Fully Nonlinear PDEs and Related Geometric Problems

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The three well known classical PDEs are Laplace equation, heat equation, and wave equation, representing the three types of PDEs with rather distinct properties: elliptic, parabolic and hyperbolic equations, respectively.

- Laplace equation – elliptic

$$\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

- Heat equation – parabolic

$$u_t = \Delta u$$

- Wave equation – hyperbolic

$$u_{tt} = \Delta u$$

In general, an elliptic PDE of the second order can be written in the form

$$F(\nabla^2 u, \nabla u, u, x) = 0. \tag{1}$$

The **ellipticity** means

$$\{F^{ij}[u]\} \equiv \{F^{ij}(\nabla^2 u, \nabla u, u, x)\} > 0 \tag{2}$$

where, if we write $F(A, \cdot, \cdot, \cdot)$ and $A = \{a_{ij}\} \in \mathcal{S}^{n\times n}$,

$$F^{ij} \equiv \frac{\partial F}{\partial a_{ij}}$$

$\mathcal{S}^{n\times n}$ is the set of n by n symmetric matrices. Equivalently, the linearized operator

$$\mathcal{L}_u = F^{ij}[u]\nabla_i\nabla_j + \text{lower order terms}$$

is elliptic.

Equation (2) is

- **linear**, if F is linear in u, ∇u and $\nabla^2 u$;
  otherwise, **nonlinear**;
- **semilinear**, if F is linear in ∇u and $\nabla^2 u$;
- **quasilinear**, if F is linear in $\nabla^2 u$;
- **fully nonlinear**, if F is not linear in $\nabla^2 u$.

Let's first look at some examples of nonlinear equations from geometry.

- The minimal surface equation
$$\operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = 0.$$
For $n = 2$,
$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0.$$
- The minimal surface equation in hyperbolic space
$$\Big(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}\Big)u_{ij} = \frac{n}{u}.$$
- The spacelike maximal surface equation in Minkowski space
$$\operatorname{div}\Big(\frac{Du}{\sqrt{1-|Du|^2}}\Big) = 0.$$
The spacelike condition
$$|Du| < 1.$$

- The Monge-Ampère equation
$$\det D^2 u = \psi.$$
For $n = 2$,
$$u_{xx}u_{yy} - u_{xy}^2 = \psi.$$
- The complex Monge-Ampère equation
$$\det \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = \psi.$$
- The prescribed Gauss curvature equation
$$\det D^2 u = K(1 + |Du|^2)^{\frac{n+2}{2}}.$$
For spacelike hypersurfaces in Minkowski space
$$\det D^2 u = K(1 - |Du|^2)^{\frac{n+2}{2}}.$$

There are important fourth order equations in geometry.

- Willmore surface equation.
- Affine maximal hypersurface equation

$$u^{ij}\frac{\partial^2}{\partial x_i \partial x_j}\det D^2 u = 0.$$

1. The Isoperimetric Inequality. Let C be a simple closed curve in $\mathbb{R}^2$. Then

$$4\pi A \leq L^2$$

where A is the enclosed area, L donotes the length of C.

More generally, let Ω be a domain in $\mathbb{R}^n$. Then

$$\omega_1(n|\Omega|)^{n-1} \leq |\partial\Omega|^n$$

where ω donotes the volume of the unit sphere in $\mathbb{R}^n$.

Proof. Consider the Neumann problem

$$\Delta u = C \ \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 1 \ \text{ on } \partial\Omega.$$

Let Γ^+ be the lower contact of (the graph of) u

$$\Gamma^+ = \{x \in \Omega : u(y) \geq u(x) + Du(x)\cdot(y-x)\}.$$

It is easy to see that

$$B_1 \subset Du(\Gamma^+)$$

where B_1 denote the unit ball in $\mathbb{R}^n$ centered at the origin. Consequently,

$$|B_1| \leq |Du(\Gamma^+)| \leq \int_{\Gamma^+} \det D^2 u \leq \frac{1}{n^n} \int_{\Gamma^+} (\Delta u)^n \leq \frac{1}{n^n} C^n |\Omega|.$$

By the Divergence Theorem,

$$C|\Omega| = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = |\partial\Omega|.$$

So

$$C = \frac{|\partial\Omega|}{|\Omega|}.$$

This completes the proof. $\square$

2. Alexandrov Theorem. An embedded closed hypersurface in $\mathbb{R}^{n+1}$ of constant mean curvature must a sphere.

The proof: Alexandrov's moving plane method, based the maximum priciple. It is also called *Alexandrov reflection principle.* The idea can be explained using curves of contant curvature in the plane.

