Complex Analytic Methods in Real One-dimensional Dynamics

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Time-discrete dynamics

Let M be a compact manifold (phase space) and consider a (smooth) map $f : M \to M$. For each $n \geq 0$, let

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

For each $x \in M$, the orbit is

$$\mathrm{orb}(x) = \{f^n(x)\}_{n=0}^{\infty}.$$

- The theory of *time-discrete dynamical systems* studies the orbit structure of maps.
- A dynamical system is *stable* if nearby dynamical systems have roughly the same orbit structure.

- A point x is called *periodic* if $f^k(x) = x$ for some $k \geq 1$ and the minimal positive integer k with this property is called its *period*.
- It happens that $orb(x)$ is dense in the phase space.

Problem.

- Describe the dynamical property of 'most' dynamical systems.
- Are these properties 'stable' in suitable sense?

Difficulty. Sensitive dependence on initial values:

$$d(x, y) \ll 1 \not\Rightarrow \sup_{n=0}^{\infty} d(f^n(x), f^n(y)) \ll 1.$$

Structural stability

Definition

Two maps $f : M \to M$ and $g : N \to N$ are called *topologically conjugate* if there is a homeomorphism $h : M \to N$ such that $g \circ h = h \circ f$.

Definition

A C^r map $f : M \to M$ is called *C^r-structurally stable* if there is a neighborhood $\mathcal{U}$ of f in the space $C^r(M, M)$ such that all the maps in $\mathcal{U}$ are topologically conjugate to each other.

Problems.

- Characterize structurally stable maps.
- Are "most" maps structurally stable?

- In 1960s, Smale posed the notion "Axiom A" and conjectured that it is essentially equivalent to structural stability. In general dimension, this was so far verified in the C^1 topology for diffeomorphsim.(Mãné, Hayashi, Wen)

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- Smale also hoped that Axiom A maps are dense, but it was soon realized that this is not true when $\dim(M) \geq 2$.
- We shall see that in the case $\dim(M) = 1$, Axiom A is essentially equivalent to structural stability and these maps are dense in the C^k topology for any $k \geq 1$.

- Even in the one-dimensional case, Jakobson (1981) showed that there is a large set of dynamical systems in the measure-theoretical sense which are not Axiom A, but are still well understood from stochastic point of view.

Physical measures

A Borel probability measure μ is called *f-invariant* if for any Borel set $A \subset M$, we have

$$\mu(f^{-1}(A)) = \mu(A).$$

The invariant measure μ is called *ergodic* if for each Borel set $A \subset M$,

$$A = f^{-1}(A) \Rightarrow \mu(A) = 0 \text{ or } 1.$$

The basin of μ is defined as

$$B(\mu) = \left\{ x \in M : \begin{array}{l} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \mu \text{ as } n \to \infty \\ \text{in the weak star topology} \end{array} \right\}.$$

If $\mathrm{Leb}(B(\mu)) > 0$ then we say that μ is a *physical measure*.

Examples.

- A periodic point x is called *hyperbolic attracting* if all eigenvalues of $Df^k(x)$ lie inside the unit circle, where k is the period. In this case, $\mu := \frac{1}{k}\sum_{i=1}^{k} \delta_{f^i(x)}$ is a physical measure since $B(\mu)$ contains a neighborhood of x.
- If μ is an ergodic invariant probability measure and $\mu \ll \mathrm{Leb}$ (an *acip*), then μ is a physical measure. Indeed, by Birkhroff's ergodic theorem, $\mu(B(\mu)) = 1$, hence $\mathrm{Leb}(B(\mu)) > 0$.

Palis Conjecture (Simplified version)

Given a compact manifold M, in measure-theoretical sense, 'most' maps $f : M \to M$ are stochastic, *that is, there exists finitely many physical measures $\mu_1, \dots, \mu_m$ such that*

- *the topological support of μ_i's are pairwise disjoint;*
- $Leb(M \setminus (\bigcup_{i=1}^{m} B(\mu_i))) = 0.$

Moreover, the statistical property of f is stable under random perturbation.

Example.

- $f : \mathbb{T} \to \mathbb{T}$, $x \mapsto 2x \mod 1$. For this map the Haar measure on $\mathbb{T}$ is the unique physical measure and its basin has full measure in $\mathbb{T}$.

