量子可积系统新进展

杨文力

西北大学

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提　纲

I. 可积模型简介

II. 非对角Bethe Ansatz方法

III.应用举例：拓扑自旋环

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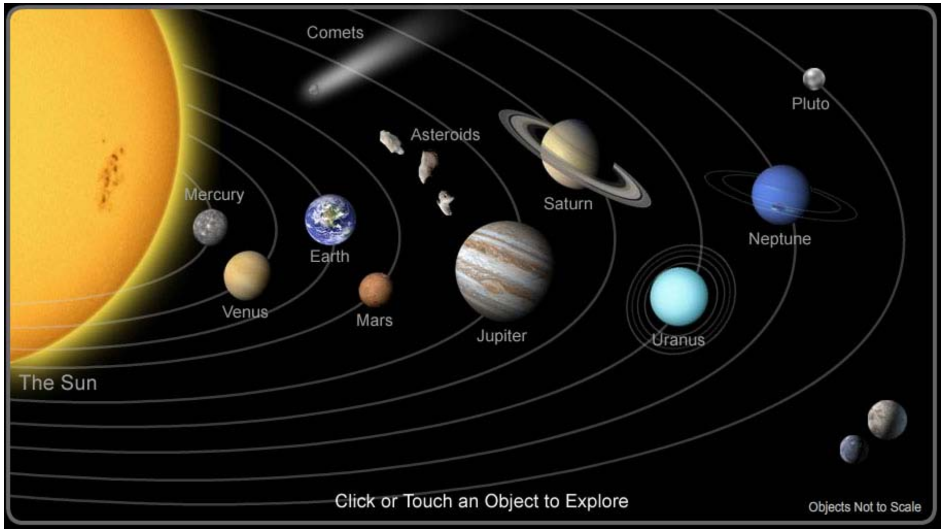
V. 总结和展望

I. 什么是可积?

牛顿二体问题



经典系统:刘维定理

守恒量的数目=自由度的数目

变量分离

$$k_i + k_j = k_i' + k_j',$$

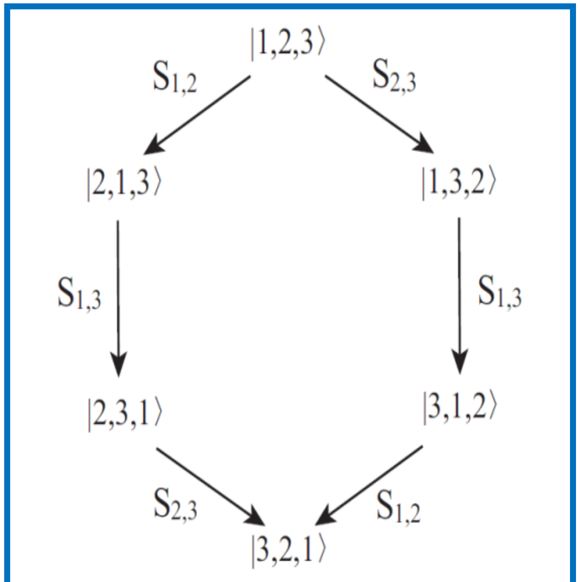
$$k_i^2 + k_j^2 = k_i'^2 + k_j'^2.$$

$$C_n = \sum_{j=1}^{N} k_j^n, \quad n = 1, \cdots, N,$$

I. 什么是可积?

量子可积模型是一类特殊的非线性量子多体系统，它可以被精确求解。

$$S_{1,2\cdots N} = S_{1,N}(k_1,k_N)S_{1,N-1}(k_1,k_{N-1})\cdots S_{1,2}(k_1,k_2)$$



因式化

Yang-Baxter 方程

$$S_{1,2}(k_1,k_2)S_{1,3}(k_1,k_3)S_{2,3}(k_2,k_3) = S_{2,3}(k_2,k_3)S_{1,3}(k_1,k_3)S_{1,2}(k_1,k_2)$$

量子可积模型的重要性

为某些重要的物理问题或概念提供基准!

- 氢原子模型 → 原子光谱
- 二维伊辛模型 → 热力学相变理论的正确性
- 自旋链模型 → Spinon元激发(分数荷)

量子可积模型的重要性

- 重要物理问题的严格基准
- 数学物理前沿领域重要分支

最近的应用：

凝聚态物理、统计物理、Yang-Mills场论、超弦理论AdS/CFT…

边界问题（拓扑量子态，开弦、D-膜）是当前重要问题，而在这类问题中U(1)对称破缺（粒子数不守恒）的非平庸边界尤其重要

U(1)对称破缺模型

U(1)对称破缺普遍存在于各类物理系统中：

- 凝聚态物理：超导、超流
- 非平衡统计物理：随机过程
- 高能物理中的场论及弦理论

精确求解

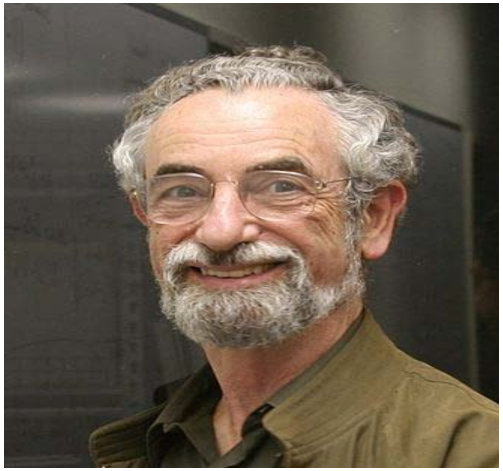
能谱问题

(非微扰)