Let C be a simple closed curve in $\mathbb{R}^2$. Locally,

$$C = \text{graph of } f.$$

The curvature of C is

$$\kappa = \frac{f''}{(1+|f'|^2)^{\frac{3}{2}}}.$$

This is an ODE for f. Suppose κ is constant. Then C must a circle, following the uniqueness of solution to the initial value problem.

3. Minkowski Type Problems. Let Σ be a hypersurface in $\mathbb{R}^{n+1}$. We use $\kappa = (\kappa_1, \ldots, \kappa_n)$ to denote the principal curvatures of Σ with respect to its interior normal. The kth Weingarten curvature W_k of Σ is defined as

$$W_k = \sigma_k(\kappa_1, \ldots, \kappa_n), \quad k = 1, \ldots, n$$

where σ_k is the kth elementary symmetric function. For $k = 1, 2$ and n, W_k corresponds to the mean, scalar and Gauss curvature, respectively.

Suppose now that Σ is a strictly convex closed hypersurface. The Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^n$ is then a diffeomorphism. Let $\mathbf{n}^{-1}$ denote its inverse which we call the inverse Gauss map.

In 1950s, A. D. Alexandrov and S. S. Chern raised the following extended version of the **Minkowski problem**

Let $1 \leq k \leq n$ be a fixed integer, and $\psi > 0$ on $\mathbb{S}^n$. Does there exist a closed strictly convex hypersurface Σ in $\mathbb{R}^{n+1}$ such that

$$W_k(\mathbf{n}^{-1}(x)) = \psi(x) \quad \forall\, x \in \mathbb{S}^n? \tag{3}$$

For $k = n$ this is the classical Minkowski problem, which was studied by Minkowski, Alexandrov, Lewy, Nirenberg, Pogorelov, Cheng-Yau, etc.

Theorem 1. *For $k = n$, a necessary and sufficient condition is*

$$\int_{\mathbb{S}^n} \frac{x}{\psi(x)} = 0. \tag{4}$$

This turns out not to be the case for $1 \le k < n$.

Theorem 2 (**P.-F. Guan, G. 2002**). *(a) For every* $1 \le k < n$ *and any nonzero real number* m*, there exists a parameter family of closed strictly convex hypersurfaces (all are small perturbations of the unit sphere) in* $\mathbb{R}^{n+1}$ *satisfying*

$$\int_{\mathbb{S}^n} \frac{x}{(W_k(\mathbf{n}^{-1}(x)))^m} \neq 0. \tag{5}$$

(b) There exists a function $f \in C^\infty(\mathbb{S}^n)$ *and a constant* $\delta > 0$ *such that for all* $t \in (0, \delta)$*, problem (3) has no solution for* $\psi := (1 + tf)^{-1}$ *while (4) is satisfied.*

A partial existence result.

Theorem 3. *Suppose ψ is invariant under an automorphic group G of $\mathbb{S}^n$ without fixed points, i.e., $\psi(g(x)) = \psi(x)$ for all $g \in G$ and $x \in \mathbb{S}^n$. Then there exists a closed strictly convex hypersurface Σ in $\mathbb{R}^{n+1}$ satisfying (3).*

For instance, if $\psi(-x) = \psi(x)$ for all $x \in \mathbb{S}^n$ the problem is solvable.

The PDE:

$$\frac{\sigma_n(\lambda)}{\sigma_{n-k}(\lambda)} = \frac{1}{\psi}$$

where $\lambda = \lambda(\nabla^2 u + ug) =$ eigenvalues of $\nabla^2 u + ug$ on $\mathbb{S}^n$.

Subsequent work: Sheng-Trudinger-Wang.

4. Plateau Type Problems. Let f be a smooth symmetric function of n ($n \geq 2$) variables, and

$$\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$$

a disjoint collection of closed smooth embedded submanifolds of dimension $(n-1)$ in $\mathbb{R}^{n+1}$.

Question. *Does there exist an immersed hypersurfaces M in $\mathbb{R}^{n+1}$ of constant curvature*

$$f(\kappa[M]) = K \tag{6}$$

with boundary

$$\partial M = \Gamma? \tag{7}$$

Here $\kappa[M] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of M and K is constant.

The Plateau problem: $f = \sigma_1$, the mean curvature of M, raised by Joseph-Louis Lagrange in 1760, named after Joseph Plateau who experimented with soap films, and solved independently by Jesse Douglas and Tibor Rado in 1930's. But there were a lot of subsequent developments and research activities, especially in geometric measure theory.