1D Dynamics

- Real 1D dynamics: iteration of interval/circle maps
- Complex 1D dynamics: iteration of holomorphic maps on Riemann surfaces
- Interplay:
  - A map defined by a real polynomial can be viewed as both a real and a complex 1D dynamical system.
  - A real analytic interval map, via *generalized renormalization*, can be reduced to a real polynomial (Sullivan, Levin-van Strien, Graczyk-Swiatek, Lyubich-Yampolsky, Levin, Hu, Shen, Smania, Clark-Treje-van Strien)

The logistic family

Jakobson's Theorem (1981)

Consider the logistic family $Q_a(x) = x^2 + a$, $-2 \leq a \leq 1/4$. Then there is a subset $\mathcal{J}$ of parameters of positive Lebesgue measure such that for $a \in \mathcal{J}$, Q_a has a unique physical measure which is ergodic and absolutely continuous.

Purely real analytic method.

Lyubich's Dichotomy Theorem (2002)

For almost every $a \in [-2, 1/4]$, either Q_a satisfies Axiom A or it has a unique physical measure which is ergodic and absolutely continuous.

Mainly complex analytic method. Solved Palis' conjecture for the logistic family. Continued by Avila-Lyubich-de Melo, Avila-Moreira, Avila-Shen-Lyubich, Avila-Lyubich

Kozlovski-Shen-van Strien's Theorem (2007)

For any $d = 2, 3 \ldots$, Axiom A maps form an open and dense subset of the space of real polynomials of degree d.

Here, a map $f : \mathbb{R} \to \mathbb{R}$ is called *Axiom A* if for each of its critical points c, $f^n(c)$ converges to an attracting periodic orbit or to infinity. For $d = 2$, the theorem is due to Lyubich (1997), Graczyk-Swiatek (1997). Mainly complex analytic method.

Corollary (Structural stability conjecture holds in 1D)

Let $M = [0,1]$ or S^1.

- *For any $k \geq 1$, if a C^k map $f : M \to M$ is C^k structurally stable then f satisfies Axiom A.*
- *C^k-structurally stable maps are open and dense in the C^k topology.*

This solves the second part of Smale's 11-th problem for the 21st century.

Complex analytic methods played important roles in the following problems:

- Milnor-Thurston's monotonicity problem (1980s)
- Conceptual proof of Feigenbaum renormalization (1988-)
- Milnor's attractor problem (1991)
- Density of hyperbolicity (1997-)
- Palis conjecture for unimodal maps

Complex analytic methods have NOT played essential roles in the following problems:

- Non-existence of wandering intervals
- Statistical property of interval maps: existence of acip, stochastic stability
- Existence of wild attractor
- Problems on interval maps with non-integral critical order

qc map

Definition

A homeomorphic $\varphi : U \to V$ between two open sets in $\mathbb{C}$ is called *K-qc* if

- φ has locally integrable partial derivatives in the sense of distribution;
- $$\left|\frac{\overline{\partial}\varphi}{\partial\varphi}\right| \leq \frac{K-1}{K+1}, \text{ a.e..}$$

- A qc map is differentiable a.e., and the classical partial derivatives coincide with the distributional ways.
- The space of all K-qc map of $\mathbb{C}$ normalized at two distinct points is compact.

A *unimodal* map is a continuous map from an interval I into itself for which there is a unique point $c \in I^o$ such that f is strictly increasing on the left of c and strictly decreasing on the right of c (or vice versa). E.g.

- $$f_c(x) = x^2 + c.$$
- $$f_c(x) = |x|^\ell + c,\ \ell > 1.$$
- $$f_c(x) = be^{-1/|x|^\ell} + c, b > 0, \ell > 1.$$

Feigenbaum renormalization

Let $f : I \to I$ be a unimodal map with a maximum at the critical point 0. f is called *Feigenbaum-renormalizable*, if there exists a closed interval J which contains 0 in its interior such that

- $f^2(J) \subset J$, $f^2(\partial J) \subset \partial J$ and $f^2 : J \to J$ is unimodal;
- the intervals J and $f(J)$, have pairwise disjoint closure.

In fact,

$$f \text{ is Feigenbaum renormalizable } \Leftrightarrow f^2(0) < 0 < f^4(0) \leq f^3(0) < f(0).$$

Let $\mathcal{R}_F f : I \to I$ denote the unimodal map which is affine conjugate to $f^2 : J \to J$ and which has a maximum at 0.