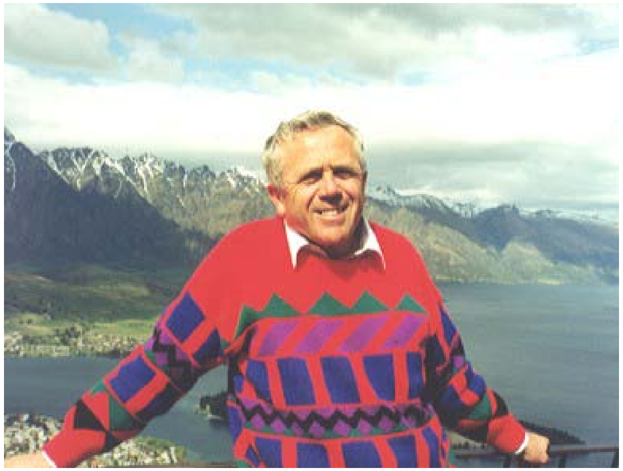
由于这类系统缺乏基准，迫切需要典型U(1)对称破缺可积模型的精确解

Bethe

Onsager

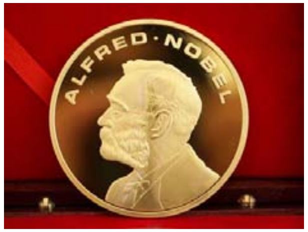


Lieb



Baxter

求解可积模型的方法



Bethe ansatz (1931)

1967年诺贝尔奖

杨振宁，一维多体(1967)

Baxter，统计模型 (1971)

2006年昂萨格奖

Yang-Baxter 方程

Faddeev学派：量子逆散射方法(1979)

在处理U(1)对称(粒子数守恒)可积模型中取得了巨大成功，但无法处理对称性破缺情况

可积系统的挑战性问题

Yang-Baxter 方程

Baxter T-Q关系

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}$$

没有参考态！

70年代末，法杰耶夫学派提出了量子逆散射方法。严格证明了具有U(1)对称的可积模型确实满足T-Q关系

40年来人们从未怀疑过T-Q关系的正确性，对于U(1)对称破缺可积系统，无论直接求解T-Q关系还是利用量子逆散射方法都未能成功

建立了求解可积模型普适的理论方法：非对角Bethe ansatz 方法

抛开本征态，直接从代数关系求出转移矩阵本征值

N阶多项式，有N+1个未知系数

$$\Lambda^{(p)}(\theta_j)\Lambda^{(p)}(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta),\quad j=1,\ldots,N.$$

$$\Lambda(u)=2f^N(u)+\cdots$$

$$\Lambda(u)=a(u)\frac{Q_1(u-\eta)}{Q_2(u)}+d(u)\frac{Q_2(u+\eta)}{Q_1(u)}+\underline{c(u)\frac{a(u)d(u)}{Q_1(u)Q_2(u)}}$$

CYSW
非齐次项

可积模型本征值问题的统一理论框架

同行专家对该方法的评价

arXiv:1307.3960v1 [math-ph] 15 Jul 2013

Complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms

S. Faldella[1], N. Kitanine[2], G. Niccoli[3]

After this paper was completed we became aware of the recent and interesting results reported in [8]. The authors construct the T-Q functional equations for the spin chains with non-diagonal boundaries[4] and thus they obtain the transfer matrix eigenvalues. An important achievement of [8] is that the equation of type (5.4) are associated to a system of Bethe equations leading to a more traditional analysis of the eigenvalue problem. It would be interesting to establish a connection

"建立了非对角边界系统的T-Q关系，是重要成就，它建立了与传统方法的桥梁"

[8] J. Cao, W. Yang, K. Shi, and Y. Wang. Off-diagonal Bethe ansatz solutions of the anisotropic spin-1/2 chains with arbitrary boundary fields. arXiv preprint arXiv:1307.2023

[9] J. Cao, W. Yang, K. Shi, and Y. Wang. Off-diagonal Bethe ansatz and exact solution a topological spin ring. arXiv preprint arXiv:1305.7328, 2013.

[10] J. Cao, W. Yang, K. Shi, and Y. Wang. Off-diagonal bethe ansatz solution of the XXX spin-chain with arbitrary boundary conditions. arXiv preprint arXiv:1306.1742, 2013.

[11] J. Cao, W. Yang, K. Shi, and Y. Wang. Spin-1/2 XYZ model revisit: general solutions via off-diagonal Bethe ansatz. arXiv preprint arXiv:1307.0280, 2013.

同行专家对该方法的评价

日　期: 2013-12-4 20:10:03

Ms. Ref. No.: NPB-D-13-00506
Title: Exact solution of the one-dimensional Hubbard model with arbitrary boundary magnetic fields
Nuclear Physics B

Dear Prof. Yang,

The Editorial Office has received the decision on your paper. For your guidance, the reviewer's comments are appended below.

arbitrary boundary magnetic fields.
As the U(1) symmetry is reserved for the charge sector, they use the coordinate Bethe ansatz to nest the problem into the one of the eigenproblem
of the inhomogeneous XXX spin with arbitrary boundary fields.
The latter has been solved by the authors in a previous publication [23] with the new off-diagonal Bethe ansatz method.

The paper is well written and the proposed result is important for condensed matter and hight energy physics. It opens the way to study new boundary effects.