For $f = \sigma_n$, the Gauss curvature, M is locally determined by

$$\det D^2 u = K(1 + |Du|^2)^{\frac{n+2}{2}}.$$

This equation is elliptic for strictly convex solutions. We require M to be *locally strictly convex*, i.e, the second fundamental form is M is positive definite everywhere.

- The second fundamental form of each Γ_k is nondegenerate everywhere. For $n = 1$ this means that the curvature of Γ_k never vanishes.
- There are topological obstructions (H. Rosenberg).

Existence results.

- Caffarelli-Nirenberg-Spruck (1980's): The Dirichlet problem is solvable over a strictly conevx domain, provided that there is a strictly convex subsolution.
- Spruck-G. (1993, 1998): On any smooth bouned domain, as long as there is a strictly convex subsolution.
- Spruck-G. (2002): If Γ bounds a locally strictly convex hypersurface, it bounds one with constant Gauss curvature.
  It was also independently proved by Trudinger-Wang (2002).
- Spruck-G. (2004): This is true for more general function f.

Some of the technical issues.

- The Dirichlet problem in general domain.
- Perron's mehtod for locally convex hypersurfaces.
- Local gradient estimates.
- Area minimizing for locally convex hypersurfaces.

Some general questions to understand.

- Global bahavior/properties of solutions
  —Liouville type theorem
  —Bernstein theorem
  —Symmetry
- Existence/expression of solutions.
  —Separation of variables, eigenfunction expansions
  —Poisson representation
  —d'Alembert's formula
  —A priori estimates

In the rest of this talk we shall mainly concerned with equations of the form

$$f(\lambda(A[u])) = \psi \tag{8}$$

on a Riemannian manifold (M^n, g), where

- f: a smooth symmetric function of n variables defined in $\Gamma \subset \mathbb{R}^n$
- Γ: a symmetric open and convex cone with vertex at the origin, $\partial\Gamma \neq \emptyset$, and $\Gamma_n \subseteq \Gamma$ where

$$\Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\}. \subseteq \Gamma. \tag{9}$$

- $A[u] = \nabla^2 u + \chi$.
- $\lambda(A) = (\lambda_1, \cdots, \lambda_n)$ denotes the eigenvalues of A.

Note that $F(A) = f(\lambda(A))$ if and only if $F(PAP^T) = F(A)$ for any orthogonal matrix P.

Examples of f. This covers a very broad class of equations.

- $f = \sigma_k^{\frac{1}{k}}$ or $(\sigma_k/\sigma_l)^{\frac{1}{k-l}}$ define on

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0,\ 1 \le j \le k\}$$

where $\sigma_k(\lambda)$ is the elementary symmetric function

$$\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}, \quad k = 1, \ldots, n.$$

In particular, $\sigma_1(\lambda) = \Delta u$, $\sigma_n(\lambda) = \det \nabla^2 u$.
- The special Lagrange equation: $f(\lambda) = \sum \tan^{-1} \lambda_i$.
- $f = \log P_k$ where

$$P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}),\ \ 1 \le k \le n$$

defined in the cone

$$\mathcal{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0\}.$$

- The inverse sum:

$$f(\lambda) = -\sum \frac{1}{\lambda_i^\alpha}, \quad \lambda \in \Gamma_n, \quad \alpha > 0. \tag{10}$$

The fundamental structure conditions [CNS1985]. To study the equations under the framework of elliptic PDE theory, we need some basic assumptions.

- *ellipticity*

$$f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \ \text{ in } \Gamma, \quad 1 \leq i \leq n, \tag{11}$$

- *concavity*

$$f \text{ is a concave function in } \Gamma \tag{12}$$

- *nondegeneracy:*

$$\inf_{\Omega} \psi > \sup_{\partial \Gamma} f \tag{13}$$

where

$$\sup_{\partial \Gamma} f \equiv \sup_{\lambda_0 \in \partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda).$$

These conditions were introduced by Caffarelli-Nirenberg-Spruck in 1985 and have become standard in the literature.

Admissible functions. A function $u \in C^2$ is called *admissible* if $\lambda(A[u]) \in \Gamma$.

