed point on $M$ where $g_{i\bar{j}} = \delta_{ij}$ and $g_{i\bar{j}} = \delta_{ij} \lambda_i$

with $\lambda_1 \geq \cdots \geq \lambda_n$,

\[
\sum f_i(|\chi_{i\bar{i},\zeta_\alpha}| + \chi_{i\bar{i},\zeta_\alpha}) \leq C_1 \lambda_1 f_\alpha, \quad \forall \ \alpha \leq n - r_0
\]

where $r_0 = \min \{ \text{rank of } \tilde{C}^+(\psi(z)) : z \in M \}$. Then

\[
\max_M |\Delta u| \leq C_1 e^{C_2(u - \inf_M u)}
\]

where $C_1$ depends on $|\nabla u|_{C^0(M)}$ and $C_2$ is a uniform constant (independent of $u$).
Results of Guan-Qiu-Yuan.

\[ F(\Delta u\omega + \gamma \sqrt{-1}\partial\bar{\partial}u + \chi(z, u, du)) = \psi, \quad \gamma < 1. \]
Thank You!