- It could happen that $\mathcal{R}_F f$ is again Feigenbaum renormalizable and then we obtain a second renormalization $\mathcal{R}_F^2$. If the procedure can be continued indefinitely then we obtain a sequence of unimodal maps $\mathcal{R}_F^n f$ and say that f is *infinitely renormalizable in the sense of Feigenbaum*.

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- By numerical experiments, Feigenbaum found that the sequence $\mathcal{R}_F^n f$ converges and moreover, the limit is independent of the map f where we start with.

- More generally, given a permutation $\sigma : \{0, 1, \ldots, s-1\} \circlearrowleft$ of order $s \geq 2$, we say that f is *σ-renormalizable* if there exists a closed interval J containing 0 in its interior such that

$$f(f^j(J)) \subset f^{\sigma(j)}(J), \ \forall 0 \leq j < s$$

and the intervals $J, f(J), \ldots, f^{s-1}(J)$ have pairwise disjoint interior. Let $\mathcal{R}_\sigma f$ denote the unimodal map affine conjugate to $f^s : J \to J$ with maximal at 0.
- The Feigenbaum renormalization corresponds to the case $s = 2$ and $\sigma = (10)$.

- We say that a unimodal map f is *infinitely renormalizable* with combinatorics $(\sigma_n)_{n=0}^{\infty}$ if $f_n = \mathcal{R}_{\sigma_{n-1}} f_{n-1}$ are well defined for all $n \geq 1$, where $f_0 = f$.
- It is called infinitely renormalizable of bounded type if $\{\sigma_n : n \geq 0\}$ is finite.

Feigenbaum's Conjecture

Consider a suitable space $\mathcal{B}$ of unimodal maps $f : [-1, 1] \circlearrowleft$ and let $\mathcal{B}_0$ be the subspace of all renormalizable maps. Then the renormalization map $\mathcal{R}_F : \mathcal{B}_0 \to \mathcal{B}$ has a unique fixed point f_*. The fixed point is hyperbolic and has codimensional one stable manifold consisting of all infinitely renormalizable maps in the sense of Feigenbaum.

Lanford gave computer-assisted proof for suitable $\mathcal{B}$.

Sullivan's conceptual proof

Theorem (Sullivan's Beau Bounds Theorem)

Let f be an infinitely renormalizable C^3 unimodal map, of bounded type, with maximum at 0 and such that $f''(0) \neq 0$. Then the sequence $\{\mathcal{R}^n f\}$ is precompact in C^1 and any possible limit map is real analytic which has a quadratic-like extension with a universal bounds.

- A quadratic-like map is a proper holomorphic map $F : U \to V$ of degree 2, where U, V are topological disks.
- Sullivan's universal bounds refers to a lower bound on $\mu = \text{mod}\,(V \setminus U)$.
- By Douday-Hubbard's straightening theorem, F is $K(\mu)$-qc conjugate to a quadratic polynomial.

Theorem (Levin-van Strien, Lyubich-Yampolsky, Graczyk-Swaitek, Shen, Clark-Treje-van Strien...)

Let f be an arbitrary real analytic interval map and let c be a non-periodic recurrent critical point. Then there is an arbitrarily small neighborhood J of c such that the first return map to J has (generalized) polynomial-like extension with universal bounds.

Theorem (Sullivan's Rigidity Theorem)

Let f, g be quadratic-like infinitely renormalizable unimodal maps of the same bounded type. Then f and g are qc conjugate.

Sullivan also proved that $\mathcal{R}^n f$ converges to a unique fixed point f_* of $\mathcal{R}$ and $\mathcal{R}^n f \to f_*$ for any infinitely renormalizable f. This proof of this part was simplified and strengthened by McMullen:

Theorem (McMullen 1994)

Let f, g be quadratic-like infinitely renormalizable unimodal maps of the same bounded type. Then $\|\mathcal{R}^n f - \mathcal{R}^n g\|_{C^0}$ converges to zero exponentially fast.

Avila-Lyubich generalized the result to unbounded type.

Theorem (Kozlovski-Shen-van Strien 2007)

Let f, g be real polynomials of the same degree d with all critical points real and non-degenerate and without neutral periodic points. If f, g are topologically conjugate in $\mathbb{R}$, then they are quasiconformally conjugate in $\mathbb{C}$.

Remark:

- The case $d = 2$ is due to Lyubich (1997), Graczyk-Swiatek (1997), which uses special geometric property of quadratic polynomials.

Rigidity Problem.