开创了研究边界效应的新方法

利用非对角Bethe ansatz 方法解决的问题

1. XXZ Spin Torus (拓扑周期）， PRL 111， 137201 （2013）
2. 非平行场中自旋链模型， Nucl. Phys. B 875, 152 (2013)
   Nucl. Phys. B 877, 152 (2013)

（挑战该领域20多年的著名问题）

3. 周期XYZ模型， Nucl. Phys. B 886, 185 (2013)

（Baxter 40年前未能解决的问题）

4. 非对角边界 Hubbard， Nucl. Phys. B 879, 98 (2013)
5. 非对角边界 t-J模型， JSTAT 04, P04031 (2014)
6. 非对角边界 SU(n)自旋链， JHEP 04 , 143 (2014)
7. 热力学性质， Nucl. Phys. B 884, 17 (2014)
8. IK模型， JHEP 06, 128 (2014)

……

一：拓扑边界问题的精确解

Hamiltonian

Phys. Rev. Lett. 111 (2013), 137201

$$H = -\sum_{j=1}^{N}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cosh\eta\,\sigma_j^z\sigma_{j+1}^z\right]$$

$$\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}$$

U(1)对称

$$\sigma_{N+1}^{\alpha} = \sigma_1^{x}\sigma_1^{\alpha}\sigma_1^{x}$$

U(1)对称破缺

Quantum Spin torus: T-Q

$$\Lambda(u) = a(u)e^{u}\frac{Q(u-\eta)}{Q(u)} - e^{-u-\eta}d(u)\frac{Q(u+\eta)}{Q(u)} - c(u)\frac{a(u)d(u)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{N} \sinh(u-\lambda_j)$$

$$c(u) = \sinh^{N}\eta\left[e^{u-N\eta+\Sigma_{j=1}^{N}(\theta_j-\lambda_j)} - e^{-u-\eta-\Sigma_{j=1}^{N}(\theta_j-\lambda_j)}\right].$$

Table 4.3 Numerical solutions of the BAEs (4.4.18) for $N=3$, $\eta=\ln 2$. E_n is the n-th eigenenergy and n indicates the number of the energy levels. The eigenvalues calculated from (4.4.19) are exactly the same to those given in Table 4.1.

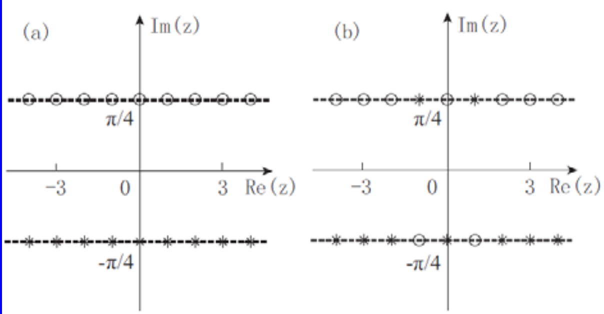
λ_1	λ_2	λ_3	E_n	n
-0.97964	-0.34657	0.28649	-3.02200	1
$-1.11092+0.50000i\pi$	$-0.34657+0.50000i\pi$	$0.41778+0.50000i\pi$	-3.02200	1
$-0.72948+0.43156i\pi$	$-0.34657+0.21409i\pi$	$0.03633+0.43156i\pi$	-1.25000	2
$-0.72948-0.43156i\pi$	$-0.34657-0.21409i\pi$	$0.03633-0.43156i\pi$	-1.25000	2
$-0.72959-0.09417i\pi$	$-0.34657-0.45213i\pi$	$0.03644-0.09417i\pi$	-1.25000	2
$-0.72959+0.09417i\pi$	$-0.34657+0.45213i\pi$	$0.03644+0.09417i\pi$	-1.25000	2
$-0.34657-0.50000i\pi$	$-0.34657-0.10517i\pi$	$-0.34657+0.10517i\pi$	5.52200	3
$-0.71841-0.50000i\pi$	$-0.34657-0.00000i\pi$	$0.02526+0.50000i\pi$	5.52200	3

The quantum spin torus:

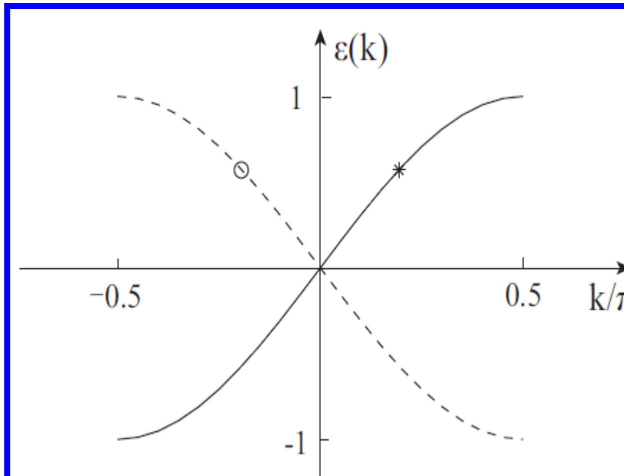
$$\eta = \frac{i\pi}{2}$$

$$H = -2\sum_{j=1}^{N-1}\left\{a_j^{\dagger}a_{j+1} + a_{j+1}^{\dagger}a_j\right\} - 2U^z\left[a_1^{\dagger}a_N^{\dagger} + a_N a_1\right]$$

PRL 111, 137201 (2013)



拓扑元激发



Quantum Spin torus: Bethe states

A convenient basis

$$\langle \theta_{p_1}, \cdots, \theta_{p_n}| = \langle 0| \prod_{j=1}^{n} C(\theta_{p_j}),$$

$$|\theta_{q_1}, \cdots, \theta_{q_n}\rangle = \prod_{j=1}^{n} B(\theta_{q_j})|0\rangle,$$

$$D(u)|\theta_{p_1}, \cdots, \theta_{p_n}\rangle = d(u) \prod_{j=1}^{n} \frac{\sinh(u-\theta_{p_j}+\eta)}{\sinh(u-\theta_{p_j})} |\theta_{p_1}, \cdots, \theta_{p_n}\rangle,$$

$$\langle \theta_{p_1}, \cdots, \theta_{p_n}|D(u) = d(u) \prod_{j=1}^{n} \frac{\sinh(u-\theta_{p_j}+\eta)}{\sinh(u-\theta_{p_j})} \langle \theta_{p_1}, \cdots, \theta_{p_n}|.$$

$$q_j, p_j \in (1, \cdots, N),\ p_1 < p_2 < \cdots < p_n \text{ and } q_1 < q_2 < \cdots < q_n,$$

$$\langle \theta_{p_1}, \cdots, \theta_{p_n}|\theta_{q_1}, \cdots, \theta_{q_m}\rangle = f_n(\theta_{p_1}, \cdots, \theta_{p_n})\delta_{m,n} \prod_{j=1}^{n} \delta_{p_j,q_j},$$

$$f_n(\theta_{p_1}, \cdots, \theta_{p_n}) = \prod_{j=1}^{n} a(\theta_{p_j}) d_{p_j}(\theta_{p_j}) \prod_{k \neq l}^{n} \frac{\sinh(\theta_{p_k}-\theta_{p_l}+\eta)}{\sinh(\theta_{p_k}-\theta_{p_l})}$$