- (11): Eq (8) is elliptic for admissible solutions.
- (12): $F(A) \equiv f(\lambda[A])$ is concave for A with $\lambda[A] \in \Gamma$.
- (13): Eq (8) will not become degenerate.
- (11) & (13) & $|\nabla^2 u| \leq C$: Eq (8) becomes uniformly elliptic.
- **Evans-Krylov theorem:** Suppose that (1) is uniformly elliptic, F is concave w.r.t. $\nabla^2 u$ and $|u|_{C^2(\bar{\Omega})} \leq C$. Then
$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C.$$
- **Schauder theory**: $C^{2,\alpha}$ estimates imply higher regularity.
- **Continuity method:** $|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C$ implies the classical solvability of the Dirichelt problem.

From this point of view, conditions (11)-(13) are fundamental to the classical solvability of equation (8).

The ultimate goal is to solve equation (8). To prove the existence of classical solutions.

- The Dirichlet problem.
- On closed manifolds.

The key is to derive global C^2 estimates. We hope to establish this for general manifolds—without curvature restrictions, and for general domians in the case of the Dirichlet problem—without assumptions on the geometric shape of ∂M, the boundary of M.

Question: Are assumptions (11)-(13) necessary? Sufficient?

- For the degenerate Monge-Ampère equation, the solution may fail to belong to $C^{1,1}(\Omega)$.
- The Dirichlet problem for $\det D^2u = 1$ in $\Omega \subset \mathbb{R}^n$ with $u = 0$ on $\partial\Omega$ does not have a solution unless Ω is strictly convex.
- **Nadirashvili *et al.*:** For nonconcave F, the solution may fail to belong to $C^{1,1}$ ($n \geq 5$).
- **CNS3**: There is an equation($n = 2$) satisfying (11)-(13), with solution in $C^\infty(B_1) \cap C(B_1)$ but not in $C^1(\overline{B_1})$.

Previous Work.

Caffarelli, Nirenberg and Spruck (1985)
Chou-Wang (2006)
Ivochkina
Ivochkina-Trudinger-Wang (degenerate case)
Kryov (1980's)
Y.-Y. Li (1990)
Trudinger (1996)
Urbas (2003)

The Dirichlet problem in $\mathbb{R}^n$.

- **Caffarelli, Nirenberg and Spruck (1985, CNS).**

Theorem 4 (**[CNS3, Acta 1985]**)**.** *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$, $\psi \in C^{k,\alpha}(\bar{\Omega})$, $\varphi \in C^{k+2,\alpha}(\partial\Omega)$, $k \geq 2$. Assume* **a)** (11)-(13)*, and in addition* **b)**, **c)**, **d)** *below. Then the Dirichelt problem*

$$\begin{cases} f(\lambda(\nabla^2 u + \chi) = \psi \;\; in\ \bar{\Omega} \\ \qquad u = \varphi \;\; on\ \partial\Omega \end{cases} \tag{14}$$

admits a unique admissible solution $u \in C^{k+2,\alpha}(\bar{\Omega})$

Moreover, if $\psi \in C^{\infty}(\bar{\Omega})$, $\varphi \in C^{\infty}(\partial\Omega)$ then $u \in C^{\infty}(\bar{\Omega})$.

The additional conditions:

b) for every $C > 0$ and $\lambda \in \Gamma$ there is a number $R = R(C, \lambda)$ such that

$$f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) \geq C \tag{15}$$

$$f(R\lambda) \geq C \tag{16}$$

c) A geometric condition on $\partial\Omega$:

$$(\kappa_1, \dots, \kappa_{n-1}, R) \in \Gamma \text{ on } \partial\Omega \text{ for some } R > 0 \tag{17}$$

where $(\kappa_1, \dots, \kappa_{n-1})$ are the principal curvatures of $\partial\Omega$

d) $\chi = 0$.

- For the Monge-Ampère equation ($f = \sigma_n^{1/n}$), **c)** means that Ω is strictly convex.

Trudinger in 1995 removed condition (15).

- **Guan (2014, for general domains).**

Theorem 5 (**Guan 2014, arXiv:1403.2133**). *Assume*

a) (11)-(13) *hold,*

e) the subsolution assumption: *there exists an admissible subsolution* $\underline{u} \in C^2(\bar{\Omega})$

$$\begin{cases} f(\lambda(\nabla^2 \underline{u} + \chi) \geq \psi \ \ \textit{in } \bar{\Omega} \\ \qquad\qquad \underline{u} = \varphi \quad \textit{on } \partial\Omega. \end{cases} \tag{18}$$

The Dirichlet problem (14) *then has a unique admissible solution* $u \in C^{k+2,\alpha}(\bar{\Omega})$*.*

Moreover, if $\psi \in C^\infty(\bar{\Omega})$, $\varphi \in C^\infty(\partial\Omega)$ *then* $u \in C^\infty(\bar{\Omega})$*.*

The proof. To derive

$$|u|_{C^2(\bar{\Omega})} \leq C.$$

Then $C^{2,\alpha}$ and higher order estimates follows from Evans-Krylov Theorem and Schauder theory; existence by the continuity method.

