几何与表示掠影

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数学所讲座

（2017.10.11）

Regular polyhedron

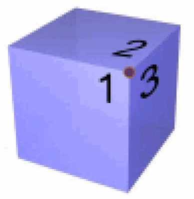
A ***regular polyhedron*** is a convex polyhedron such that all faces are regular polygons and all are identical = “most symmetric” polyhedrons

The following is attributed to Theaetetus (415—369 B.C)

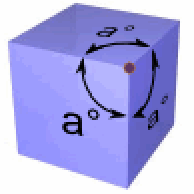
Theorem

There are exactly 5 regular polyhedron (solids) in $\mathbb{R}^3$.

Euclid's Elements (Book XIII)

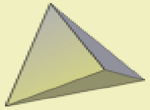
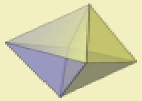
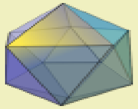
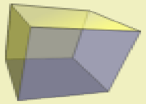
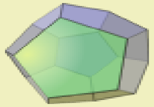


At each vertex **at least 3 faces** meet



When we add up the internal angles that meet at a vertex, it must be less than 360 degrees

A regular n-gon has internal angle 180-360/n, which is bigger than 120 if n>5, hence get n=3,4,5.

3 triangles meet	180°	tetrahedron	
4 triangles meet	240°	octahedron	
5 triangles meet	300°	icosahedron	
3 squares meet	270°	cube	
3 pentagons meet	324°	dodecahedron	

Floruit 300 BC

Statue in Oxford

Topological proof using Euler's formula

Leonhard Euler
15/04/1707 – 18/09/1783

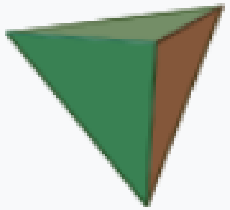
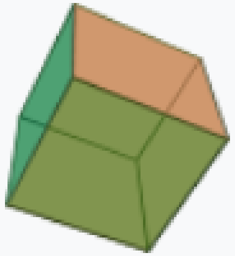
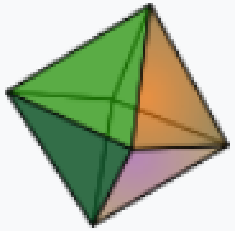
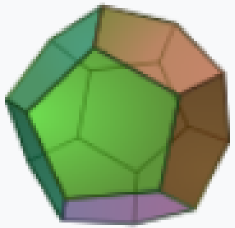
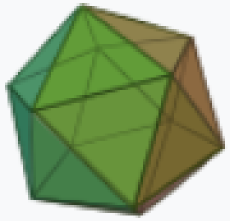
$V = \#$ of vertex, $E = \#$ of edges, $F = \#$ of faces

Assume each face is a regular n-gon, and each vertex is shared by m faces.

Euler's Formula: $V - E + F = 2$

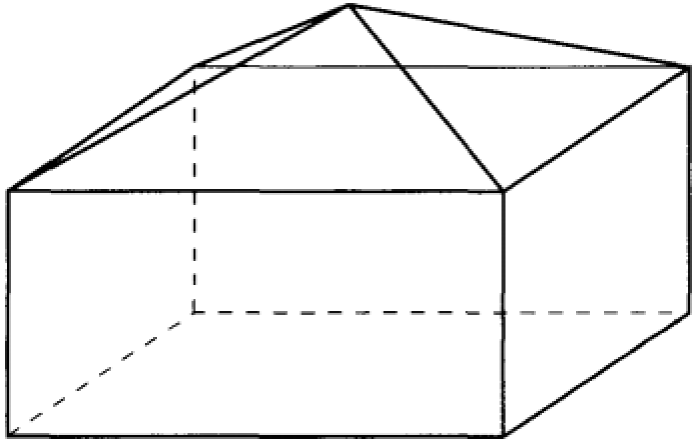
Two ways to count # of edges: $2E = nF$, $2E = mV$

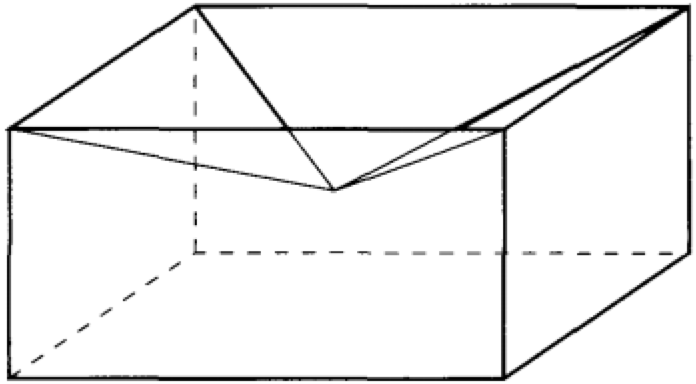
This gives $1/m + 1/n = 1/2 + 1/E$, which has only 5 possibilities:

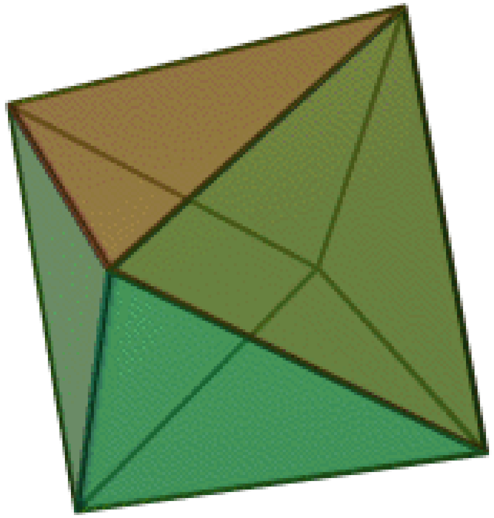
Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Augustin-Louis Cauchy
(21/08/1789 – 23/05/1857)

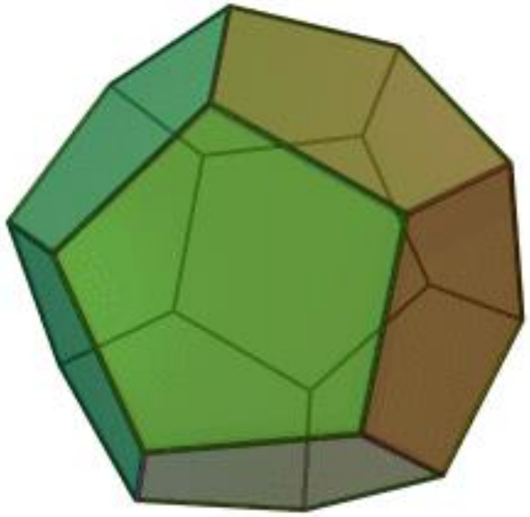
Theorem. *If two 3-dimensional convex polyhedra P and P' are combinatorially equivalent with corresponding facets being congruent, then also the angles between corresponding pairs of adjacent facets are equal (and thus P is congruent to P').*



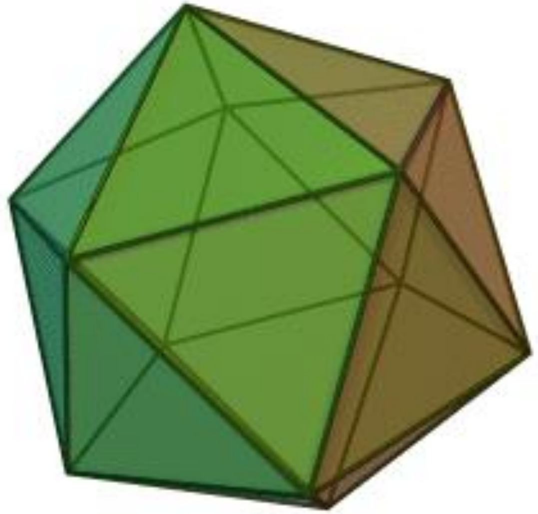




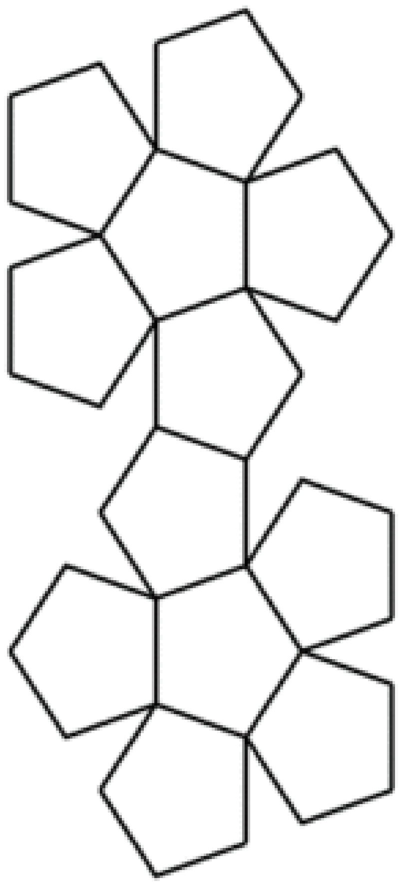
Octahedron

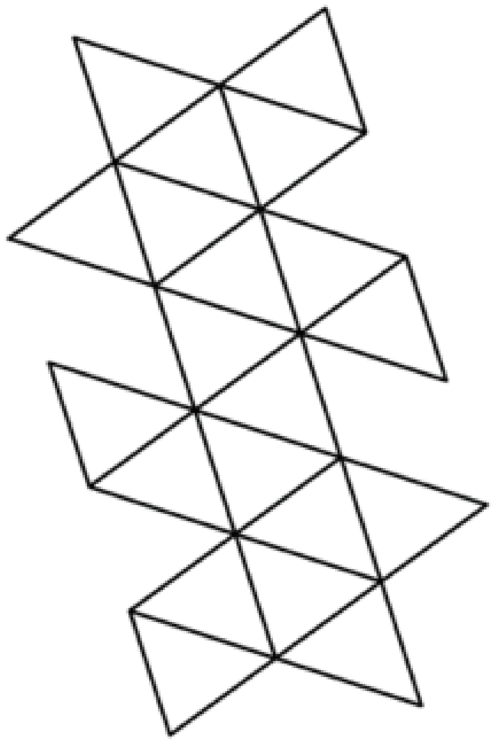


Dodecahedron

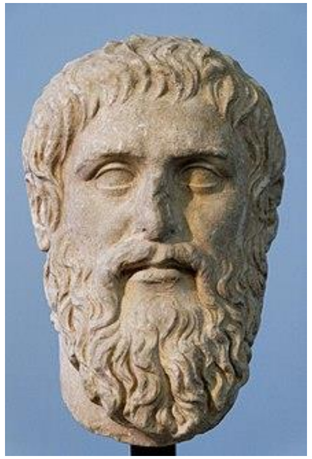


Icosahedron

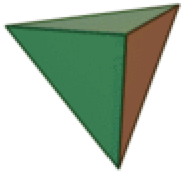




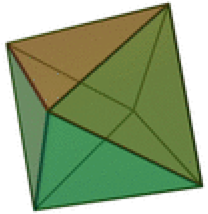
Plato’s Theory of Everything: *Timaeus*



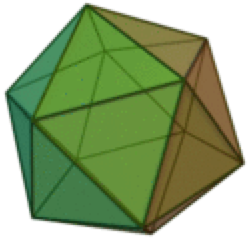
428/427 or 424/423 – 348/347 BC



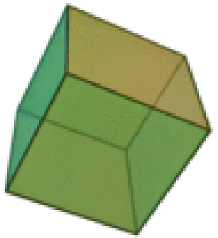
Fire



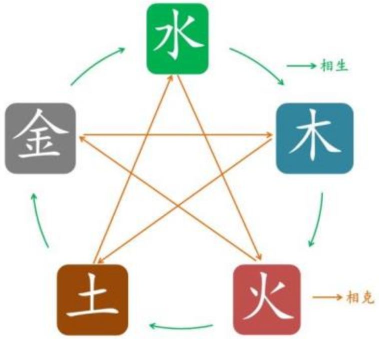
Air



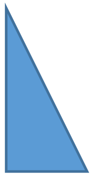
Water



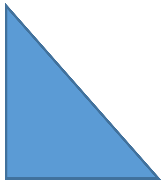
Earth



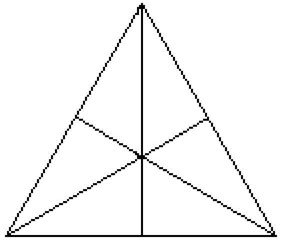
天有五行，水火金木土，分时化育，以成万物。其神谓之五帝

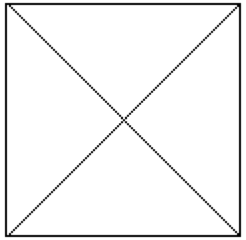


Type 1



Type 2





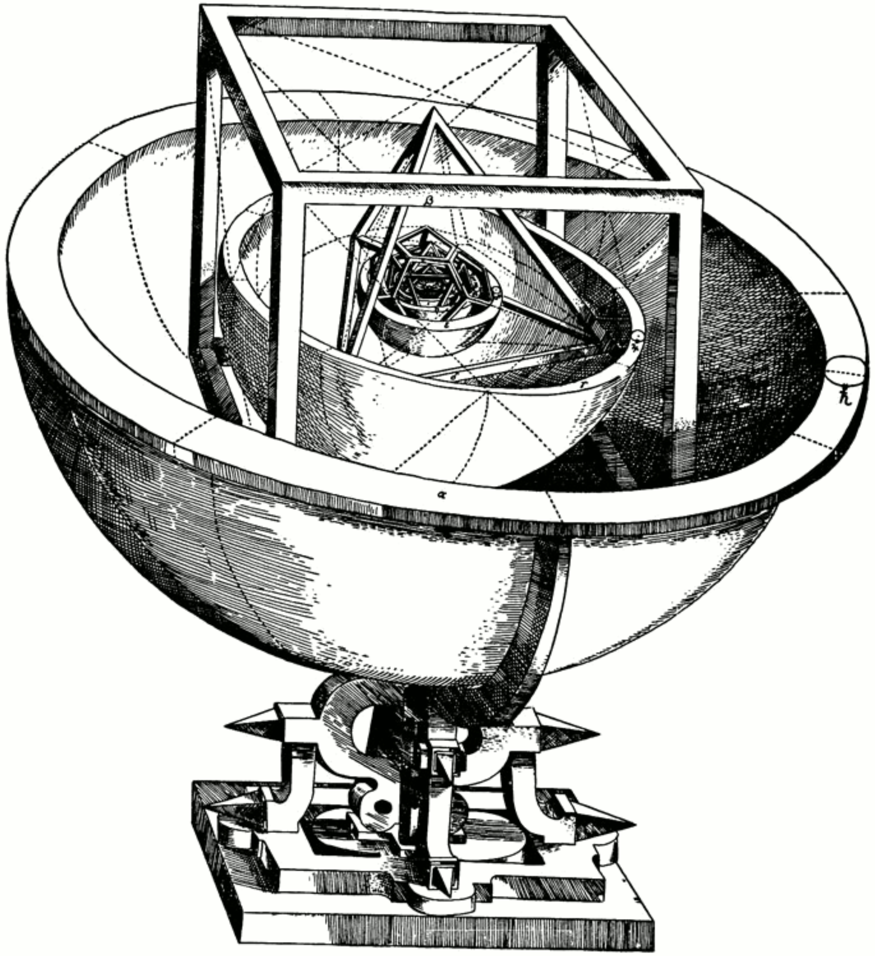
			number of triangles	
			type 1	type 2
tetrahedron	plasma	("fire")	24	0
octahedron	gas	("air")	48	0
icosahedron	liquid	("water")	120	0
hexahedron	solid	("earth")	0	24