1. *Any two combinatorially equivalent rational maps f, g are qc conjugate.*
2. *A qc conjugacy is conformal a.e. on the Julia set unless f is a Lattés example.*

Remark:

- If f, g are C^1 conjugate, then all the corresponding periodic points have the same multiplier which would imply that f, g are conformally conjugate.
- A Lattés example is a rational map which is doubly convered by an integral torus endomorphism.

Lyubich's contribution

Theorem (Lyubich 2002)

- *The Feigenbaum conjecture holds if we take $\mathcal{B}$ be quadratic-like unimodal maps.*
- *Extension of Feigenbaum conjecture to all (real) combinatorics.*

In particular, Lyubich gave a simple proof of McMullen's result:

- Let V be a small neighborhood of $[-1, 1]$ and let $\mathcal{B}$ denote the Banach space of all bounded holomorphic maps from V to $\mathbb{C}$.
- The subspace of $\mathcal{B}$ consisting of all infinitely renormalizable maps is a complex submanifold, denoted by $\mathcal{B}^s$.
- By Sullivan's Beau bounds Theorem, Theorem, there exists N such that $\mathcal{R}^N$ maps $\mathcal{B}^s(f_*, 2\varepsilon)$ into $\mathcal{B}^s(f_*, \varepsilon)$. Thus $\mathcal{R}^N|\mathcal{B}^s(f_*, \varepsilon)$ is contracting due to the Schwarz Lemma.

- Another important contribution of Lyubich is existence of unstable direction of f_*.
- He deduced it from a Rigidity Theorem via a *small orbits* argument: If there were no unstable direction, then there would be f such that $\mathcal{R}^n f$ converges to f_* slowly. But f is hybrid equivalent to f_* and this is impossible.
- Using his result on renormalization, Lyubich proved that in the quadratic family, almost every map is NOT infinitely renormalizable.

Smania's approach in the multimodal case

Theorem (Smania, to appear)

Extension of the Feigenbaum conjecture to multimodal case with bounded combinatorics.

Smania's approach is based on solving the following cohomologous equation:

$$\alpha(f(z)) - Df(z)\alpha(z) = v(z).$$

Corollary (Smania)

In the real cubic family $f_{a,b}(x) = x^3 - 3a^2x + b$, the set of parameters corresponding to infinitely renormalizable maps of bounded type has Lebesgue measure zero.

Martens' result

- The Feigenbaum conjecture is expected to hold for unimodal maps with critical order $\ell > 1$: maps with

$$f(x) - f(c) = \phi(|x - c|^{\ell}).$$

- But the methods discussed above only applies to the cases where ℓ is an even integer.
- For general ℓ, Martens (1996) proved existence of a renormalization fixed point.
- He also claimed that the fixed point has a stable manifold of codimension ≤ 1.

More on Lyubich's Dichotomy Theorem

- To obtain the Dichotomy Theorem, Lyubich also proved that among all non-infinitely renormalizable, non-hyperbolic maps, almost every map has enough expansion, so that acip exists.
- The one-dimensional feature of the parameter space of the quadratic family ($x \mapsto x^2 + c$) is essentially used in the proof:
  - the fine structure of the one-dimensional parameter space (para-puzzle), developed by Douady-Hubbard
  - quasiconformality of a transition map from phase space to parameter space.
- It is hard to extend the argument to the multimodal case, e.g. $f_{a,b}(x) = x^3 - 3a^2x + b$.
- Maybe some generalization of Smania's work applies here.

Kneading sequence

Associate to a unimodal map f there is a sequence $\mathbf{i} = \{i_n\}_{n=1}^{\infty} \in \{L, c, R\}^{\mathbb{Z}^+}$ with

$$i_n = \begin{cases} L & \text{if } f^n(c) < c; \\ c & \text{if } f^n(c) = c; \\ R & \text{if } f^n(c) > c. \end{cases}$$

This sequence determines the relative order the post-critical orbit $\{f^n(c)\}_{n=0}^{\infty}$ and thus the folding pattern of iterates of f.

Topological entropy

- Let $\ell_n(f)$ denote the number of maximal intervals of monotonicity of f^n. Define the *entropy* of f to be

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log \ell_n(f),$$

  which measures the topological complexity of the dynamical system f.
- $h_{top}(f)$ is determined by the kneading sequence $\mathbf{i}(f)$.