Theorem 6 (**Guan, 2014**)**.** *Let $u \in C^4(M) \cap C^2(\bar{M})$ be an admissible solution of the Dirichlet problem* (14)*. Suppose* (11)-(13) *hold and that there exists an admissible subsolution $\underline{u} \in C^2(\bar{M})$:*

$$\begin{cases} f(\lambda[\nabla^2 \underline{u} + \chi]) \geq \psi \quad \text{in } \bar{M}, \\ \qquad\qquad \underline{u} = \varphi \quad \text{on } \partial M. \end{cases} \tag{19}$$

Then

$$\max_{\bar{M}} |\nabla^2 u| \leq C. \tag{20}$$

The proof consists of two steps:

- a maximum principle for $|\nabla^2 u|$

$$\max_{\bar{M}} |\nabla^2 u| \leq C\Big(1 + \max_{\bar{M}} |\nabla u|^2 + \max_{\partial M} |\nabla^2 u|\Big) \tag{21}$$

- the boundary estimate

$$\max_{\partial M} |\nabla^2 u| \leq C. \tag{22}$$

The concavity and subsoltuion. For $\sigma > \sup_{\partial\Gamma} f$, let

$$\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$$

and suppose $\Gamma^\sigma \neq \emptyset$. By (11) and (12) the level set

$$\partial\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) = \sigma\}$$

i.e., boundary of Γ^σ, is a smooth convex hypersurface.

For $\lambda \in \partial\Gamma^\sigma$ let

$$\nu_\lambda = \frac{Df(\lambda)}{|Df(\lambda)|}$$

be the unit normal to $\partial\Gamma^\sigma$ and $T_\lambda\partial\Gamma^\sigma$ denote the tangent plane of $\partial\Gamma^\sigma$ at λ.

Theorem 7. *Let $\mu \in \Gamma$ and $\beta > 0$. There exists uniform constant $\varepsilon > 0$ such that when $|\nu_\mu - \nu_\lambda| \geq \beta$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon \sum f_i(\lambda) + \varepsilon. \tag{23}$$

- Used in global estimates for $|\nabla^2 u|$ too.
- Apply Theorem 7 to $\mu = \lambda[\underline{u}]$, $\lambda = \lambda[u]$.

Equations on closed Riemannian manifolds. Let (M^n, g) be a Riemannian manifold without boundary.

Y.-Y. Li (1990) first studied equation (14) for $\chi = g$ on closed Riemannian manifolds with nonnegative sectional curvature. John Urbas (2002) removed the curvature assumption.

Their main extra assumptions are

$$\lim_{|\lambda|\to\infty} \sum f_i = \infty$$

and

$$\lim_{|\lambda|\to\infty} \sum f_i(1+\lambda_i^2) = \infty$$

respectively.

• **The notaion of subsolutions on closed Riemannian manifolds.** When M is closed, the subsolution assumption does not make sense. Indeed, a subsolution must be a solution, or there is no solution. This is a consequence of the maximum principle.

The tangent cone at infinity (Guan, DJM 2014). Define

$$S^\sigma_\mu = \{\lambda \in \partial\Gamma^\sigma : \nu_\lambda \cdot (\mu - \lambda) \leq 0\}$$

and

$$\mathcal{C}^+_\sigma = \{\mu \in \mathbb{R}^n : S^\sigma_\mu \text{ is compact}\}.$$

which we call the *tangent cone at infinity* to Γ^σ.

Theorem 8 (**Guan, DJM 2014**)**. a)** *$\mathcal{C}_\sigma^+$ is open.*

b) *Assume* (11)-(13) *and that there exists $\underline{u} \in C^2(\bar{M})$ with*

$$\lambda(\nabla^2 \underline{u} + \chi)(x) \in \mathcal{C}^+_{\psi(x)}, \quad \forall\, x \in \bar{M}. \tag{24}$$

Then

$$\max_{\bar{M}} |\nabla^2 u| \leq C\Big(1 + \max_{\partial M} |\nabla^2 u|\Big) \tag{25}$$