Chemical Action: 1 water + 1 fire = 3 air

Kepler's Mysterium Cosmographicum

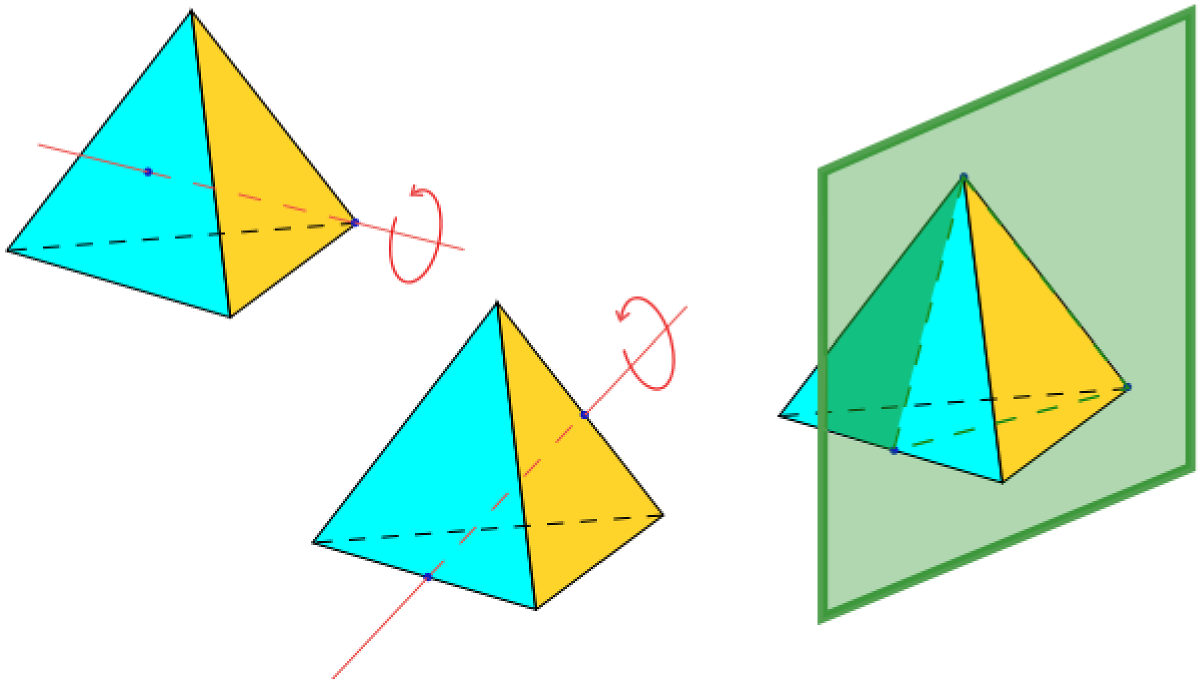
Johannes Kepler
(27/12/1571 – 15/11/1630)

the octahedron	between	Mercury and Venus,
the icosahedron	between	Venus and Earth,
the dodecahedron	between	Earth and Mars,
the tetrahedron	between	Mars and Jupiter
and the cube	between	Jupiter and Saturn



Kepler's Platonic solid model of the Solar system from *Mysterium Cosmographicum*

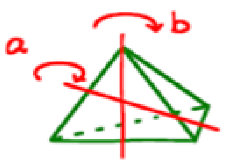
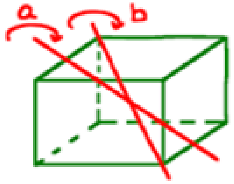
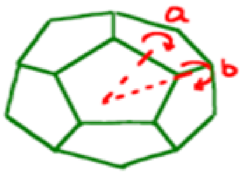
Symmetries



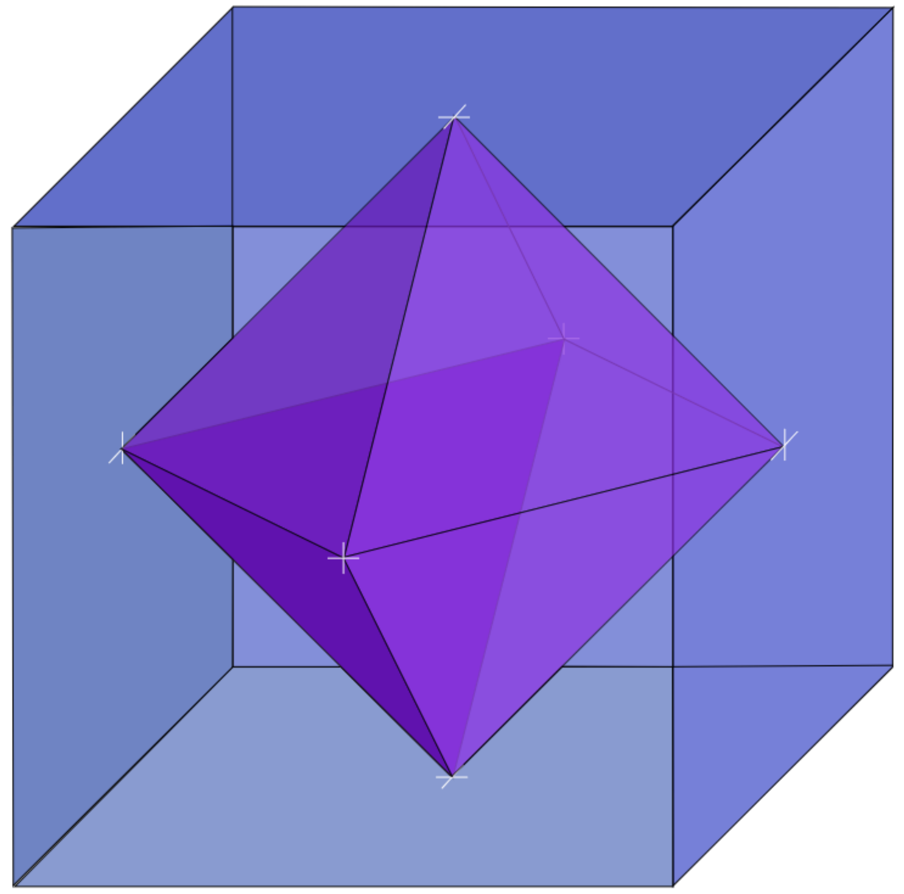
The proper rotations, (order-3 rotation on a vertex and face, and order-2 on two edges) and reflection plane (through two faces and one edge) in the symmetry group of the regular tetrahedron

Symmetries of Platonic solids

tetrahedron	A_4
cube	S_4
octahedron	S_4
dodecahedron	A_5
icosahedron	A_5

$A_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$	12	
$S_4 = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1 \rangle$	24	
$A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$	60	

Dual solids



- The tetrahedron is self-dual.
- The cube and the octahedron form a dual pair.
- The dodecahedron and the icosahedron form a dual pair.

Finite subgroups in $SO(3, \mathbb{R})$

Let $\Gamma \subset SO(3, \mathbb{R})$ be a finite subgroup.

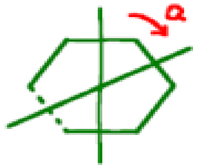
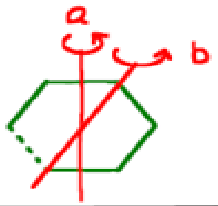
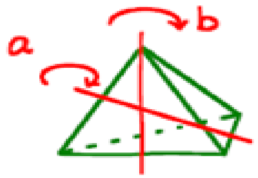
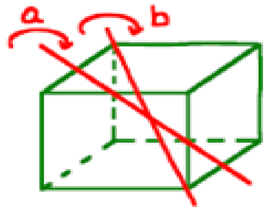
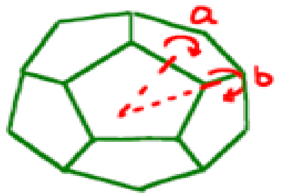
- every $1 \neq \gamma \in \Gamma$ has two fixed points on S^3 (axe of rotation).
- The set P of all such fixed points is Γ-invariant.

$$|\Gamma| - 1 = \frac{1}{2} \sum_{p \in P} (|\mathrm{Stab}_\Gamma(p)| - 1)$$

- Note that $|\mathrm{Stab}_\Gamma(p)| = \dfrac{|\Gamma|}{|\Gamma \cdot p|}$
- Let $P = O_1 \sqcup O_2 \sqcup \cdots O_r$ be the Γ-orbits, $|\Gamma| - 1 = \dfrac{1}{2} \displaystyle\sum_{i=1}^{r} (|\Gamma| - \dfrac{|\Gamma|}{|\mathrm{Stab}_\Gamma(p)|})$

$$\boxed{2 - \frac{2}{|\Gamma|} = \sum_{i=1}^{r} (1 - \frac{1}{a_i}), a_i \textit{ divides } |\Gamma|}$$

Finite subgroups in $SO(3,\mathbb{R})$: lists

$C_n = \langle a \mid a^n = 1 \rangle$	n	
$D_n = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$	$2n$	
$A_4 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$	12	
$S_4 = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1 \rangle$	24	
$A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$	60	

Connecting $SU(2)$ to $\mathrm{SO}(3, \mathbb{R})$

$$SU(2) = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\} \simeq S^3$$

Its Lie algebra $\mathfrak{su}(2) \simeq \mathbb{R}^3$ has an inner product: $\langle U, V \rangle = \frac{1}{2}\mathrm{Tr}(UV^*)$

For any $A \in SU(2)$, define $\phi_A \in \mathrm{End}(\mathfrak{su}(2))$ by $\phi_A(U) = A\bar{U}A^{-1}$

- ϕ_A preserves the inner product, hence

$$\boxed{\phi : SU(2) \to SO(3, \mathbb{R})}$$

- This is a 2:1 map of Lie groups, with kernal $= \{\pm 1\}$

Finite subgroups in $SU(2)$

C_n	$\langle a \mid a^n = 1 \rangle$	n	Cyclic
D_{2n}^*	$\langle a, b \mid a^2 = b^2 = (ab)^n \rangle$	$4n$	dihedral
A_4^*	$\langle a, b \mid a^2 = b^3 = (ab)^3 \rangle$	24	Binary tetrahedral
S_4^*	$\langle a, b \mid a^2 = b^3 = (ab)^4 \rangle$	48	Binary octahedral
A_5^*	$\langle a, b \mid a^2 = b^3 = (ab)^5 \rangle$	120	Binary icosahedral

Finite subgroups in $SL(2)$

- $SU(2) \subset SL(2) = SL(2, \mathbb{C})$ is a subgroup.

Christian Felix Klein
(25/04/1849 – 22/06/1925)

- For any finite subgroup $\Gamma \subset SL(2)$, it preserves a Hermitian inner product on $\mathbb{C}^2$ by averaging, hence it is isomorphic to a subgroup of $SU(2)$

$$\boxed{\text{finite subgroups of } SL(2) = \text{finite subgroups of } SU(2).}$$

Representation of finite groups

Definition

A **representation** of a group Γ is a complex vector space V with a homomorphism $\rho : \Gamma \to \mathrm{GL}(V)$, i.e. Γ acts linearly (via ρ) on V. The representation (V, ρ) is **irreducible** if $\nexists 0 \neq W \subsetneqq V$ such that W is Γ-invariant.

- Only consider $\dim V < \infty$ and Γ finite.
- Every representation of Γ is a direct sum of irreducible representations.
- Write $\mathrm{Irr}(\Gamma)$ the set of all irred. representations of Γ. Then $|\mathrm{Irr}(\Gamma)| = \sharp$ conjugacy classes of Γ.

McKay graph

- $\Gamma \subset SL(2)$ a finite subgroup
- $V_0, \cdots, V_r$ the irred. rep. of Γ
- $W := \mathbb{C}^2$ is the natural rep. of Γ

$$\boxed{W \otimes V_i = \oplus_j V_j^{\oplus a_{ij}} \text{ as } \Gamma - \textit{representations.}}$$

Note that $W^* \simeq W$ (as $\Gamma \subset SL(2)$), we have

$$a_{ij} = \dim \mathrm{Hom}_\Gamma(V_j, W \otimes V_i) = \dim \mathrm{Hom}_\Gamma(V_j \otimes W, V_i) = a_{ji}$$

McKay graph:

- vertice = irred. rep. of Γ, labelled with its dim.
- two vertices V_i, V_j are connected by a_{ij} edges.