Monotonicity of entropy

Let $Q_c(x) = x^2 + c$. Milnor-Thurston (1977) asked whether $\Phi(c) := h_{top}(Q_c)$ is a monotone decreasing function.

Theorem (Milnor-Thurston, Sullivan, ...)

The function Φ is monotone decreasing.

- Φ is a continuous function from $\mathbb{R}$ onto $[0, \log 2]$.
- $\Phi(c) = \log 2$ when $c \leq -2$ and $\Phi(c) = 0$ when $c \geq 0$.
- We say that Q_c is *hyperbolic* if Q_c has an attracting cycle. Near a hyperbolic parameter c_0, Q is constant.
- Guckenheimer (1980): Φ is Hölder continuous.
- Dobbs-Mihalache(2016 preprint) $\Phi'(c) = 0$ holds for a.e. c. $\Rightarrow \Phi$ is not Lipschitz.

- Several proofs of the theorem exist, all using complex techniques. *Important: Q_c is a holomorphic map from $\mathbb{C}$ to $\mathbb{C}$.*
- quasiconformal mappings, Teichmüller theory

Approach 1: rigidity

Theorem (Sullivan)

Suppose Q_c and $Q_{\hat{c}}$ are two quadratic maps with the same kneading sequence and with $Q_c^q(0) = Q_{\hat{c}}^q(0) = 0$. Then $c = \hat{c}$.

$\Rightarrow$ monotonicity of entropy (Milnor-Thurston's kneading theory)

Outline of proof of Sullivan's theorem.

- By the classical Böttcher's theorem, Q_{a_1} and Q_{a_2} are conformally conjugate near ∞ and also near their critical orbits;
- Via lifting and taking convergent subsequence, we obtain a qc conjugacy which is conformal in the basin of ∞ and the orbit of 0, i.e. outside the Julia set.
- Prove that the Julia set has Lebesgue measure zero, so that the qc conjugacy is indeed conformal (Weil's Lemma).

Approach 2: Teichmüller Theory

- Given a finite set $P \subset \mathbb{C}$, we say that two qc maps

$$\varphi_i : (\mathbb{C}, P) \to (\mathbb{C}, \varphi_i(P))$$

  are *Teichmuller equivalent* if $\varphi_2 \circ \varphi_1^{-1}$ is homotopic rel $\varphi_1(P)$ to the affine map sending $\varphi_1(P)$ to $\varphi_2(P)$. The Teichmüller space $\mathcal{T}_P$ is the set of all qc map mod the Teichmuller equivalence relation.
- The space $\mathcal{T}_P$ is a complex-analytic manifold of dimension $\#P - 2$.
- It is also endowed with a *Teichmüller metric* such that any holomorphic endomorphism does not increase the metric.
- The cotangent space of $\mathcal{T}_P$ at $[\varphi]$ is naturally identified with the space of all integrable meromorphic quadratic differential in $\widehat{\mathbb{C}}$ with (simple) poles in $\varphi(P) \cup \{\infty\}$, endowed with L^1 norm.

Thurston's algorithm

- Suppose $Q(x) = Q_c(x) = x^2 + c$ satisfies $Q_c^q(0) = 0$ and let $P = \{Q_c^j(0) : j \geq 0\}$.
- Given any qc map $\varphi : (\mathbb{C}, P) \to (\mathbb{C}, \varphi(P))$ there exists a quadratic map $\tilde{Q}$ and qc map $\psi : (\mathbb{C}, P) \to (\mathbb{C}, \psi(P))$ such that
$$\tilde{Q} \circ \psi = \varphi \circ Q.$$
- The map $\varphi \mapsto \psi$ descends to a holomorphic map
$$\sigma : \mathcal{T}_P \to \mathcal{T}_P$$
which fixes the $[id]$. It is non-decreasing with respect to the Teichmüller metric.
- Sullivan's rigidity theorem follows once we prove that σ has no non-trivial fixed point. In particular, once we prove that σ is strictly contracting.

Extremal mapping

Outline of Milnor-Thurston's proof.

- Assume that σ_f has a non-trivial fixed point $[\varphi]$ and assume φ is an extremal mapping in the equivalence class, i.e. a K-qc map in the homotopy class with minimal K.
- By a theorem of Teichmüller asserts that there is an integrable meromorphic quadratic $v(z)dz^2$ with poles in $P \cup \{\infty\}$ such that
$$\frac{\overline{\partial}\varphi}{\partial\varphi} = k\frac{\overline{v(z)}}{|v(z)|}.$$
- However, $[\sigma_f(\varphi)] = [\varphi] \Rightarrow Q^*(v(z)dz^2) = v(z)dz^2 \Rightarrow v(z)$ has infinitely many poles, a contradiction!