The enlarged cone $\widetilde{\mathcal{C}_\sigma^+}$. We now construct a larger cone from $\mathcal{C}_\sigma^+$. Note that the unit normal vector of any supporting hyperplan to Γ^σ belongs to $\overline{\Gamma_n}$. We define $\tilde{\mathcal{C}}_\sigma^+$ to be the region in $\mathbb{R}^n$ bounded by those supporting hyperplans to $\mathcal{C}_\sigma^+$ with unit normal vector in $\partial\Gamma_n$; so $\tilde{\mathcal{C}}_\sigma^+ = \mathbb{R}^n$ if there are no such supporting planes. Clearly, if $\tilde{\mathcal{C}}_\sigma^+ \neq \mathbb{R}^n$ it is an open symmetric convex cone with vertix at $a\mathbf{1}$ for some $a \in \mathbb{R}^n$ where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Moreover, $\mu + \Gamma_n \subset \tilde{\mathcal{C}}_\sigma^+$ for $\mu \in \tilde{\mathcal{C}}_\sigma^+$ and $\tilde{\mathcal{C}}_\sigma^+ \subset \tilde{\mathcal{C}}_\rho^+$ if $\sigma \geq \rho$.

Theorem 9 (Guan, 2015). *Let $\mu \in \tilde{\mathcal{C}}_\sigma^+$ and $d(\mu)$ denote the distance from μ to $\partial\tilde{\mathcal{C}}_\sigma^+$. Then there exist $\delta, \varepsilon > 0$ such that for any $\lambda \in \partial\Gamma^\sigma$, either*

$$f_i(\lambda) \geq \delta \sum f_k(\lambda) \tag{26}$$

or

$$f_i(\lambda)(\mu_i - \lambda_i) \geq \varepsilon \sum f_i(\lambda) + \varepsilon. \tag{27}$$

Theorem 10 (**Guan, 2015**). *Assume* (11)-(13) *and that there exists* $\underline{u} \in C^2(\bar{M})$ *with*

$$\lambda(\nabla^2 \underline{u} + \chi)(x) \in \widetilde{\mathcal{C}^+_{\psi(x)}}, \quad \forall\, x \in \bar{M}. \tag{28}$$

Then

$$\max_{\bar{M}} |\nabla^2 u| \leq C\Big(1 + \max_{\partial M} |\nabla^2 u|\Big). \tag{29}$$

Szekelyhidi (2015) introduced the another nontion of generalized subsolutions:

$$(\lambda(\nabla^2 \underline{u} + \chi)(x) + \Gamma_n) \cap \partial\Gamma^{\psi(x)} \text{ is compact } \ \forall\, x \in \bar{M}. \tag{30}$$

and proved the same estimates under a little stronger assumptions. The following result clarify the ralations.

Theorem 11 (**Guan, 2016**). **a)** *For Type I cone,* $\mathcal{C}_\sigma^+ = \tilde{\mathcal{C}_\sigma^+}$. **b)** *Assumptions* (28) *and s* (30) *are equivalent.*

According to CNS, a cone Γ is Type 1 if each λ_i-axis belongs to $\partial\Gamma$. For instance, Γ_k $(k \geq 2)$ are Type 1.

- Canonical χ: $\chi = 0$ or $\chi = g$. For instance,

$$\det(\nabla^2 u + g) = K(x)(-2u - |\nabla u|^2)\det g \tag{31}$$

  is the Darboux equation (isometric embedding).
- $\nabla^2 u + \mathrm{Ric}_g$: the Bakry-Emery Ricci tensor of the Riemannian measure space $(M^n, g, e^{-u}dg)$.
- Te Ricci soliton equation: $\nabla^2 u + \mathrm{Ric}_g = \lambda g$
- $\nabla^2 u + ug$ on $\mathbb{S}^n$. (In classical geometry. Minkowski problem, extensions proposed by Alexandrov, Chern; Christoffel-Minkowski problem.)
- In conformal geometry. The Schouten tensor of $(M^n, e^{2u}g)$

$$\chi = du \otimes du - \frac{1}{2}|\nabla u|^2 g + S_g \tag{32}$$

  where S_g is the Schouten tensor of (M^n, g); the Ricci tensor

$$A[u] = \nabla^2 u + \gamma \Delta u g + du \otimes du - \frac{1}{2}|\nabla u|^2 g + \mathrm{Ric}_g.$$

- In optimal transportation.
- Conformal deformation of metrics on Hermitian manifolds.
  $A[u] := \pm \mathrm{Ric}_u = \sqrt{-1}\partial\bar{\partial} u \pm \mathrm{Ric}.$

More general equations. For the more general equation

$$F(D^2u + \chi) = \psi,$$

we assume the function F to be defined in an open convex cone Γ in $\mathcal{S}^{n\times n}$, the (inner product) space of $n \times n$ symmetric matrices, with vertex at 0, $\Gamma^+ \subseteq \Gamma \neq \mathcal{S}^{n\times n}$ where Γ^+ denotes the cone of positive matrices, and to satisfy the fundamental structure conditions:

(**a**) the *ellipticity* condition.