Example: Γ is cyclic

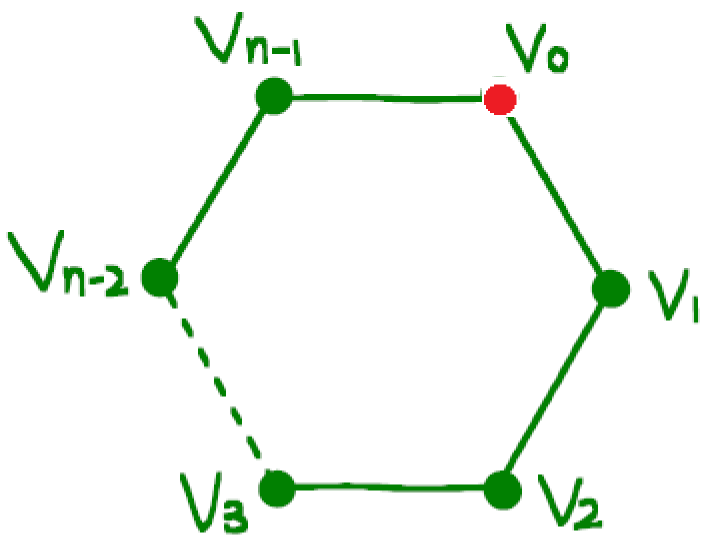
- Consider the case $\Gamma = C_n$: a cyclic group of order n.
- Let $\xi = e^{\frac{2\pi i}{n}}$. Take one generator σ of Γ, then $\sigma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$
- For $0 \leq k \leq n-1$, let

$$\rho_k : \Gamma = C_n \to \mathrm{GL}(1) = \mathbb{C}^* : \sigma \mapsto \xi^k.$$

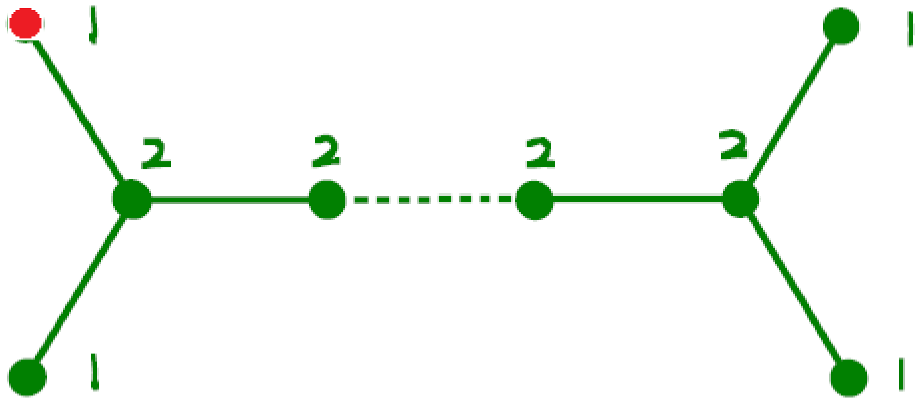
- Irred. representations are $V_0, \cdots, V_{n-1}$.

W is decomposed as $V_1 \oplus V_{n-1}$

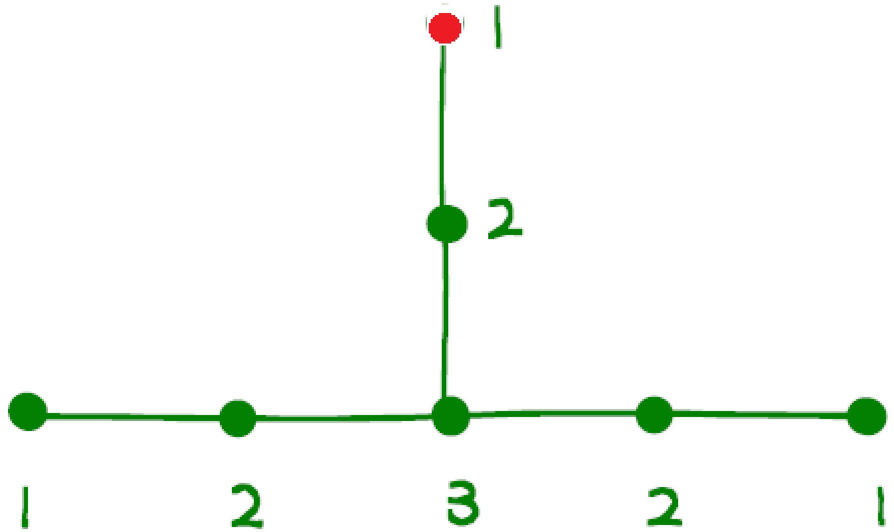
$$\boxed{W \otimes V_i = (V_1 \oplus V_{n-1}) \otimes V_i = V_{i-1} \oplus V_{i+1}}$$



McKay graphes for $\Gamma = D_{2n}^*$ and A_4^*

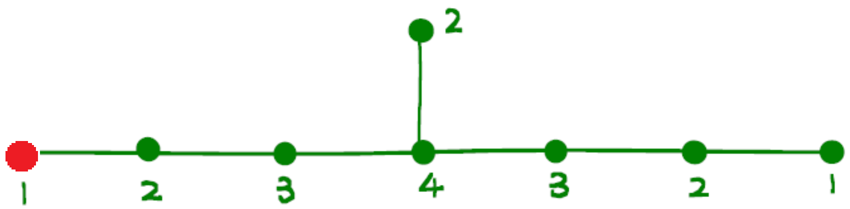


McKay graph for $\Gamma = D_{2n}^*$

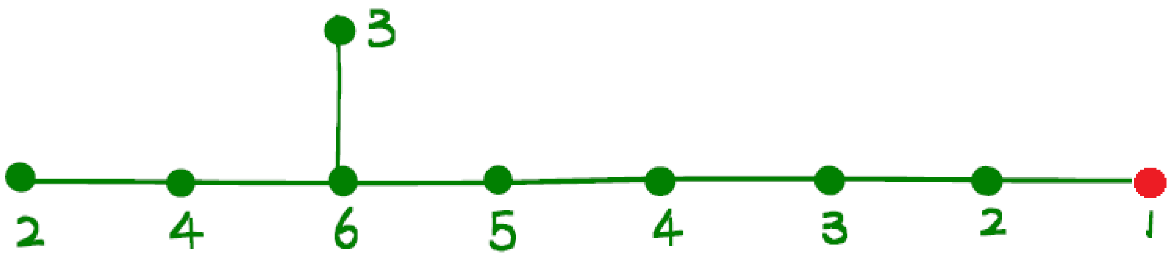


McKay graph for $\Gamma = A_4^*$

McKay graphes for $\Gamma = S_4^*$ and A_5^*



McKay graph for $\Gamma = S_4^*$



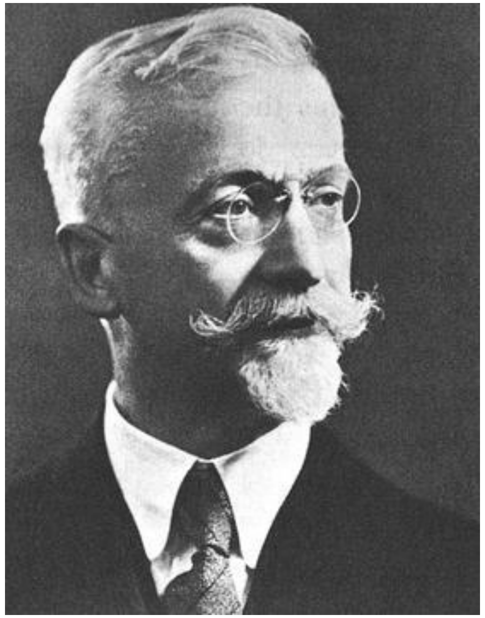
McKay graph for $\Gamma = A_5^*$

Complex Lie algebras

A **(complex) Lie algebra** is a complex vector space $\mathfrak{g}$ together with a non-associative, alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$ called the Lie bracket, satisfying the Jacobi identity. It is called **simple** if it has no non-trivial ideals and is not abelian.

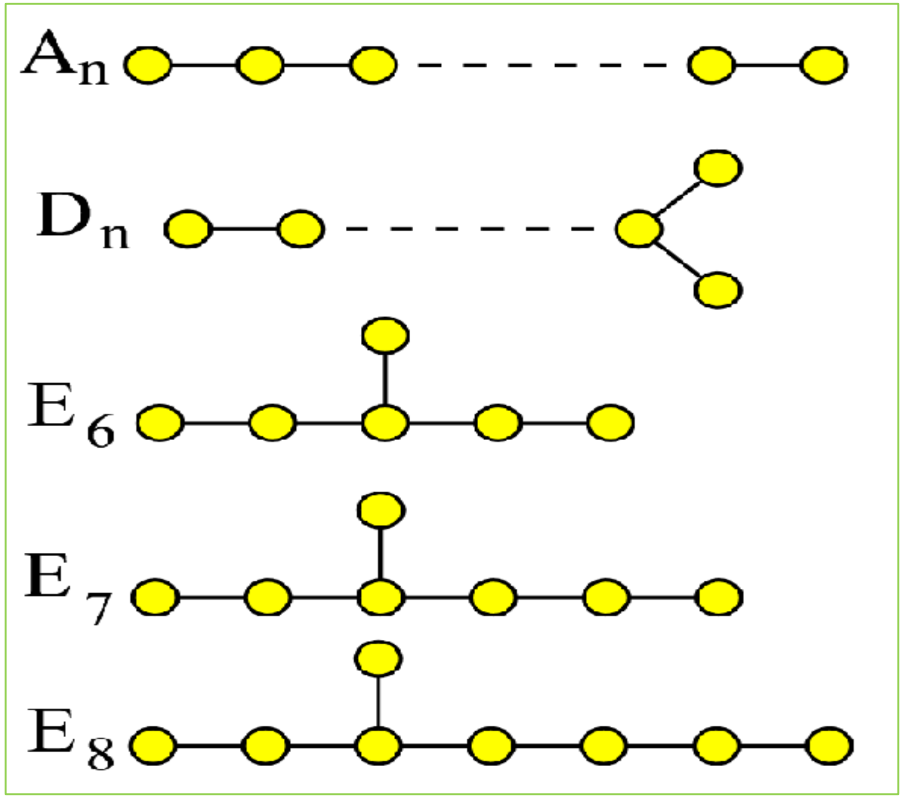
Examples: $\mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$.

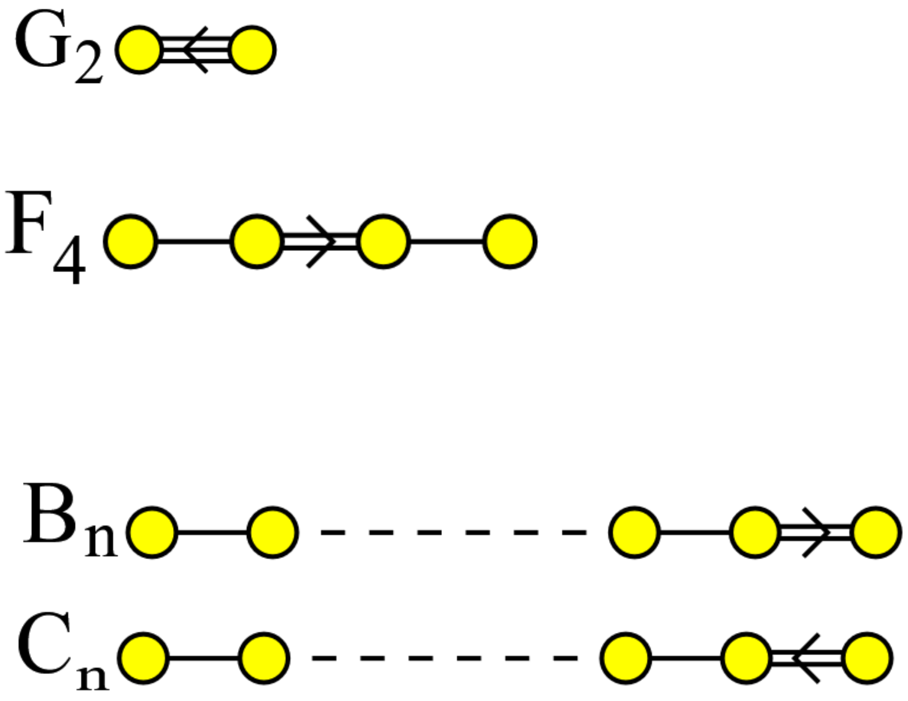
Wilhelm Karl Joseph Killing
(10/05/1847 – 11/02/1923)



Élie Joseph Cartan
(09/04/1869 – 06/05/1951

Dynkin diagrams





Eugene Borisovich Dynkin
(11/05/1924 – 14/11/2014)

McKay graphes are exactly affine Dynkin diagram of type ADE!
What's the relation of them with Lie algebras?

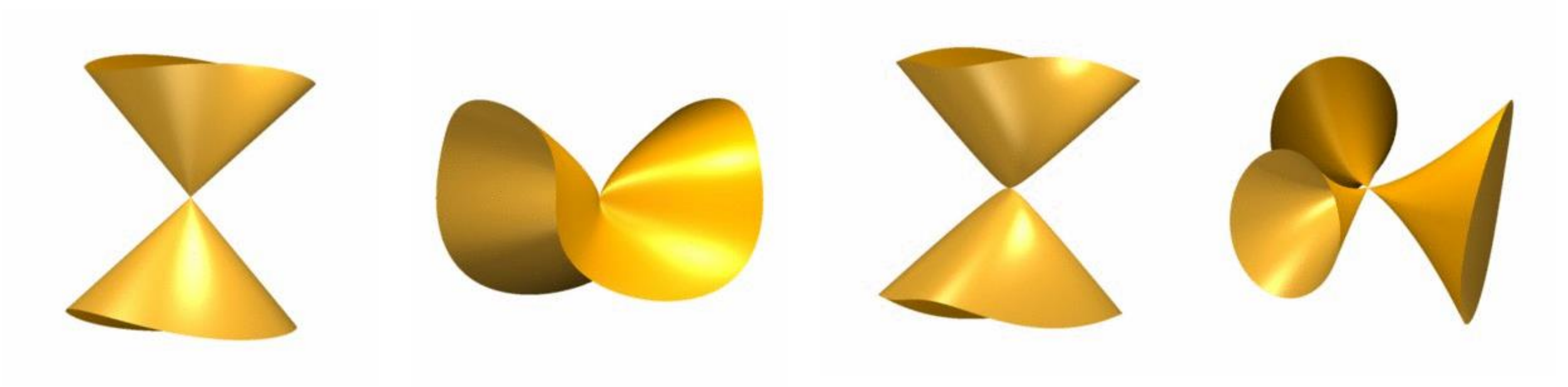
Simple surface singularities

For $\Gamma \subset SL(2)$ finite subgroup, the quotient $\mathbb{C}^2/\Gamma$ (set of Γ-orbits) is an affine algebraic variety $\mathrm{Spec}(\mathbb{C}[u,v]^\Gamma)$. It turns out they all can be realized as a hypersurface in $\mathbb{C}^3$.

Example: $\Gamma = C_n$ with generator $\sigma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$, then $\mathbb{C}[u,v]^\Gamma = \mathbb{C}[uv, u^n, v^n]$, hence $\mathbb{C}^2/\Gamma$ is the hypersurface in $\mathbb{C}^3$ defined by $x^n = yz$.

cyclic	n	$x^n + yz = 0$	A_{n-1}
binary dihedral	$4(n-2)$	$x^{n-1} + xy^2 + z^2 = 0$	D_n
binary tetrahedral	24	$x^4 + y^3 + z^2 = 0$	E_6
binary octahedral	48	$x^3y + y^3 + z^2 = 0$	E_7
binary icosahedral	120	$x^5 + y^3 + z^2 = 0$	E_8

Examples: A_1, A_2, A_3 and D_4



Minimal resolutions

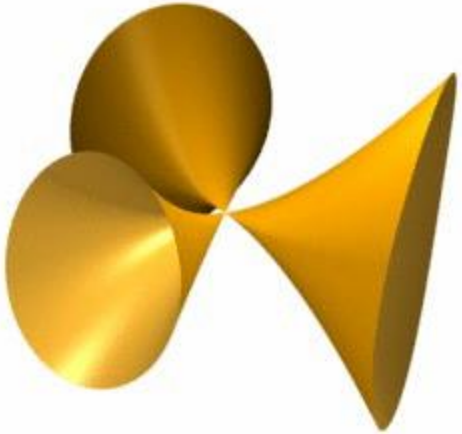
- Every surface S admits a unique **minimal resolution** $\pi : \tilde{S} \to S$.