Orthogonal and complete basis

Niccoli 12
CYSW13

Quantum Spin torus : Bethe states

Expansion of the eigenvector

$$\langle \Psi | = \sum_{n=0}^{N} \sum_{p} \chi_n(\theta_{p_1}, \cdots, \theta_{p_n}) \langle \theta_{p_1}, \cdots, \theta_{p_n} |$$

$$\chi_n(\theta_{p_1}, \cdots, \theta_{p_n}) = \frac{\prod_{j=1}^{n} \Lambda(\theta_{p_j})}{f_n(\theta_{p_1}, \cdots, \theta_{p_n})}$$

$$| \Psi \rangle = \sum_{n=0}^{N} \sum_{p} \chi_n(\theta_{p_1}, \cdots, \theta_{p_n}) | \theta_{p_1}, \cdots, \theta_{p_n} \rangle$$

SoV state by Niccoli

Reference state

$$|\lambda_1, \cdots, \lambda_N\rangle = \prod_{j=1}^{N} \frac{D(\lambda_j)}{d(\lambda_j)} |\omega\rangle$$

$$|\omega\rangle = \sum_{n=0}^{N} \sum_{p} f_n^{-1}(\theta_{p_1}, \cdots, \theta_{p_n}) e^{\sum_{l=1}^{n} \theta_{p_l}} \prod_{l=1}^{n} a(\theta_{p_l}) |\theta_{p_1}, \cdots, \theta_{p_n}\rangle$$

$$\langle \theta_{p_1}, \cdots, \theta_{p_n} | \lambda_1, \cdots, \lambda_N \rangle = \langle \theta_{p_1}, \cdots, \theta_{p_n} | \Psi \rangle = F_n(\theta_{p_1}, \cdots, \theta_{p_n})$$

Why?

影响和评价

本文对我们的理论结果进行了数值的直接验证

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **46** (2013) 442002 (7pp) doi:10.1088/1751-8113/46/44/442002

FAST TRACK COMMUNICATION

An inhomogeneous T-Q equation for the open XXX chain with general boundary terms: completeness and arbitrary spin

Rafael I Nepomechie

Physics Department, PO Box 248046, University of Miami, Coral Gables, FL 33124, USA

Abstract

An inhomogeneous T-Q equation has recently been proposed by Cao, Yang, Shi and Wang for the open spin-1/2 XXX chain with general (nondiagonal) boundary terms. We argue that a simplified version of this equation describes

摘要中指出非齐次T-Q关系最近由我们提出

1. Introduction

The open spin-1/2 XXZ quantum spin chain with general (nondiagonal) boundary terms has long been known to be integrable [1–3]. Nevertheless, the Bethe ansatz solution of this model has remained elusive[1]. One of the difficulties is that a reference state (simple eigenstate of the transfer matrix) is not available.

An important advance was recently made by Cao, Yang, Shi and Wang (CYSW) [12, 13] (see also [14, 15]), who proposed to use instead an *inhomogeneous* T-Q equation. Although

可积性很早已被证明，但精确解一直以来未能找到，最近一个重要进展…

2. The CYSW solution

Following [12], we consider an open spin-1/2 XXX chain of length N, with the Hamiltonian

以我们名字命名该精确解，并在整个第二节中介绍我们的工作

[12] Cao J, Yang W, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz solution of the XXX spin-chain with arbitrary boundary conditions *Nucl. Phys.* B **875** 152 (arXiv:1306.1742 [math-ph])

[13] Cao J, Yang W-L, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz solutions of the anisotropic spin-1/2 chains with arbitrary boundary fields, arXiv:1307.2023 [cond-mat.stat-mech]

[14] Cao J, Yang W-L, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz and exact solution a topological spin ring *Phys. Rev. Lett.* **111** 137201 (arXiv:1305.7328 [cond-mat.stat-mech])

[15] Cao J, Yang W-L, Shi K-J and Wang Y 2013 Spin-1/2 XYZ model revisit: general solutions via off-diagonal Bethe ansatz, arXiv:1307.0280 [cond-mat.stat-mech]

二：非平行边界场中自旋链的精确解

$$H=\sum_{n=1}^{N-1}(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\Delta\sigma_n^z\sigma_{n+1}^z)+\vec{h}_1\cdot\vec{\sigma}_1+\vec{h}_N\cdot\vec{\sigma}_N$$

重要性： **凝聚态和统计物理、规范场理论、弦理论等领域**

困　难：

- ✓ **非平行场破坏了U(1)对称性；**
- ✓ **精确解已困扰该领域二十多年；**

我们的工作： **精确求解了这一问题**

Nucl. Phys. B 877 (2013), 152; Nucl. Phys. B 875 (2013), 152.