(**b**) the *concavity* condition.

Equations on complex manifolds.

Let (M^n, ω) be a compact Hermitian manifold of complex dimension $n \geq 2$ with smooth boundary ∂M which may be empty (M is closed) and let, for a function $u \in C^2(M)$, $\chi[u] := \chi(\cdot, du(\cdot), u(\cdot))$ be a real $(1,1)$ form on M, and define

$$\chi_u := \chi[u] + \sqrt{-1}\partial\bar{\partial}u.$$

The equation:

$$f(\lambda(\chi_u)) = \psi(z, du, u) \text{ in } M \tag{33}$$

where $\lambda(\chi_u) = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of χ_u with respect to the metric ω.

This covers most of the important equations in complex geometry.

- In local coordinates:

$$\chi_u = \sqrt{-1}(u_{i\bar{j}} + \chi_{i\bar{j}})dz_i \wedge d\bar{z}_j.$$

- **Guan-Nie:** $\chi_u := \chi[u] + \sqrt{-1}\partial\bar{\partial}u$ where χ depends linearly on du.
- **Guan-Qiu-Yuan:** $A[u] = \Delta u\omega - \gamma\sqrt{-1}\partial\bar{\partial}u + \chi$ where $\chi = \chi(du)$.

- **The complex Monge-Ampère equation:**

$$\chi_u^n = \psi^n \omega^n \tag{34}$$

corresponds to equation (8) for $f = \sigma_n^{1/n}$. It plays a central role in Kähler geometry.

  - **Calabi-Yau Theorem.** Yau's proof of Calabi conjecture; Aubin indepedently for $c_1(M) < 0$.
  - **Extension to Hermitian case.**
    - Cherrier: $n = 2$;
    - Tosatti-Weinkove.

- **The Dirichlet problem.**

  – Caffarelli-Kohn-Nirenberg-Spruck: strongly pseudoconvex domain in $\mathbb{R}^n$;

  – Guan (1998): general domain in $\mathbb{C}^n$;

  – Guan adn Qun Li (2010): on Hermitian manifolds.
- **Some applications.**

  – Chern-Levine-Nirenberg conjecture: Pengfei Guan

  – Donaldson's conjecture on geodesics in the space of Kähler metrics: Mabuchi, Donaldson, Xiuxiong Chen, many others .....

  – Pluricomplex Green functions.

  – Totally real submanifolds.

There have also been increasing interests in other fully nonlinear equations from Kähler geometry.

• **Donaldson:**

$$\chi_u^n = \psi \chi_u^{n-1} \wedge \omega, \ \chi_u > 0, \tag{35}$$

proposed in the setting of moment maps, where he assumes χ is also Kähler and ψ is the Kähler class invariant

$$\psi = c_1 = \frac{\int_M \chi^n}{\int_M \chi^{n-} \wedge \omega}.$$

This corresponds to $f = \sigma_n/\sigma_{n-1}$. The equation is also closely related to the lower bound and properness of the Mobuchi energy.

• **Conformal metrics on Hermitian manifolds.** Let $\alpha = e^{\pm u}\omega$ be a conformal metric on M. The Chern-Ricci form is given by

$$\pm\mathrm{Ric}_\alpha = \pm\sqrt{-1}\partial\bar{\partial}\log\alpha^n = \sqrt{-1}\partial\bar{\partial}u \pm \mathrm{Ric}_\omega$$

Consequently, the problem of determining a metric in the conformal class of ω with special properties of the Chern-Ricci form leads to the following equation for $\chi = \pm\mathrm{Ric}_\omega$

$$F\left(\frac{\chi_u}{e^{\pm u}\omega}\right) = \psi \quad \text{on } M. \tag{36}$$

The sign in front of Ric_α is determined by requiring $\chi_u \in \Gamma$. The negative sign case will be much more difficult to study.

For $F(A) = (\det A)^{1/n}$, this equation is related to a conjecture of Yau on the holomorphic sectional curvature of a Kähler manifolds which was recently solved by Wu-Yau and Tosatti-Yang.