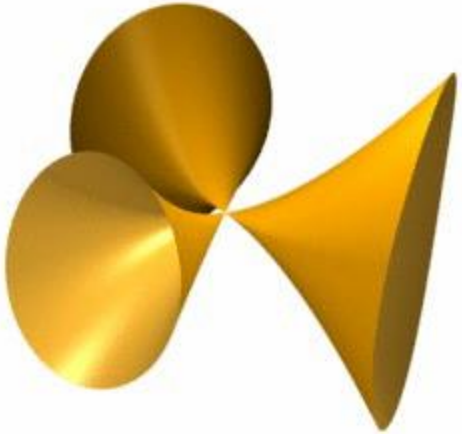
- When $S = \mathbb{C}^2/\Gamma$ for a finite subgroup $\Gamma \subset SL(2)$, the exceptional locus of $\pi : \tilde{S} \to S$ is a union of $\mathbb{P}^1$s, each with self-intersection -2.

$$\mathbb{C}^2/\pm 1 = \{(a, b, c) | a^2 + bc = 0\} \quad \longleftarrow \quad T^*(\mathbb{P}^1).$$

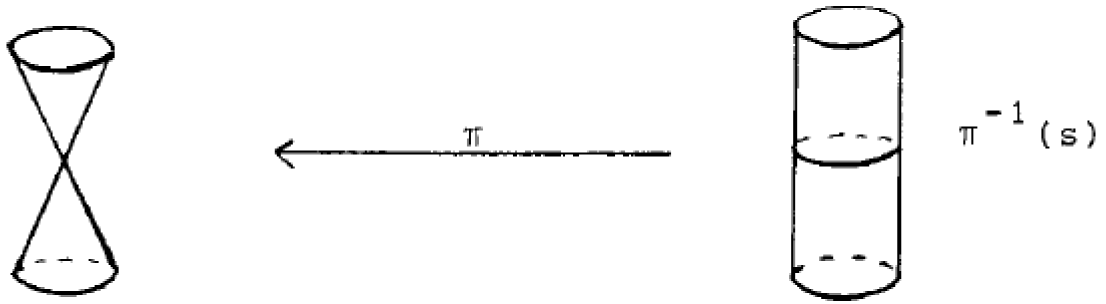
Examples of minimal resolutions



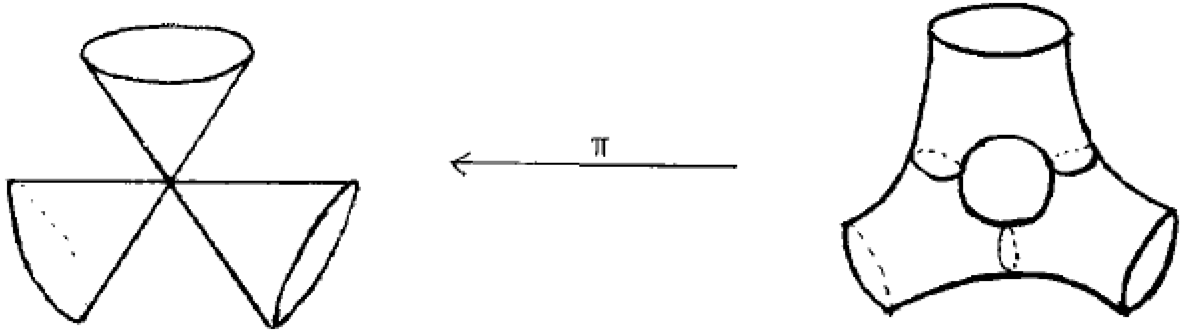


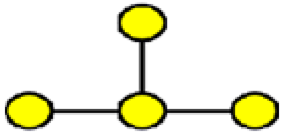



Central fibres and dual graph



$S = \{x^2+yz = o\}$



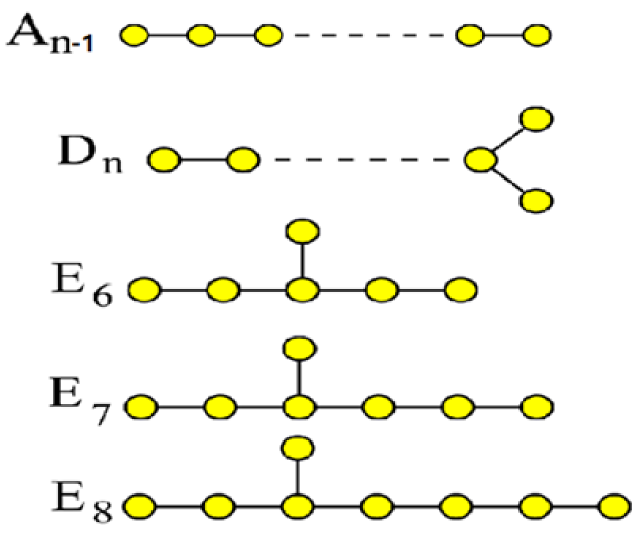


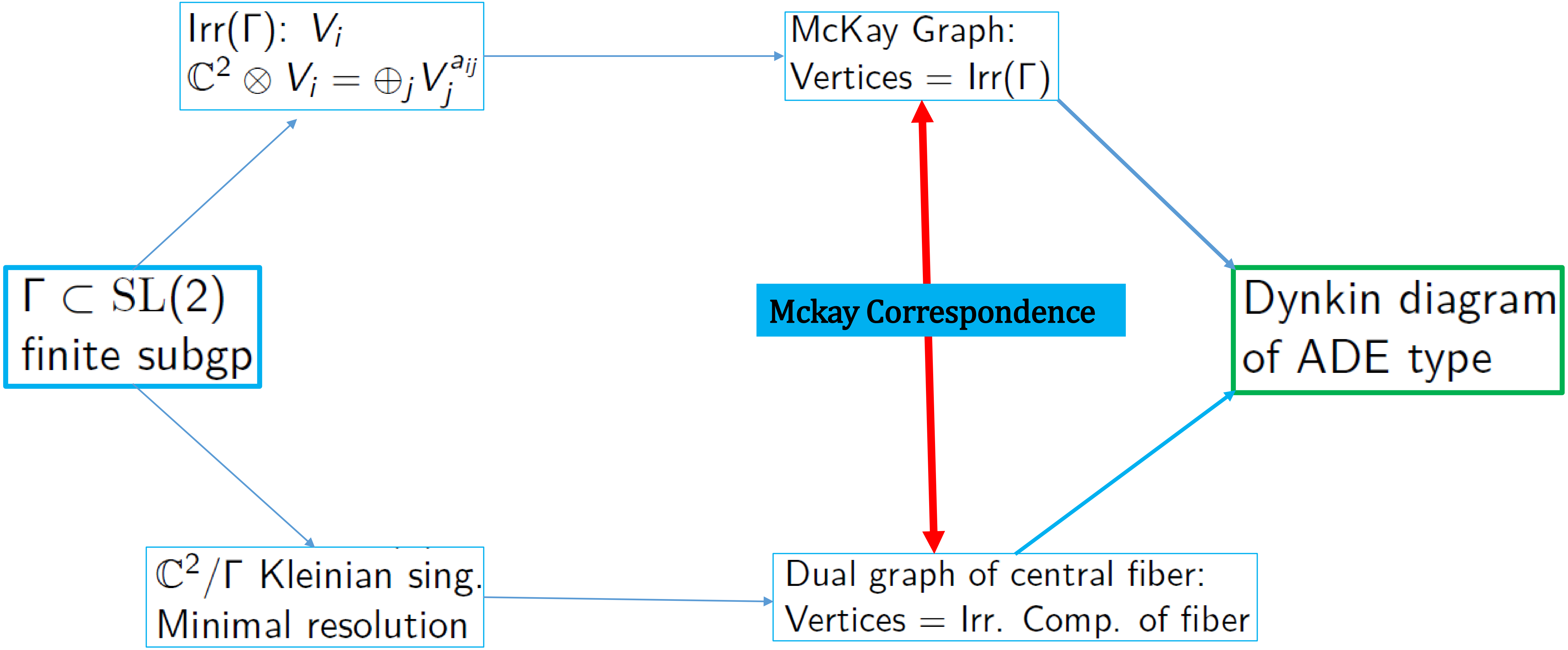
$S = \{x(y^2-x^2)+z^2 = 0\}$

$\pi^{-1}(s)$ consists of four circles

Dynkin diagram again!

cyclic	n	$x^n + yz = 0$
binary dihedral	$4(n-2)$	$x^{n-1} + xy^2 + z^2 = 0$
binary tetrahedral	24	$x^4 + y^3 + z^2 = 0$
binary octahedral	48	$x^3y + y^3 + z^2 = 0$
binary icosahedral	120	$x^5 + y^3 + z^2 = 0$





$\mathrm{Irr}(\Gamma)$: V_i
$\mathbb{C}^2 \otimes V_i = \oplus_j V_j^{a_{ij}}$
McKay Graph:
Vertices $= \mathrm{Irr}(\Gamma)$
$\Gamma \subset \mathrm{SL}(2)$
finite subgp
Mckay Correspondence
Dynkin diagram
of ADE type
$\mathbb{C}^2/\Gamma$ Kleinian sing.
Minimal resolution
Dual graph of central fiber:
Vertices $=$ Irr. Comp. of fiber

McKay Correspondence

$\Gamma \subset \mathrm{SL}(2)$ finite subgroup

Representation

Geometry

$\boxed{\{\text{non-trivial irred. rep. of } \Gamma\}} \Longleftrightarrow \boxed{\{\text{Irred. components of minimal resol. of } \mathbb{C}^2/\Gamma\}}$

How to realize this correspondence geometrically?

Representation and resolution

Let $S = \mathbb{C}^2/\Gamma$ and $\pi : \tilde{S} \to S$ the minimal resolution.

- Let $S_0 = S \setminus \{0\}$, then its fundamental group is Γ.
- A representation R of Γ gives a vector bundle F_0 over S_0.
- Hence we get a vector bundle $\pi^*(F_0)$ on $\tilde{S} \setminus E$.
- This $\pi^*(F_0)$ extends to a vector bundle $V(R)$ on $\tilde{S}$.
- $R = R_i$ is a non-trivial irreducible representation of Γ,
- C_i is the exceptional component corresponding to R_i in the Dynkin diagram

Theorem (Gonzalez-Sprinberg; Verdier '83)

$c_1(V(R_i)) \in H^2(\tilde{S}, \mathbb{Z})$ is dual to the class $[C_i] \in H_2(\tilde{S}, \mathbb{Z})$

Going to higher dimension?

Finite subgroups in $\mathrm{SL}(3,\mathbb{C})$ is done by Miller, Blichfeldt and Dickson (1916-1917), completed by Yau-Yu (1993)

- four infinite series of finite subgroups
- eight sporadic finite subgroups

PROBLEM: The notion of *minimal resolution* does not exist for higher dimension!

OBSERVATION: $\Gamma \subset \mathrm{SL}(3)$, hence Γ preserves the volume form on $\mathbb{C}^3$, hence $\mathbb{C}^3/\Gamma$ has a nonwhere vanishing volume form, i.e. its canonical sheaf is trivial (Calabi-Yau property).

Crepant resolution

Let $\Gamma \subset \mathrm{SL}(n)$ be a finite subgroup.

Definition

A resolution $\pi : Z \to \mathbb{C}^n/\Gamma$ is said **crepant** if K_Z is trivial.

- For $\mathbb{C}^2/\Gamma$, crepant resolution $=$ minimal resolution, hence exists and unique!

Conjecture (Dixon, Harvey, Vafa, Witten 1985)

If $Z \to \mathbb{C}^3/\Gamma$ is a crepant resolution, then

$$\chi_{top}(Z) = \sharp \text{ conjugacy classes of } \Gamma = |\mathrm{Irrd}(\Gamma)|.$$

Existence of crepant resolution

- For $n = 4$, $\mathbb{C}^4/\pm 1$ has terminal singularities, hence it has no crepant resolution!
- In general, the crepant resolution is not unique (if exist).

Theorem (Ito, Markushevich, Reid, Roan 1994-1997)

For any $\Gamma \subset \mathrm{SL}(3)$ finite subgroup, $\mathbb{C}^3/\Gamma$ admits a crepant resolution!

- The proof is a case-by-case construction.
- they verified physicists' Euler number conjecture for these resolutions (only)!

McKay correspondence à Reid

Conjecture (Reid 1992)

Let $\Gamma \subset \mathrm{SL}(n)$ be a finite subgroup and $Z \to \mathbb{C}^n/\Gamma$ a crepant resolution. Then there exists "natural" bijections:

$$\text{Irred. Rep. of } \Gamma \to \text{ a basis of } H^*(Z, \mathbb{Z})$$

$$\text{Conjugacy classes of } \Gamma \to \text{ a basis of } H_*(Z, \mathbb{Z})$$

SLOGAN: representation theory of Γ = homology theory of Z.

Proof of Reid's conjecture

[**Batyrev** 1999] Reid's conjecture on Euler numbers holds, by using non-Archimedean integrals.

By using motivic integration of Kontsevich, we have

Theorem (Denef-Loeser 2002)

If $\pi : Z \to \mathbb{C}^n/\Gamma$ is a crepant resolution, then the following holds in the completion of localised Grothendieck group of $\mathbb{C}$-varieties.

$$[\pi^{-1}(0)] = \sum_{\gamma \in \mathrm{Conj}(\Gamma)} \mathbb{L}^{n-\omega\gamma}$$

By using $\mathbb{C}^*$-action, Z is homotopic to $\pi^{-1}(0)$, hence (co)homology of Z is determined by conjugacy classes of Γ.

Derived McKay correspondence

[**Ito and Nakamura**] introduced the Γ-Hilbert scheme as a preferred partial resolution of $\mathbb{C}^n/\Gamma$.