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta)=\frac{\delta(\theta_j)\,\sinh\eta\,\sinh\eta}{\sinh(\eta-2\theta_j)\,\sinh(\eta+2\theta_j)},\quad j=1,\ldots,N,$$

$$\begin{aligned}\delta(u) &= -2^4\frac{\sinh(2u-2\eta)\sinh(2u+2\eta)}{\sinh^2\eta}\sinh(u+\alpha_-)\sinh(u-\alpha_-)\cosh(u+\beta_-)\cosh(u-\beta_-)\\ &\quad\times\sinh(u+\alpha_+)\sinh(u-\alpha_+)\cosh(u+\beta_+)\cosh(u-\beta_+)\\ &\quad\times\prod_{l=1}^{N}\frac{\sinh(u+\theta_l+\eta)\sinh(u-\theta_l+\eta)\sinh(u+\theta_l-\eta)\sinh(u-\theta_l-\eta)}{\sinh^4(\eta)}.\end{aligned}\tag{4.3}$$

$$\begin{aligned}\Lambda(-u-\eta) &= \Lambda(u),\quad \Lambda(u+i\pi)=\Lambda(u),\\ \Lambda(0) &= -2^3\sinh\alpha_-\cosh\beta_-\sinh\alpha_+\cosh\beta_+\cosh\eta\\ &\quad\times\prod_{l=1}^{N}\frac{\sinh(\eta-\theta_l)\,\sinh(\eta+\theta_l)}{\sinh^2\eta},\\ \Lambda(\tfrac{i\pi}{2}) &= -2^3\cosh\alpha_-\sinh\beta_-\cosh\alpha_+\sinh\beta_+\cosh\eta\\ &\quad\times\prod_{l=1}^{N}\frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\,\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh^2\eta},\end{aligned}$$

$$\lim_{u\to\pm\infty}\Lambda(u) = -\frac{\cosh(\theta_--\theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta}\times\mathrm{id}+\ldots.$$

$$\begin{aligned}\Lambda(u) &= \bar{a}(u)\frac{Q_1(u-\eta)}{Q_2(u)}+\bar{d}(u)\frac{Q_2(u+\eta)}{Q_1(u)}\\ &\quad+\frac{2\bar{c}\sinh(2u)\sinh(2u+2\eta)}{Q_1(u)Q_2(u)}\bar{A}(u)\bar{A}(-u-\eta),\end{aligned}$$

Even N

$$\begin{aligned}\bar{A}(u) &= \prod_{l=1}^{N}\frac{\sinh(u-\theta_l+\eta)\,\sinh(u+\theta_l+\eta)}{\sinh^2\eta},\\ \bar{a}(u) &= -2^2\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\sinh(u-\alpha_-)\cosh(u-\beta_-)\\ &\quad\times\sinh(u-\alpha_+)\cosh(u-\beta_+)\bar{A}(u),\\ \bar{d}(u) &= \bar{a}(-u-\eta).\end{aligned}$$

$$\begin{aligned}Q_1(u) &= \prod_{j=1}^{N}\frac{\sinh(u-\mu_j)}{\sinh(\eta)},\\ Q_2(u) &= \prod_{j=1}^{N}\frac{\sinh(u+\mu_j+\eta)}{\sinh(\eta)}=Q_1(-u-\eta).\end{aligned}$$

$$\frac{2\bar{c}\sinh(2\mu_j)\sinh(2\mu_j+2\eta)\,\bar{A}(\mu_j)\bar{A}(-\mu_j-\eta)}{\bar{d}(\mu_j)Q_2(\mu_j)Q_2(\mu_j+\eta)} = -1,\ \ j=1,\ldots,N,$$

BAE

$$\bar{c} = \cosh((N+1)\eta + \alpha_- + \beta_- + \alpha_+ + \beta_+ + 2\sum_{j=1}^{N}\mu_j) - \cosh(\theta_- - \theta_+).$$

$$(N-1-2M)\eta = \alpha_- + \beta_- + \alpha_+ + \beta_+ \pm (\theta_- - \theta_+).$$

Local vacuum

$$(\mu_l, -\mu_l - \eta), \qquad (\mu_l, -\mu_l - 2\eta).$$

$$\Lambda(u) = \bar{a}(u)\frac{\bar{Q}(u-\eta)}{\bar{Q}(u)} + \bar{d}(u)\frac{\bar{Q}(u+\eta)}{\bar{Q}(u)},$$

$$\bar{Q}(u) = \prod_{j=1}^{M}\frac{\sinh(u-\mu_l)\sinh(u+\mu_l+\eta)}{\sinh^2\eta}.$$

Cao et al, Nepomechie 2003

Table 1: Numerical solutions of the BAEs for the $N = 4$ case with the parameters: $\eta = 0.5$, $\alpha_+ = 1$, $\alpha_- = 0.8$, $\beta_+ = 0.4$, $\beta_- = 0.3$, $\theta_+ = 0.7i$ and $\theta_- = 0.9i$. E is the eigevalues of the Hamiltonian. The eigenvalues are exactly the same as those from the exact diagonalization.