• **Balanced metrics with prescribed volume form** Recall that a Hermitian metric ω is *balanced* if $d(\omega^{n-1}) = 0$ and *Gauduchon* if $\partial\bar{\partial}(\omega^{n-1}) = 0$. A well know result due to Gauduchon asserts that any conformal class of Hermitian metrics on a compact (closed) complex manifold contains a Gauduchon metric. Fu-Wang-Wu introduced an equation of prescribed volume for balanced metrics, which can be described as below following Tosatti-Weinkove.

Let ω_0 be a balanced metric on a closed Hermitian manifold (M, ω). We seek a balanced metric η such that

$$\eta^{n-1} = \omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}(u\omega^{n-2})$$

for some $u \in C^\infty(M)$ and prescribed volume

$$\frac{\eta^n}{\omega^n} = \psi \text{ on } M. \tag{37}$$

This is a Monge-Ampère type equation of $(n-1, n-1)$ forms. Nevertheless, by the Hodge star duality approach of Tosatti-Weinkove, equation (37) can be converted to an equation of form

(38) for $F(A) = P_{n-1}(A)$ where χ depends linearly on u and the gradient of u; More precisely,

$$\chi = (\mathrm{tr}\tilde{\chi} - (n-1)\tilde{\chi}$$

where

$$\tilde{\chi} = \frac{1}{(n-1)!}\star(\omega_0^{n-1} - \sqrt{-1}\partial u\bar{\partial}(\omega^{n-2}) + \sqrt{-1}\bar{\partial}u\partial(\omega^{n-2}) + \sqrt{-1}u\partial\bar{\partial}(\omega^{n-2}))$$

and $\star$ denotes the Hodge star-operator.

• **Gauduchon conjecture.**

Let complex dimension $n \geq 2$ and let Ω be a closed real $(1,1)$ form on M with $[\Omega] = c_1^{BC}(M)$ in the Bott-Chern cohomology group $H^{1,1}_{BC}(M, \mathbb{R})$. In 1984, Gauduchon conjectured that there exists a Gauduchon metric $\tilde{\omega}$ on M with Chern-Ricc curvature $Ric_{\tilde{\omega}} = \Omega$.

This is a natural extension of the Calabi conjecture for Kähler manifolds solved by Yau. It was discovered by Tosatti and Weinkove and independently by Popovicithat the Gauduchon conjecture reduces to solving a Monge-Ampère type equation of the form

$$\det(\Phi_u) = e^{F+b} \det(\omega^{n-1}) \text{ in } M \tag{38}$$

with

$$\Phi_u = \omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-2} + \mathfrak{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}\omega^{n-2}) > 0$$

and $\sup_M u = 0$, where ω_0 is any Hermitian metric (positive definite $(1,1)$ form) and ω a Gauduchon metric. This equation is equivalent to one for $f = P_{n-1}$ with linear dependences of χ on du.

Results of Guan-Xiaolan Nie.

We study equation (38) on closed Hermitain manifold with χ depending linearly on du. Our result applies to P_{n-1} giving the estimates need in proving Gaudochun's conjecture.

Theorem 12. *Let $\psi \in C^2(M)$ and $u \in C^4(M)$ be an admissible solution of (8). Suppose that there exists a function $\underline{u} \in C^2(M)$ satisfying*

$$\lambda(\chi_{\underline{u}}(z)) \in \tilde{\mathcal{C}}^{+}_{\psi(z)} \ \ \forall\, z \in M, \tag{39}$$

and that at any fixed point on M where $g_{i\bar{j}} = \delta_{ij}$ and $\mathfrak{g}_{i\bar{j}} = \delta_{ij}\lambda_i$ with $\lambda_1 \geq \cdots \geq \lambda_n$,

$$\sum f_i(|\chi_{i\bar{1},\zeta_\alpha}| + \chi_{i\bar{i},\zeta_\alpha\bar{1}}|) \leq C\lambda_1 f_\alpha, \ \ \forall\, \alpha \leq n - r_0 \tag{40}$$

where $r_0 = \min\left\{\text{rank of } \tilde{\mathcal{C}}^{+}_{\psi(z)} : z \in M\right\}$. Then

$$\max_M |\Delta u| \leq C_1 e^{C_2(u - \inf_M u)} \tag{41}$$

where C_1 depends on $|\nabla u|_{C^0(M)}$ and C_2 is a uniform constant (independent of u).

Results of Guan-Qiu-Yuan.

$$F(\Delta u \omega + \gamma\sqrt{-1}\partial\bar{\partial}u + \chi(z, u, du)) = \psi, \quad \gamma < 1.$$

Thank You!