Let $N = |\Gamma|$, then $\Gamma - \mathrm{Hilb}(\mathbb{C}^n)$ is the main component of $\mathrm{Hilb}^N(\mathbb{C}^n)^\Gamma$.

Theorem (Bridgeland-King-Reid 2001)

Let $Z = \Gamma - \mathrm{Hilb}(\mathbb{C}^3)$, then the natural map $\pi : Z \to \mathbb{C}^3/\Gamma$ is a crepant resolution, and we have

$$D^b(\mathrm{Coh}(Z)) \simeq D^b(\mathrm{Coh}^\Gamma(\mathbb{C}^3)).$$

The method only works for dim. 3. No similar results in higher dimension.

Multiplicative McKay correspondence

For a crepant resolution $Z \to \mathbb{C}^n/\Gamma$, the group structure of $H^*(Z)$ is determined by Γ, but how to determine the ring structure of $H^*(Z)$?

- *Chen-Ruan, 2004* defined orbifold cohomology $H^*_{CR}([\mathbb{C}^n/\Gamma])$.
- $\exists$ quantum cohomology $QH^*_q(Z)$, a deformation of $H^*(Z)$.

Conjecture (Ruan's cohomological crepant resolution conjecture)

For a suitable choice of q_0, we have $QH^*_{q_0}(Z) \simeq H^*_{CR}([\mathbb{C}^n/\Gamma])$

SLOGAN: string topology of $\mathbb{C}^n/\Gamma$ is part of quantum theory of Z.

Symplectic case

Assume $\Gamma \subset \mathrm{Sp}(2n)$ a finite subgroup.

- the quotient $\mathbb{C}^{2n}/\Gamma$ is symplectic on its smooth locus.
- Crepant resolution = symplectic resolution.
- Ruan conjectured there is no quantum correction.

Let $\pi : Z \to \mathbb{C}^{2n}/\Gamma$ be a crepant resolution, then

Kaledin 2002: Reid's conjecture made explicit.

Ginzburg-Kaledin 2004: $H^*(Z) \simeq H^*_{CR}([\mathbb{C}^{2n}/\Gamma])$ as rings.

Bezrukavnikov-Kaledin 2004: $D^b(\mathrm{Coh}(Z)) \simeq D^b(\mathrm{Coh}^{\Gamma}(\mathbb{C}^{2n}))$.

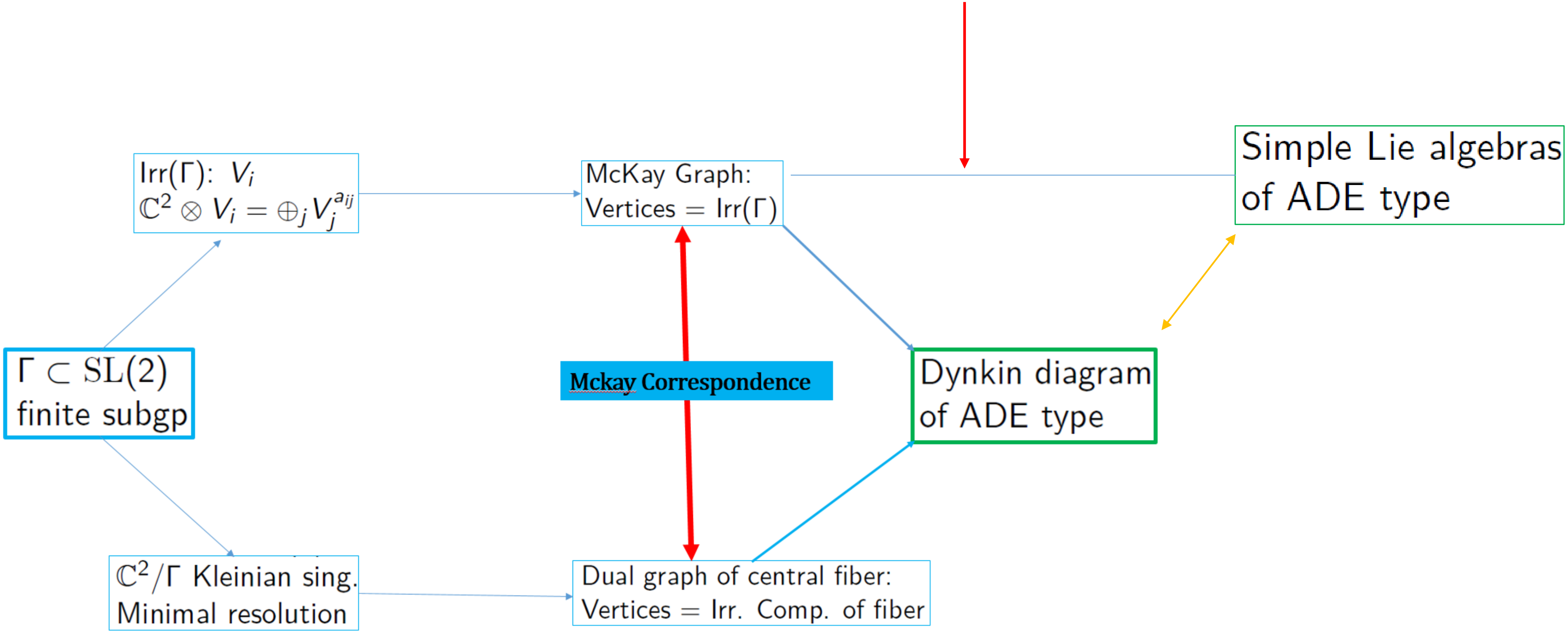
More directions

- Global quotient X/Γ or orbifolds or DM-stacks...
- Motivic McKay correspondence on Chow groups/rings
- Crepant Resolution Conjecture for Donaldson-Thomas Invariants
- The Crepant Resolution Conjecture of Bryan-Graber-Ruan
- $\cdots$


BACK TO
EARTH

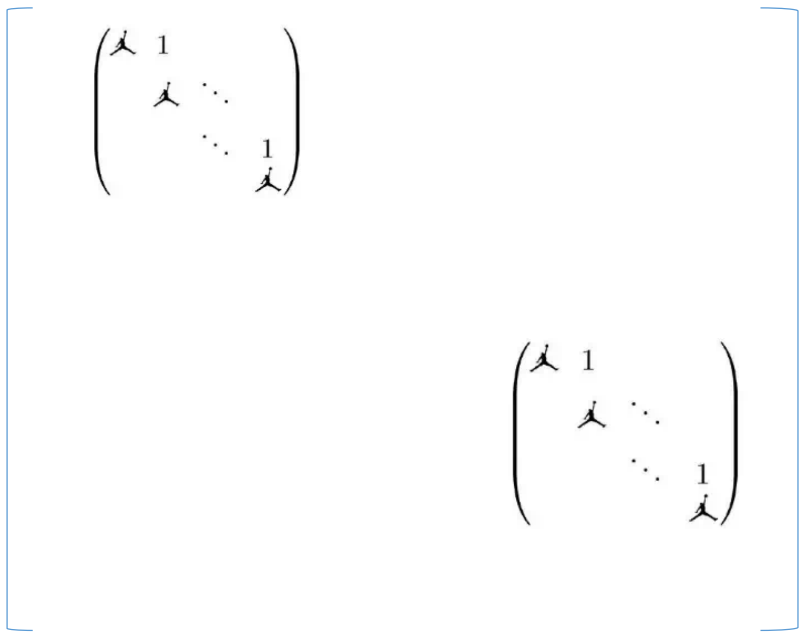
McKay graphes are exactly affine Dynkin diagram of type ADE!

What's the relation of them with Lie algebras?



Nilpotent orbits

For classical types Nilpotent orbits = Conjugacy classes of nilpotent matrices



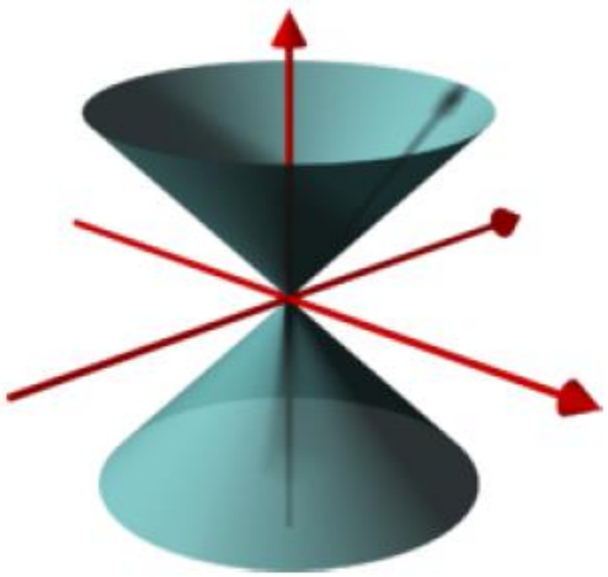
They are classified by sizes of Jordan blocks, i.e. partition of numbers

$sl_2(\mathbb{C})$

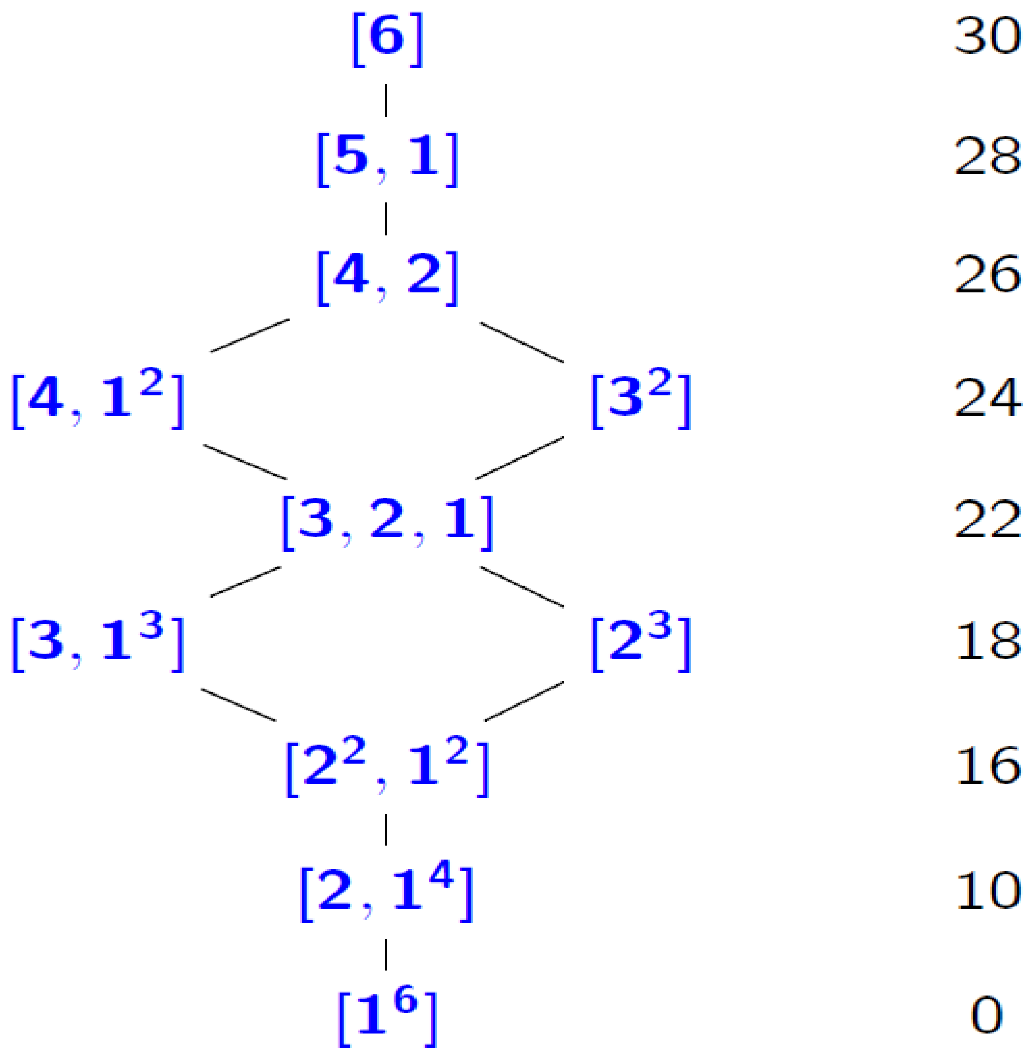
Let $A \in sl_2(\mathbb{C})$ be as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

A is nilpotent iff $A^2 = 0$ iff $a^2 + bc = 0$. The nilpotent cone consists of two orbits: $[2], [1,1]$.



Hasse diagram of A_5



Springer resolutions

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Springer showed that the moment of $T^*(G/B)$ gives a resolution $\pi : T^*(G/B) \to \mathcal{N}$.

Let $\mathfrak{g} = \mathfrak{sl}(n)$, then G/B is the complete flag variety

$$F = \{(V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n) | \dim V_i = i\}.$$

$$\boxed{T^*F \simeq \{(V_\bullet, A) \in F \times \mathfrak{sl}(n) | A(V_i) \subset V_{i-1}\}}$$

$\mathcal{N}$ A

Fibers over subregular orbits

- $\mathcal{N} = \{A \in \mathfrak{sl}(n) | A^n = 0\}$
- $\mathcal{O}_{[n]} = \{A \in \mathcal{N} | \mathrm{rk}(A) = n-1\}$
- $\mathcal{O}_{[n-1,1]} = \{A \in \mathcal{N} | \mathrm{rk}(A) = n-2\}$

- $\mathcal{N}$ is smooth along $\mathcal{O}_{[n]}$ and $\bar{\mathcal{O}}_{[n-1,1]}$ is its singular locus.

For $A \in \mathcal{O}_{[n-1,1]}$, its fiber $\pi^{-1}(A) = \{V_\bullet \in F | AV_i \subset V_{i-1}\}$.

$$AV_i \subset V_{i-1} \text{ hence, } A^i V_i = 0 \text{ and } A^{n-i} V_n \subset V_i.$$

$$\boxed{\mathrm{Im}(A^{n-i}) \subseteq V_i \subseteq \mathrm{Ker}(A^i)}$$

$\dim = i-1$ $\dim = i+1$

It turns out $\pi^{-1}(A)$ consists of $n-1$ copies of $\mathbb{P}^1$, given by

$$L_k := \{\mathrm{Im}(A^{n-1}) \subset \mathrm{Im}(A^{n-2}) \subset \cdots \mathrm{Im}(A^k) \subset V_{n-k} \subset \mathrm{Ker}(A^{n-k}) \subset \cdots \mathrm{Ker}(A^{n-2}) \subset \mathbb{C}^n$$

Observation: $L_k \cap L_j$ is one point if $|k-j| = 1$ and empty otherwise.