μ_1	μ_2	μ_3	μ_4	E	n
$-0.3330 - 0.4622i$	$-0.3330 + 0.4622i$	$-0.2506 - 0.1242i$	$-0.2506 + 0.1242i$	-6.8670	1
$-1.0988 - 1.5708i$	$-0.6931 + 1.5708i$	$-0.2500 - 0.1095i$	$-0.2500 + 0.1095i$	-4.8468	2
$-1.0171 + 0.0000i$	$-0.2501 - 0.0969i$	$-0.2501 + 0.0969i$	$-0.2457 + 1.5708i$	-3.9266	3
$-2.0689 - 1.5708i$	$-1.1954 - 0.0000i$	$-0.2500 - 0.0945i$	$-0.2500 + 0.0945i$	-3.2170	4
$-1.1075 - 1.5708i$	$-0.6883 + 1.5708i$	$-0.2500 - 0.2762i$	$-0.2500 + 0.2762i$	-1.6077	5
$-1.0558 - 0.0000i$	$-0.2497 - 0.2356i$	$-0.2497 + 0.2356i$	$-0.2458 - 1.5708i$	-1.2212	6
$-2.1748 + 1.5708i$	$-1.2064 + 0.0000i$	$-0.2499 - 0.2179i$	$-0.2499 + 0.2179i$	-0.6645	7
$-0.6387 - 0.5472i$	$-0.6387 + 0.5472i$	$-0.1043 + 0.5711i$	$-0.1043 + 2.5705i$	0.5747	8
$-1.2297 - 1.5708i$	$-0.5885 + 1.5708i$	$-0.2591 + 0.9163i$	$-0.2591 + 2.2253i$	1.5474	9
$-1.1140 + 0.0000i$	$-0.2950 - 0.6757i$	$-0.2950 + 0.6757i$	$0.1594 - 1.5708i$	1.8547	10
$-3.2361 - 1.5708i$	$-1.2319 - 0.0000i$	$-0.2509 - 0.4386i$	$-0.2509 + 0.4386i$	2.0475	11
$-2.9114 - 3.1416i$	$-0.9120 - 1.5708i$	$-0.7921 + 1.5708i$	$1.9542 + 1.5708i$	2.2990	12
$-1.1848 + 0.0000i$	$-1.0107 + 1.5708i$	$-0.7379 - 1.5708i$	$0.7058 - 0.0000i$	2.8030	13
$-1.1547 + 0.0000i$	$-0.9587 - 0.0000i$	$-0.4986 - 1.5708i$	$-0.0459 + 0.0000i$	3.3542	14
$-1.5932 + 0.0000i$	$-1.2867 + 0.0000i$	$-0.4656 + 1.5708i$	$0.1759 - 1.5708i$	3.8081	15
$-1.5191 - 0.0000i$	$-1.2970 - 0.0000i$	$-1.0092 - 1.5708i$	$0.8832 + 1.5708i$	4.0622	16

Table 2: Numerical solutions of the BAEs for the $N = 3$ case with the parameters: $\eta = 0.5$, $\alpha_+ = 1$, $\alpha_- = 0.8$, $\beta_+ = 0.4$, $\beta_- = 0.3$, $\theta_+ = 0.7i$ and $\theta_- = 0.9i$. E is the eigevalues of the Hamiltonian. The eigenvalues are exactly the same as those from the exact diagonalization.

μ_1	μ_2	μ_3	μ_4	E	n
$-0.5276 - 0.3652i$	$-0.5276 + 0.3652i$	$-0.2481 - 0.1756i$	$-0.2481 + 0.1756i$	-4.8590	1
$-2.9056 + 0.0000i$	$-1.1969 - 0.0000i$	$-0.2500 - 0.1261i$	$-0.2500 + 0.1261i$	-3.5939	2
$-0.6974 - 0.5166i$	$-0.6974 + 0.5166i$	$-0.2826 - 0.4381i$	$-0.2826 + 0.4381i$	-0.1251	3
$-0.9296 - 0.0000i$	$-0.2637 - 0.4333i$	$-0.2637 + 0.4333i$	$0.6919 + 1.5708i$	-0.0479	4
$-0.9741 + 4.7124i$	$-0.7424 - 4.7124i$	$-0.5001 - 0.5664i$	$-0.5001 + 0.5664i$	1.1449	5
$-1.1498 + 0.0000i$	$-0.5212 - 2.5079i$	$-0.5212 - 0.6337i$	$0.0230 + 1.5708i$	1.8855	6
$-1.1060 - 0.1659i$	$-1.1060 + 0.1659i$	$-0.5216 - 1.5708i$	$0.3535 + 0.0000i$	2.5676	7
$-1.5205 + 0.0000i$	$-1.2965 - 0.0000i$	$-1.1030 + 1.5708i$	$0.8003 - 1.5708i$	3.0278	8

Thermodynamic limit

We choose a sequence of the crossing parameter

$$\eta_m = -\frac{\alpha_- + \alpha_+ \pm (\theta_- - \theta_+) + 2\pi i m}{N+1}$$

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$$\left[\frac{\sinh(\lambda_j - i\frac{\theta}{2})}{\sinh(\lambda_j + i\frac{\theta}{2})}\right]^{2N} \frac{\sinh(2\lambda_j - i\theta)}{\sinh(2\lambda_j + i\theta)} \frac{\sinh(\lambda_j + ia_+)}{\sinh(\lambda_j - ia_+)}$$
$$\times \frac{\sinh(\lambda_j + ia_-)}{\sinh(\lambda_j - ia_-)} \frac{\cosh(\lambda_j + \beta + i\frac{\theta}{2})}{\cosh(\lambda_j + \beta - i\frac{\theta}{2})} \frac{\cosh(\lambda_j - \beta + i\frac{\theta}{2})}{\cosh(\lambda_j - \beta - i\frac{\theta}{2})}$$
$$= -\prod_{l=1}^{N} \frac{\sinh(\lambda_j - \lambda_l - i\theta)\sinh(\lambda_j + \lambda_l - i\theta)}{\sinh(\lambda_j - \lambda_l + i\theta)\sinh(\lambda_j + \lambda_l + i\theta)},$$

Nepomechie & Ravanini 03

The key point is that M=N gives a complete set of solutions!

Thermodynamic limit

Surface energy

$$\epsilon_b = \epsilon_b^0 + I_1(a_+) + I_1(a_-) + I_2(\beta_+) + I_2(\beta_-)$$

$$e_b^0 = -\sin\theta \int_{-\infty}^{\infty} \frac{\hat{a}_1(\omega)}{1+\hat{a}_2(\omega)} [\hat{a}_2(\omega/2) - 1] d\omega - \cos\theta,$$

$$I_1(\alpha) = \sin\theta \int_{-\infty}^{\infty} \frac{\hat{a}_1(\omega)}{1+\hat{a}_2(\omega)} \hat{a}_{2\alpha/\theta}(\omega) d\omega - \sin\theta \cot(\alpha - \theta/2)$$

$$I_2(\beta) = -\sin\theta \int_{-\infty}^{\infty} \frac{\hat{a}_1(\omega)}{1+\hat{a}_2(\omega)} \cos(\beta\omega)\hat{b}(\omega) d\omega.$$

For all quantities, the corrections are in the order of $O(N^{-2})$

Remark: No degenerate points for XXX chain, but we can always take proper limits of XXZ or XYZ!

影响和评价

本文给出了非平行场这一“可怕”问题的精确解

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Surface Critical Phenomena and Scaling in the Eight-Vertex Model

M. T. Batchelor and Y. K. Zhou*

significant confirmation of the scaling relations [6,7,16] $\alpha_s = \alpha_b + \nu$ and $\alpha_1 = \alpha_b - 1$ between bulk and surface critical exponents. The derivation of other surface exponents awaits the diagonalization of the transfer matrix, which remains a formidable open problem.

非平行场自旋链转移矩阵的精确解仍然是一个“可怕”的未解决问题

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Off-diagonal Bethe ansatz solutions of the anisotropic spin-$\frac{1}{2}$ chains with arbitrary boundary fields

Junpeng Cao [a], Wen-Li Yang [b,*], Kangjie Shi [b], Yupeng Wang [a,*]

Abstract

The anisotropic spin-$\frac{1}{2}$ chains with arbitrary boundary fields are diagonalized with the off-diagonal Bethe ansatz method. Based on the properties of the R-matrix and the K-matrices, an operator product identity of the transfer matrix is constructed at some special points of the spectral parameter. Combining with the asymptotic behavior (for XXZ case) or the quasi-periodicity properties (for XYZ case) of the transfer matrix, the extended $T-Q$ ansatzes and the corresponding Bethe ansatz equations are derived.
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Keywords: Spin chain; Reflection equation; Bethe ansatz; $T-Q$ relation

* Corresponding authors.
E-mail addresses: wlyang@nwu.edu.cn (W.-L. Yang), yupeng@iphy.ac.cn (Y. Wang).

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Batchelor 教授:可积领域著名专家，澳大利亚国立大学理论物理系主任，英国物理学会J. Phys. A 杂志主编。

影响和评价

本文利用我们的结果构成了相应的本征态

Symmetry, Integrability and Geometry: Methods and Applications SIGMA 9 (2013), 072, 12 pages

Heisenberg XXX Model with General Boundaries: Eigenvectors from Algebraic Bethe Ansatz

Samuel BELLIARD [†‡] *and Nicolas CRAMPÉ* [†‡]

[†] *Laboratoire Charles Coulomb L2C, UMR 5221, CNRS, F-34095 Montpellier, France*

[‡] *Laboratoire Charles Coulomb L2C, UMR 5221, Université Montpellier 2, F-34095 Montpellier, France*

E-mail: *samuel.belliard@univ-montp2.fr*, *nicolas.crampe@univ-montp2.fr*

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在摘要中指出，此文的工作完全基于最近我们的一系列工作

Abstract. We propose a generalization of the algebraic Bethe ansatz to obtain the eigenvectors of the Heisenberg spin chain with general boundaries associated to the eigenvalues and the Bethe equations found recently by Cao et al. The ansatz takes the usual form of

or for the transfer matrix. This is due to the fact that the boundaries break the U(1)-symmetry (i.e. the total spin is not anymore conserved). This problem is quite general for integrable models without U(1)-symmetry and led numerous researchers to believe that usual methods cannot work in this case and to develop different approaches to study their spectrum over the last 30 years. Basically, one can consider two families of approaches:

在过去30年间无数学者致力于此方面研究而鲜有进展

Recently, progress on the generalization of T-Q relation formalism has been performed by Cao, Yang, Shi and Wang [10] where the eigenvalues and the Bethe equations of Hamiltonian (1) are obtained. The main feature consists in adding a new term in the eigenvalues and the Bethe equations in comparison to the usual ones obtained for cyclic or diagonal boundaries (see also [24] for this idea). Then, this result has been simplified by Nepomechie [32] that shows numerical

直至我们的工作才取得突破…

The paper is organized as follows. In Section 2, we fix the notations and recall the eigenvalues and the Bethe equations obtained in [10]. Then, in Section 3, we give our conjecture for the

系统介绍我们的工作，并基于我们的结论而展开一系列讨论

影响和评价

Nepomechie教授：可积领域著名专家

IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **47** (2014) 032001 (9pp) doi:10.1088/1751-8113/47/3/032001

Fast Track Communications

Boundary energy of the open XXX chain with a non-diagonal boundary term

Rafael I Nepomechie and Chunguang Wang

1. Introduction

Ever since the open spin-1/2 XXX (isotropic) quantum spin chain with non-diagonal boundary terms was shown to be integrable [1–3], the challenge has been to find its general Bethe ansatz solution. Significant progress has been made recently on this problem. The breakthrough was the realization that the Baxter T-Q equation for this model should have an inhomogeneous term [4] (see also [5]). A simplified version of this solution was subsequently shown to produce

[4] Cao J, Yang W-L, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz solution of the XXX spin-chain with arbitrary boundary conditions *Nucl. Phys.* B **875** 152–65 (arXiv:1306.1742 [math-ph])
[5] Cao J, Yang W-L, Shi K and Wang Y 2013 Off-diagonal Bethe ansatz and exact solution of a topological spin ring *Phys. Rev. Lett.* **111** 137201 (arXiv:1305.7328 [cond-mat.stat-mech])

“该问题的精确解挑战该领域多年，最近有重要进展，突破是在Baxter的T-Q关系中加入一个非齐次项[4][5]。”