Its dual graph is: 

Dynkin diagram again!

Simply-laced cases

Egbert Valentin Brieskorn
(7/07/1936–11/07/2013)

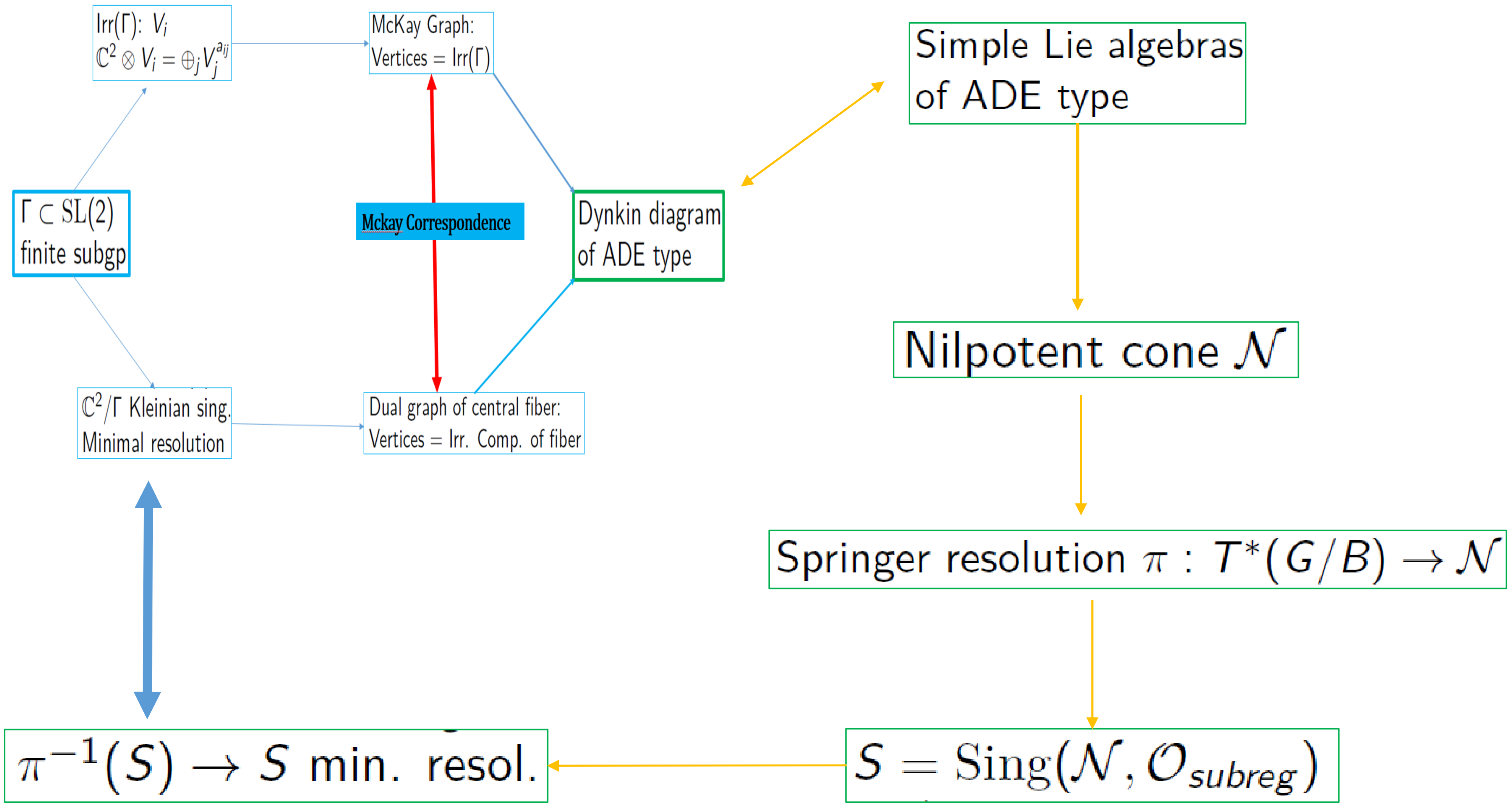
The singular locus of $\mathcal{N}$ is of codimension 2, the closure of $\mathcal{O}_{\mathrm{subreg}}$.

Theorem (Brieskorn, 1970)

Suppose that $\mathfrak{g}$ is simple of *ADE* type. Then

$$\mathrm{Sing}(\mathcal{N}, \mathcal{O}_{\mathrm{subreg}})$$

is a simple surface singularity of the same type.

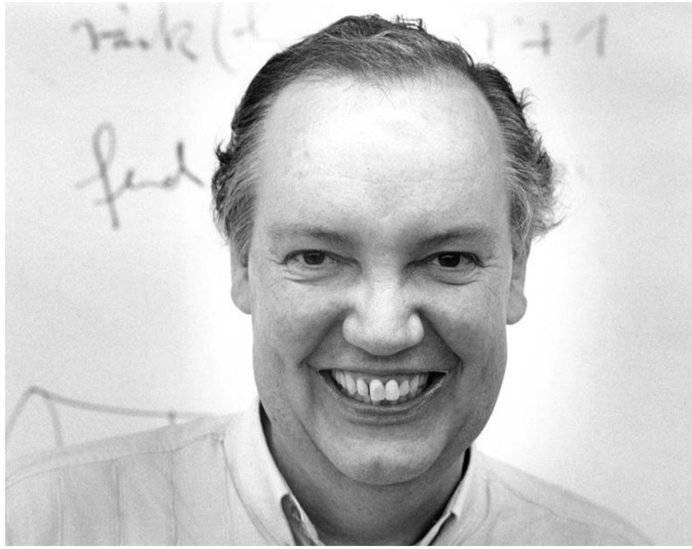


Irr(Γ): V_i
$\mathbb{C}^2 \otimes V_i = \oplus_j V_j^{a_{ij}}$
McKay Graph:
Vertices = Irr(Γ)
Simple Lie algebras of ADE type
$\Gamma \subset \mathrm{SL}(2)$ finite subgp
Mckay Correspondence
Dynkin diagram of ADE type
Nilpotent cone $\mathcal{N}$
$\mathbb{C}^2/\Gamma$ Kleinian sing.
Minimal resolution
Dual graph of central fiber:
Vertices = Irr. Comp. of fiber
Springer resolution $\pi : T^*(G/B) \to \mathcal{N}$
$\pi^{-1}(S) \to S$ min. resol.
$S = \mathrm{Sing}(\mathcal{N}, \mathcal{O}_{subreg})$

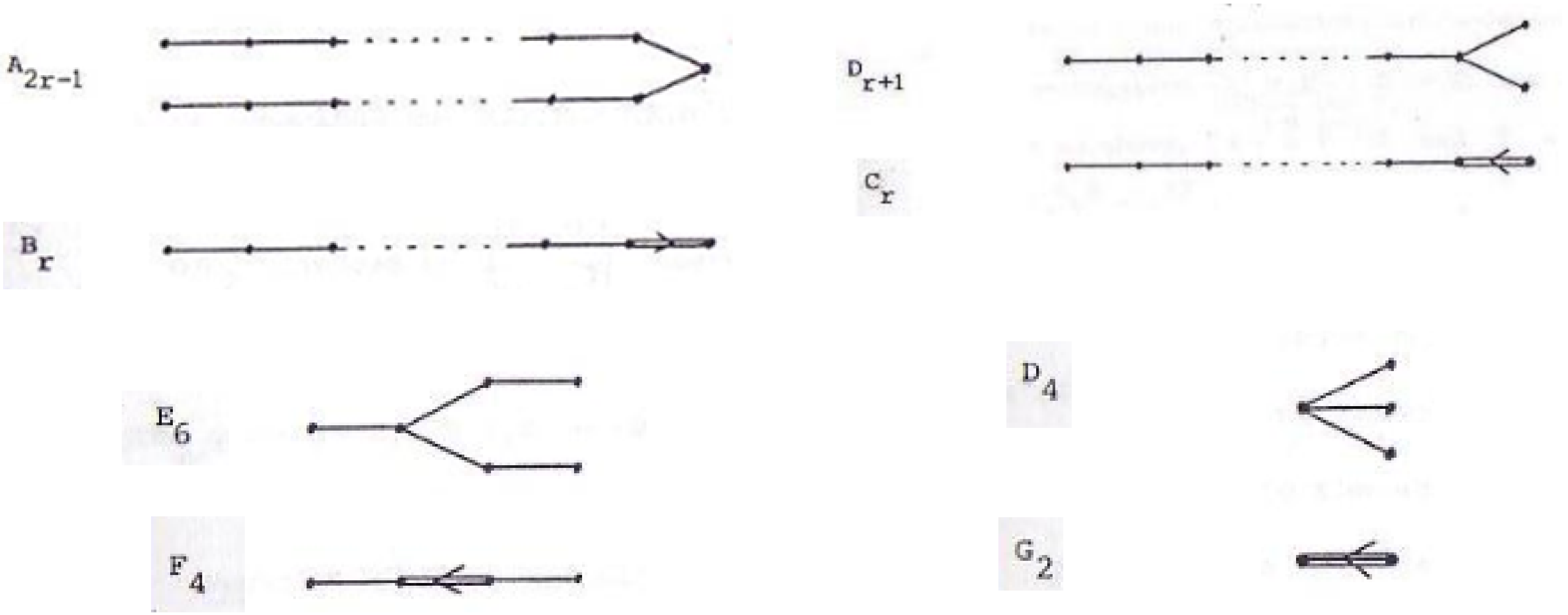
Non simply-laced cases

Slodowy explained what happens for non simply-laced types:

$$\begin{aligned}
B_n = A_{2n-1}^{+} &:= A_{2n-1} \text{ with } \mathfrak{S}_2\text{-action}\\
C_n = D_{n+1}^{+} &:= D_{n+1} \text{ with } \mathfrak{S}_2\text{-action}\\
F_4 = E_6^{+} &:= E_6 \text{ with } \mathfrak{S}_2\text{-action}\\
G_2 = D_4^{++} &:= D_4 \text{ with } \mathfrak{S}_3\text{-action}
\end{aligned}$$



Peter Slodowy
(12/10/1948-19/11/2002)



Singularities of nilp. orbits

$$\begin{tikzcd}
 & \mathcal{N} \arrow[d, dash, "{[\mathit{Brisekorn}]}"', "{[\mathit{Slodowy}]}"] & \\
 & \mathcal{O}_{\text{subreg}} \arrow[d, dash, "?"'] & \\
 & \mathcal{O}_{\mathrm{sub-subreg}} \arrow[dl, dash, "?"] \arrow[dr, dash, "?"'] & \\
\cdots \arrow[d, dash] & & \cdots \arrow[d, dash] \\
\cdots \arrow[dr, dash, "?"] & & \cdots \arrow[dl, dash, "?"'] \\
 & \mathcal{O}_{\mathrm{sup-min}} \arrow[d, dash, "?"'] & \\
 & \mathcal{O}_{\mathrm{min}} \arrow[d, dash] & \\
 & \mathbf{0} &
\end{tikzcd}$$

[**Kraft-Procesi**] Case of all classical types

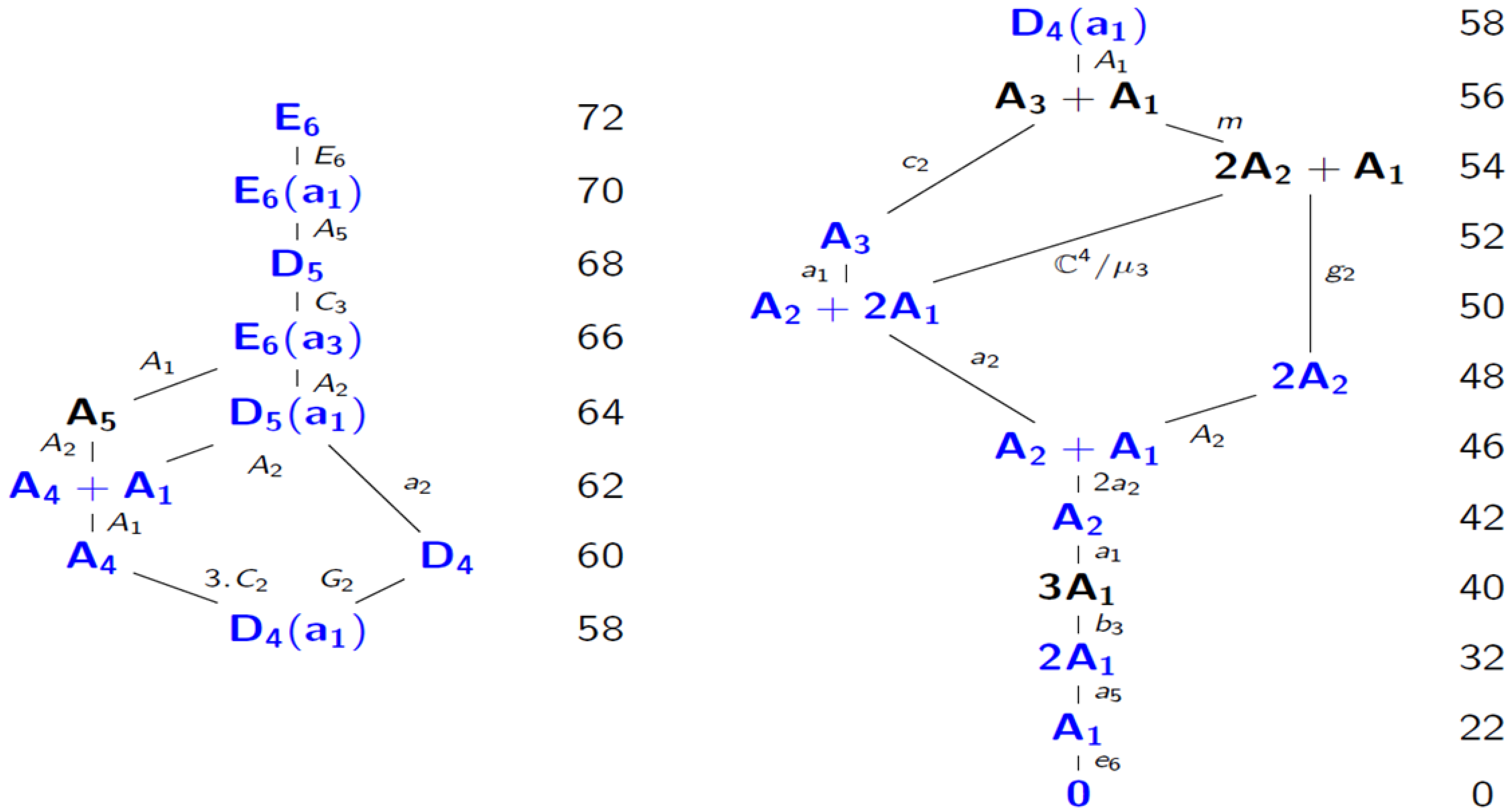
[**Kraft**] Case of $\mathfrak{g}_2$.