V. 总结和展望

Yang-Baxter Equation & Reflection Equation

⇓

Operator product identities

$$t(\theta_j)t(\theta_j-\eta)=a(\theta_j)d(\theta_j)$$

\+ **Asymptotic behavior of the polynomial**

⇓

$$\Lambda(u)=a(u)\frac{Q_1(u-\eta)}{Q_2(u)}+d(u)\frac{Q_2(u+\eta)}{Q_1(u)}+c(u)\frac{a(u)d(u)}{Q_1(u)Q_2(u)}$$

Regularity ⇒ **Bethe ansatz equations**

V. 总结和展望

High rank systems: An, Bn, Cn, Dn

$$t^{(1)}(\theta_j)t^{(r)}(\theta_j-\eta)\sim t^{(r+1)}(\theta_j),\quad r=1,\cdots,N-1.$$

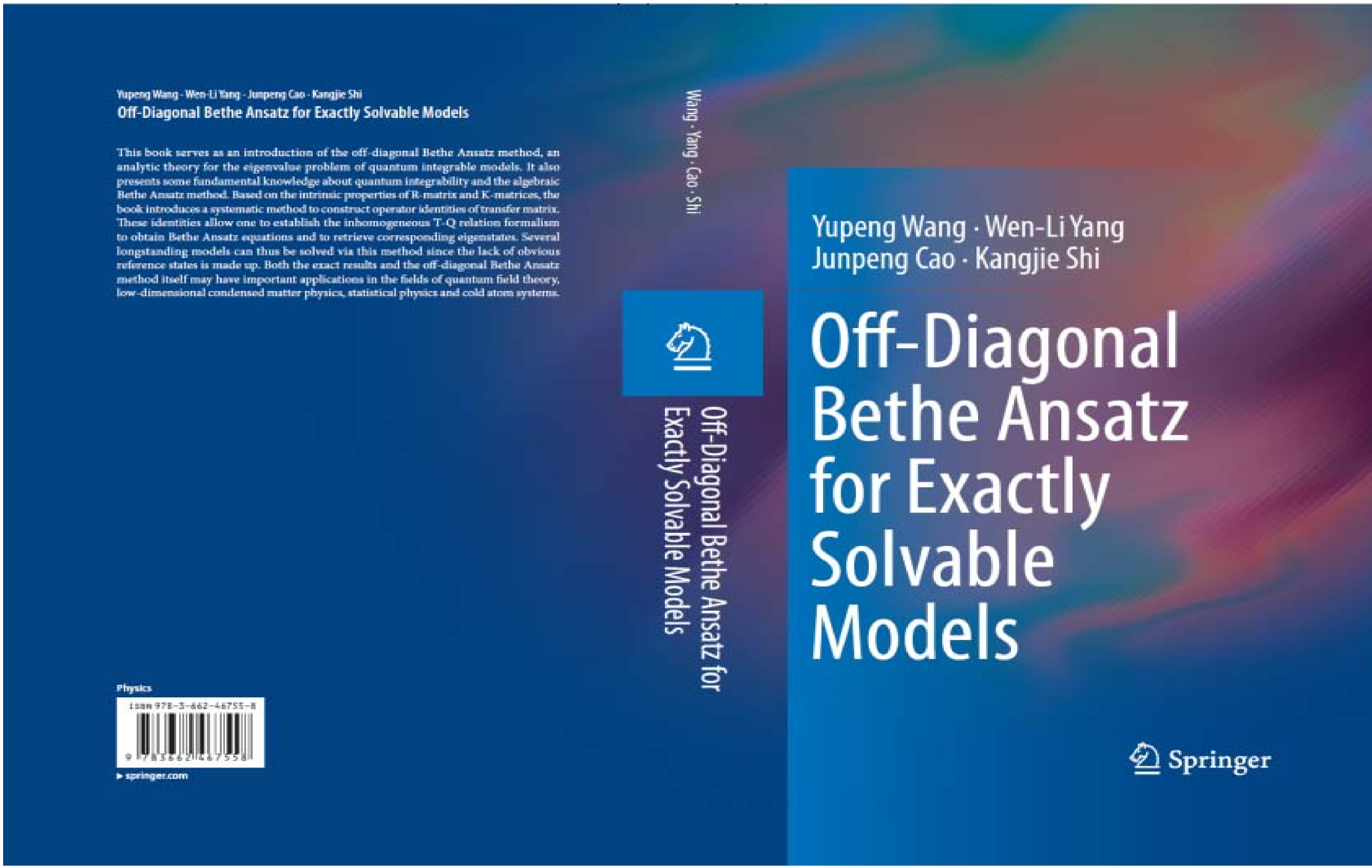
Nested T-Q

The method works for all boundary conditions!

Makes the “unsolvable models” solvable!

Provides an unified Bethe ansatz framework!

V. 总结和展望



Yupeng Wang · Wen-Li Yang · Junpeng Cao · Kangjie Shi
Off-Diagonal Bethe Ansatz for Exactly Solvable Models
This book serves as an introduction of the off-diagonal Bethe Ansatz method, an analytic theory for the eigenvalue problem of quantum integrable models. It also presents some fundamental knowledge about quantum integrability and the algebraic Bethe Ansatz method. Based on the intrinsic properties of R-matrix and K-matrices, the book introduces a systematic method to construct operator identities of transfer matrix. These identities allow one to establish the inhomogeneous T-Q relation formalism to obtain Bethe Ansatz equations and to retrieve corresponding eigenstates. Several longstanding models can thus be solved via this method since the lack of obvious reference states is made up. Both the exact results and the off-diagonal Bethe Ansatz method itself may have important applications in the fields of quantum field theory, low-dimensional condensed matter physics, statistical physics and cold atom systems.
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