[**Fu-Juteau-Levy-Sommers**] All exceptional types.

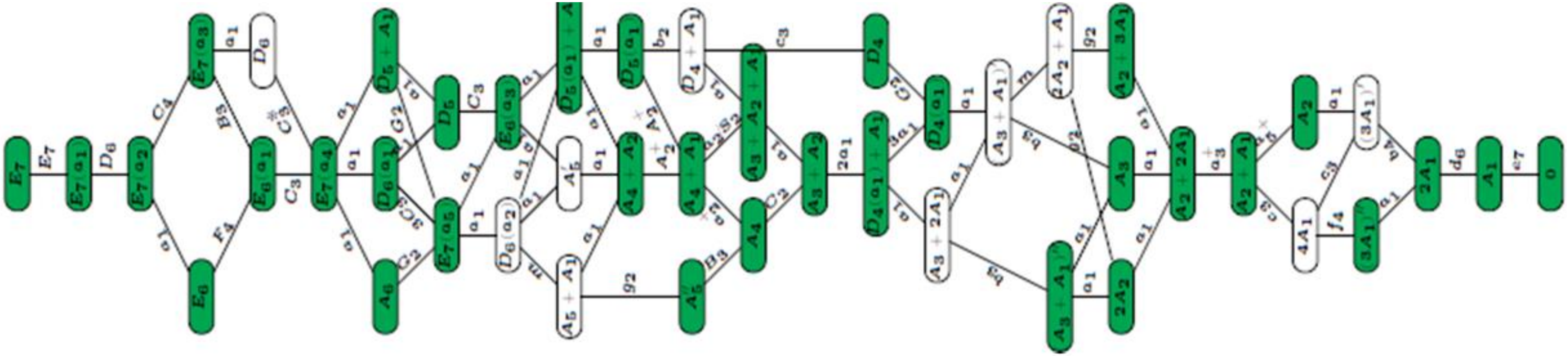
Example A_5 (Kraft-Procesi)

$$
\begin{tikzcd}
 & {[\mathbf{6}]} \arrow[d, dash, "A_5"] & & 30 \\
 & {[\mathbf{5}, \mathbf{1}]} \arrow[d, dash, "A_3"] & & 28 \\
 & {[\mathbf{4}, \mathbf{2}]} \arrow[dl, dash, "A_1"'] \arrow[dr, dash, "A_1"] & & 26 \\
{[\mathbf{4}, \mathbf{1}^2]} \arrow[dr, dash, "A_2"'] & & {[\mathbf{3}^2]} \arrow[dl, dash, "A_2"] & 24 \\
 & {[\mathbf{3}, \mathbf{2}, \mathbf{1}]} \arrow[dl, dash, "a_2"'] \arrow[dr, dash, "a_2"] & & 22 \\
{[\mathbf{3}, \mathbf{1}^3]} \arrow[dr, dash, "a_1"'] & & {[\mathbf{2}^3]} \arrow[dl, dash, "a_1"] & 18 \\
 & {[\mathbf{2}^2, \mathbf{1}^2]} \arrow[d, dash, "a_3"] & & 16 \\
 & {[\mathbf{2}, \mathbf{1}^4]} \arrow[d, dash, "a_5"] & & 10 \\
 & {[\mathbf{1}^6]} & & 0
\end{tikzcd}
$$

Example: E_6 (FJLS)



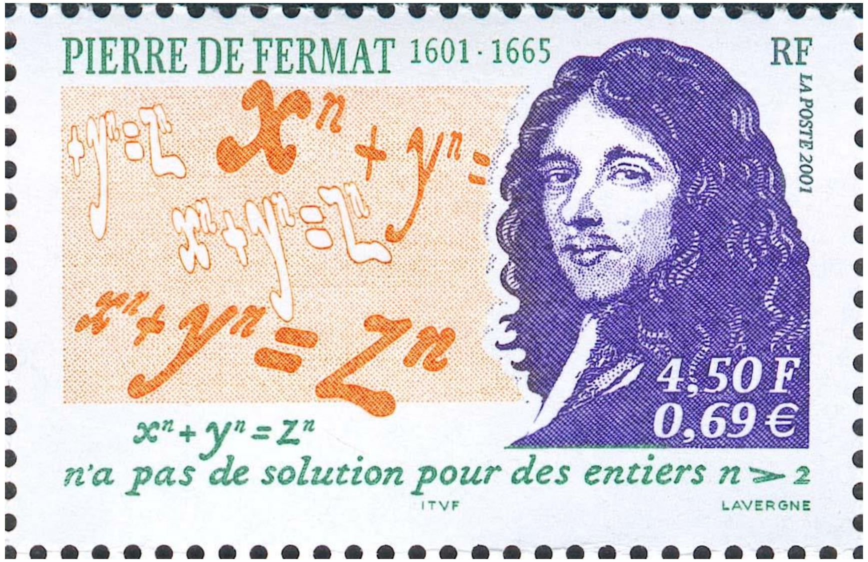
E_7 (FJLS)



E_8?

Pierre de Fermat
(17/08/1601-12/01/1665)

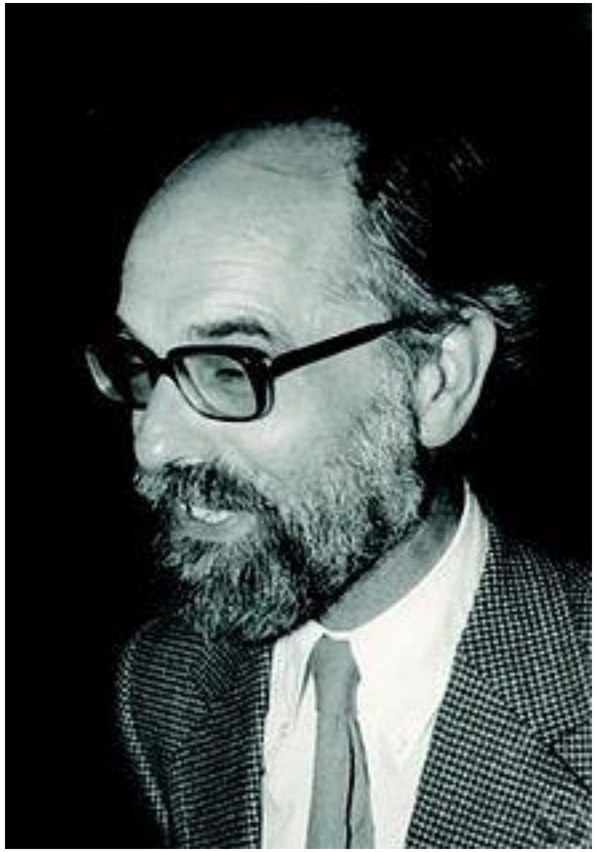
I have discovered a truly marvelous *picture* of this, which this margin is too narrow to contain.



Springer fibers

- Consider the Springer resolution $\pi : T^*(G/B) \to \mathcal{N}$.
- For $e \in \mathcal{N}$, denote by $\mathcal{B}_e = \pi^{-1}(e)$, called *Springer fiber*.

- $\mathcal{B}_e$ is connected by Zariski's main theorem ($\mathcal{N}$ is normal).
- Spaltenstein: $\mathcal{B}_e$ is of pure dim. $= \frac{1}{2}(\dim \mathcal{N} - \dim G \cdot e)$.
- $\mathcal{B}_e$ can have many components with very complicated configuration.



Tonny Albert Springer
(13/02/1926 – 7/12/2011)

Springer correspondence

$G \times \mathbb{C}^*$ acts on $\mathcal{N}$, hence its stabilizer $\tilde{G}_e$ acts on $\mathcal{B}_e$.
Let $A_e = \pi_0(\tilde{G}_e)$, a finite group, which acts on $H^*(\mathcal{B}_e)$.

Decomposition into irred. representations:

$$H^*(\mathcal{B}_e) = \oplus_{\rho \in \mathrm{Irr}(A_e)} V_\rho \otimes H^*(\mathcal{B}_e)_\rho$$

$H^*(\mathcal{B}_e)_\rho$ is the ρ-isotypical component.

Write $H(\mathcal{B}_e) = H_{top}(\mathcal{B}_e)$, the subgroup generated by irreducible components of $\mathcal{B}_e$. A_e acts on it by permuting components via monodromy.

Theorem (Springer 1976)

The group $H(\mathcal{B}_e)$ is naturally a representation of the Weyl group W, and
1) $H(\mathcal{B}_e)_\rho$ (if non-zero) is an irred. rep. of W.
2) Any irred. rep. of W arises from this way.
3) Get an injection

$$\mathrm{Irr}(W) \to \{(\text{nilp. orbits}, \text{Irred. Rep. of component groups})\}$$

Special case: $\mathfrak{sl}(n)$

In this case, $W = \mathcal{S}_n$.

A_e is in general not trivial, but only trivial rep. in $\mathrm{Irr}(W)$ appears. The Springer correspondence reads: $H(\mathcal{B}_e)$ is an irreducible representation of $\mathcal{S}_n$. This gives the correspondence

$$\boxed{\mathrm{Conj}(\mathcal{S}_n) \leftrightarrow \text{Partitions of } n \leftrightarrow \{H(\mathcal{B}_e) | \mathcal{O}_e \text{ nilp. orbit}\} \leftrightarrow \mathrm{Irr}(\mathcal{S}_n)}$$

- $[n] \leftrightarrow$ trivial representation
- $[n-1, 1] \leftrightarrow$ reflection representation
- $[1^n] \leftrightarrow$ sign representation

Steinberg variety

The Steinberg variety is defined by $Z := T^*(G/B) \times_{\mathcal{N}} T^*(G/B)$

$\boxed{\textbf{Fact: } Z = \sqcup_{w \in W} T^*_{Y_w}(G/B \times G/B))}$

$$p_{ij} : T^*(G/B) \times_{\mathcal{N}} T^*(G/B) \times_{\mathcal{N}} T^*(G/B) \to Z$$

Convolution algebra structure on $H^*(Z)$:

$$H^*(Z) \times H^*(Z) \to H^*(Z)$$

$$(\alpha, \beta) \mapsto (p_{13})_*(p_{12}^*\alpha \cdot p_{23}^*\beta)$$

Need Borel-Moore (co)homology

Example: $\mathfrak{sl}(2)$

The Steinberg variety $Z = T^*\mathbb{P}^1 \times_{\mathcal{N}} T^*\mathbb{P}^1$ has two components: $\Delta_{T^*\mathbb{P}^1}, \mathbb{P}^1 \times \mathbb{P}^1$. The group $H(Z)$ is of dimension 2, generated by $[\Delta_{T^*\mathbb{P}^1}], [\mathbb{P}^1 \times \mathbb{P}^1]$

Convolution product:

- $[\Delta_{T^*\mathbb{P}^1}]$ acts as identity.
- $[\mathbb{P}^1 \times \mathbb{P}^1] \cdot [\mathbb{P}^1 \times \mathbb{P}^1] = -2[\mathbb{P}^1 \times \mathbb{P}^1]$

If we set $T = [\mathbb{P}^1 \times \mathbb{P}^1] + [\Delta_{T^*\mathbb{P}^1}]$, then $T \cdot T = [\Delta_{T^*\mathbb{P}^1}]$.

This gives an isomorphism of algebras: $H(Z) \simeq \mathbb{C}[\mathcal{S}_2]$.

Note: Although $H(Z)$ is parametrized by W (so as $\mathbb{C}[W]$), but the algebra isomorphism if not the trivial one!

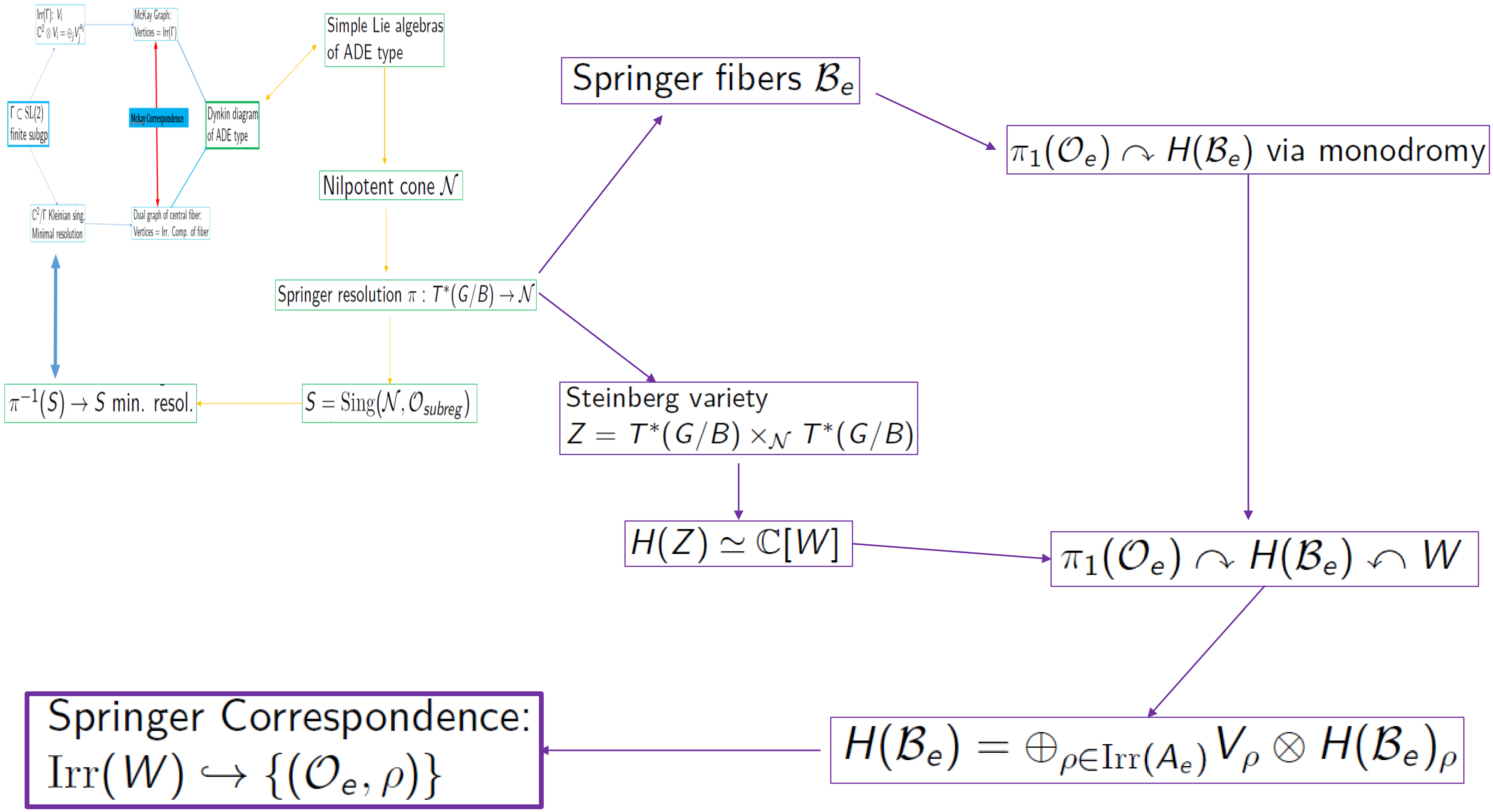
Lagrangian construction of the Weyl group

Theorem (Ginzburg)

There exists a canonical isomorphism of algebras:

$$H(Z) \simeq \mathbb{C}[W]$$

Note that as correspondences, we have $Z \circ \mathcal{B}_e = \mathcal{B}_e = \mathcal{B}_e \circ Z$. This shows that $H^*(\mathcal{B}_e)$ has a $H(Z)$-bi-module structure! This is why W acts on $H^*(\mathcal{B}_e)$!



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finite subgp
Mckay Correspondence
Dynkin diagram
of ADE type
$\mathbb{C}^2/\Gamma$ Kleinian sing.
Minimal resolution
Dual graph of central fiber:
Vertices = Irr. Comp. of fiber
Simple Lie algebras
of ADE type
Nilpotent cone $\mathcal{N}$
Springer resolution $\pi : T^*(G/B) \to \mathcal{N}$
$S = \mathrm{Sing}(\mathcal{N}, \mathcal{O}_{subreg})$
$\pi^{-1}(S) \to S$ min. resol.
Springer fibers $\mathcal{B}_e$
$\pi_1(\mathcal{O}_e) \curvearrowright H(\mathcal{B}_e)$ via monodromy
Steinberg variety
$Z = T^*(G/B) \times_{\mathcal{N}} T^*(G/B)$
$H(Z) \simeq \mathbb{C}[W]$
$\pi_1(\mathcal{O}_e) \curvearrowright H(\mathcal{B}_e) \curvearrowleft W$
$H(\mathcal{B}_e) = \oplus_{\rho \in \mathrm{Irr}(A_e)} V_\rho \otimes H(\mathcal{B}_e)_\rho$
Springer Correspondence:
$\mathrm{Irr}(W) \hookrightarrow \{(\mathcal{O}_e, \rho)\}$

Mosaic on perverse sheaves

- $D^b(X) \subset D^b(\mathrm{Sh}(X))$: full subcategory of constructible complexes
- Perverse sheaf $\mathcal{F} \in D^b(X)$ such that

$$\boxed{\dim \mathrm{supp} H^i(\mathcal{F}) \leq -i, \quad \dim \mathrm{supp} H^i(\mathcal{F}^v) \leq -i}$$

- $\mathrm{Perv}(X) \subset D^b(X)$: full subcategory of perverse sheaves
- Simple objects in $\mathrm{Perv}(X)$ are called *Intersection complexes*, denoted by $\mathrm{IC}(\bar{Y}, \mathcal{L})$.
- $\mathrm{IC}(\bar{Y}, \mathcal{L})$ is an extension of the local system $\mathcal{L}$ on (smooth) Y. Deligne has a construction via push forward and truncations.

Convolution algebra is Yoneda product

Let $\mu : M \to N$ be a proper map with M smooth of dimension d.
Let $\mathcal{C}_M = \mathbb{C}_M[d]$ Let $Z = M \times_N M$, then

Theorem

There exists a (non-necessarily grading preserving) isomorphism of algebras:

$$H_*(Z) \simeq Ext^*_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M).$$

The LHS is the convolution algebra, and RHS is the Yoneda products.

Problem: Compute the push-forward $\mu_*\mathcal{C}_M$.

Decomposition theorem

Theorem (Deligne 1968,1972)

Let $f : X \to Y$ be a smooth proper morphism, then

- $Rf_*\mathbb{Q}_X \simeq \oplus_{q\geq 0} R^q f_*\mathbb{Q}_X[-q]$
- *the sheaves $R^q f_*\mathbb{Q}_X$ are semi-simple local systems.*

Theorem (Beilinson-Bernstein-Deligne, Gabber)

Let $\mu : M \to N$ be a projective morphism and $X \subset M$ a smooth locally closed subvariety. Then in $D^b(N)$, we have

$$\mu_*\mathrm{IC}(\bar{X}, \mathbb{C}_X) = \oplus_{(i,Y,\chi)} L_{Y,\chi}(i) \otimes \mathrm{IC}(\bar{Y}, \chi)[i].$$

- Y*: locally closed subvariety of* N
- χ*: an irred. local system on* Y
- $L_{Y,\chi}(i)$*: finite dim. vector spaces*

Semismall morphisms

A morphism $\mu : X \to Y$ is called *semismall* if there exists a stratification $Y = \sqcup Y_\alpha$ such that

(i) $\mu^{-1}(Y_\alpha) \to Y_\alpha$ is locally trivial topological fibration

(ii) $2\mathrm{codim}\mu^{-1}(Y_\alpha) \geq \mathrm{codim}\, Y_\alpha$.

Relevant strata: Y_α such that $2\mathrm{codim}\mu^{-1}(Y_\alpha) = \mathrm{codim} Y_\alpha$.

Theorem (Borho-MacPherson 1981)

Assume $\mu : X \to Y$ is semismall with X smooth irreducible, then

$$\mu_*(\mathcal{C}_X) = \oplus_{\phi=(\alpha,\chi_\phi)} L_\phi \otimes \mathrm{IC}(\bar{Y}_\alpha, \chi_\phi),$$

where α is such that Y_α is relevant, L_ϕ is a vector space and χ_ϕ is an irreducible local system.

$\boxed{\textbf{Question: } \text{What is } L_\alpha\text{?}}$

- View the irreducible local system χ_ϕ as an irred. rep. χ of $\pi_1(Y_\alpha)$.
- Let $F_\alpha \subset X$ be the fiber of μ over a point of Y_α.
- $\pi_1(Y_\alpha)$ acts on $H(F_\alpha \subset X)$ by monodromy, hence gives

$$H(F_\alpha) = \oplus_{\chi \in \mathrm{Irr}(\pi_1(Y_\alpha))} H(F_\alpha)_{\chi_\phi}$$

$$\boxed{L_\phi = H(F_\alpha)_\chi}$$

Corollary

Let $\mu : X \to Y$ be a semismall proj. morphism. Let $Z = X \times_Y X$, then

$$H(Z) \simeq \oplus_{\phi=(\alpha,\chi_\phi)} \mathrm{End}(H(F_\alpha)_{\chi_\phi}).$$

Symplectic resolutions are semismall

A *symplectic resolution* is a proj. resolution $\mu : X \to Y$ such that X is symplectic.

- They are crepant resolutions.
- Springer resolution is a symplectic resolution.

Theorem (Kaledin 2006)

Symplectic resolutions are semismall.

Springer correspondence again

- The Springer resolution $\pi : T^*(G/B) \to \mathcal{N}$ is semismall.
- The Springer fibers are denoted by $\mathcal{B}_x$.
- The Steinberg variety is $Z := T^*(G/B) \times_{\mathcal{N}} T^*(G/B)$.
- As algebras, we have $H(Z) \simeq \mathbb{C}[W]$.

By previous results, we have

$$\mathbb{C}[W] \simeq \oplus_{(x,\rho)} \mathrm{End}(H(\mathcal{B}_x)_\rho),$$

where $x \in \mathcal{N}$ runs over nilpotent orbits, and ρ is an irred. representation of $\pi_1(\mathcal{O}_x)$.

Further developments

- Consider the union of symplectic resolutions

$$\sqcup_{\underline{d}|d_1+\cdots+d_n=N} T^*\mathcal{F}_{\underline{d}} \to X := \{A \in \mathfrak{sl}(N) | A^n = 0\}.$$

  This gives geometrical construction of the universal enveloping algebra of $\mathfrak{sl}(n)$. The fiber homology $H(\mathcal{F}_x)$ gives all simple $\mathfrak{sl}(n)$-modules.

- **Lusztig 1988**: Equivariant cohomology: on $H^*_{\tilde{G}_e}(\mathcal{B}_e)$, there exists an action of graded affine Hecke algebra.

- **Kazhdan-Lusztig 1987**: On the $\tilde{G}_e$-equivariant K-group of $\mathcal{B}_e$, there exists an action of the affine Hecke algebra.

McKay vs. Springer

McKay

$$\begin{tikzcd} Z \arrow[d, "\pi"', "\text{crepant}"'] & \supset Z_0 \arrow[d] & \text{central fiber} \\ \mathbb{C}^n/\Gamma & \ni 0 & \end{tikzcd}$$

(co)Homology of Z (or Z_0) is determined by Γ

Springer

$$\begin{tikzcd} T^*(G/B) \arrow[d, "\mu"', "\text{symplectic}"'] & \supset \mathcal{B}_x \arrow[d] & \text{Springer fibers} \\ \mathcal{N} & \ni x & \end{tikzcd}$$

irred. rep. of Weyl gp W are given by (top) homologies of $\mathcal{B}_x$.

McKay meets Springer?

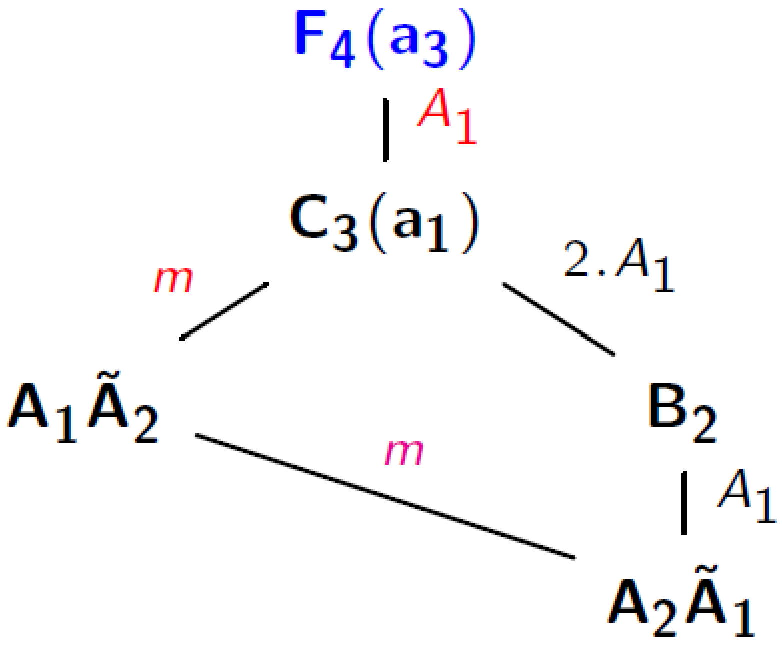
- Let $\Gamma \subset SL(2)$ be a finite subgroup and $S := \mathbb{C}^2/\Gamma$
- Let $\tilde{S} \to S$ be the minimal resolution.
- The Hilbert-Chow resolution gives a crepant (and symplectic) resolution:

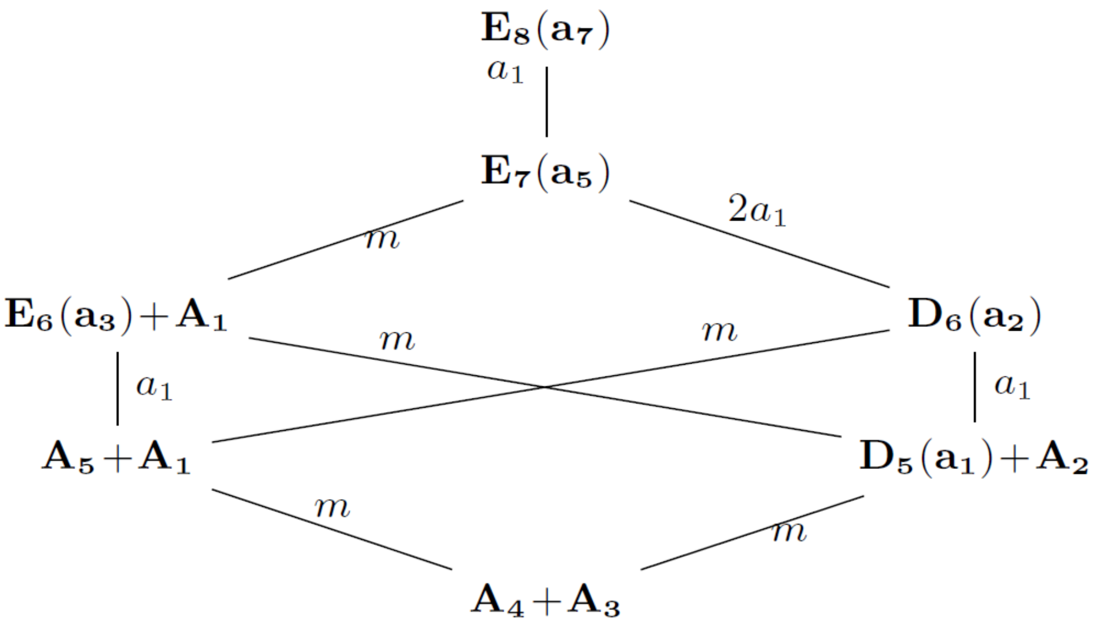
$$\mathrm{Hilb}^{[n]}(\tilde{S}) \to \mathrm{Sym}^n(\tilde{S}) \to \mathrm{Sym}^n(S) = \mathbb{C}^{2n}/(\Gamma \wr \mathcal{S}_n)$$

[*Fu 2007*] $\mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$ is isomorphic to the transverse slice from $\mathcal{O}_{[2,2,2]}$ to $\mathcal{O}_{[4,2]}$ in $\mathfrak{sp}(6)$!

Then the crepant resolution of $\mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$ coincides with the Springer resolution of $\mathcal{O}_{[4,2]}$ restricting to this slice!

More examples (FJLS)





Slice isomorphic to $(\mathbb{C}^3 \oplus (\mathbb{C}^3)^*)/\mathcal{S}_4$

Slice isomorphic to $(\mathbb{C}^4 \oplus (\mathbb{C}^4)^*)/\mathcal{S}_5$


Thanks!