A problem on extremal mappings

Let $\alpha : 0 = a_1 < a_2 < \cdots < a_q$ and $\beta : 0 < b_1 < b_2 < \cdots < b_q$.
For $\theta \in (0, 2\pi)$, let

$$S_\theta = \{re^{it} : 0 < t < \theta\}.$$

Let $K(\theta) = K_{\alpha,\beta}(\theta)$ denote the minimal number K for which there exists a K-qc map $h : S_\theta \to S_\theta$ such that

$$h(a_j) = b_j, h(a_j e^{i\theta}) = b_j e^{i\theta}.$$

Problem. *Is $K(\theta)$ monotone decreasing?*

- An affirmative answer to this question will implies monotonicity of entropy for the family $c \mapsto |x|^\ell + c$ for each $\ell \geq 2$.
- $K(\theta/n) \leq K(\theta)$ for each integer $n \geq 1$.
- Yes if $q = 2$. (Cui)

Approach 3: Transversality

Theorem (Tsujii 1998)

Suppose 0 *is of period* q *for* Q_a*. Then*

$$\frac{\left.\frac{dQ_t^q(0)}{dt}\right|_{t=a}}{DQ_a^{q-1}(a)} = \sum_{n=0}^{q-1} \frac{1}{DQ_a^n(a)} > 0.$$

$\Rightarrow a \mapsto \ell_n(Q_a) \nearrow \Rightarrow \Phi(a) \nearrow$.

- Transversality appears in other problems of complex dynamics. Epstein and Levin, among others, worked in this problem.
- Tsujii's proof still uses global analytic structure of the maps

Outline of proof of Tsujii's theorem.

- Write $f = Q_a$ and $P = \{f^j(0)\}$. Let Ω denote the space of integrable meromorphic quadratic differentials with poles in $P \cup \{\infty\}$.
- The co-derivative of $\sigma : \mathcal{T}_P \to \mathcal{T}_P$ at $[id]$ is given by the *pushforward operator*

$$f_*(v(z)dz^2) = \left(\sum_{w \in Q^{-1}(z)} \frac{v(w)}{f'(w)^2} \right) dz^2.$$

- $$\int_V |f^*(v(z)dz^2)| \leq \int_{f^{-1}V} |v(z)dz^2|.$$

- $$\det(Id - f_*) = \sum_{n=0}^{q-1} \frac{1}{Df^n(f(0))}.$$
- Let $V_0 = B(0, R)$ with $R >> 1$ and let $V_n = f^{-n}(V_0)$. Then
$$\text{area}(V_n \setminus V_{n+1}) \to 0.$$
- For each $v \in \Omega$,
$$\int_{V_0 \setminus V_1} |f_*^n(v)| \leq \int_{V_n \setminus V_{n+1}} |v| \to 0.$$

$\Rightarrow$ the spectrum radius of f_* is strictly less than 1.

Approach 4: algebraic

Theorem (Gleason Lemma)

Assume that 0 *is a periodic point of* $Q_c(x) = x^2 + c$ *of periodic* q. *Then*

$$\left.\frac{dQ_t^q(0)}{dt}\right|_{t=c} \neq 0.$$

The proof works also for $Q_c(x) = |x|^\ell + c$ with $\ell > 1$ an odd integer and $c \in \mathbb{R}$. But, it *does not give the sign*.

Outline of proof. Let $\mathbf{A}$ be the set of algebraic integers. Then $c \in \mathbf{A}$. Note that

$$\mathbf{A} \cap \mathbb{Q} = \mathbb{Z}.$$

Let

$$\Delta_k = \left. \frac{dQ_t^k(0)}{dt} \right|_{t=c}.$$

Then $\Delta_1 = 1$ and $\Delta_{k+1} = 2Q_c^k(0)\Delta_k + 1$. If $Q_c^q(0) = 0$ for some $q \geq 1$, then $c \in \mathbf{A}$. By induction,

$$\Delta_k \equiv 1 \mod 2\mathbf{A}, \forall k.$$

Thus $\Delta_k \neq 0$.

New results (joint with Levin and van Strien)

We develop method/language to deal with maps which allows partial complex analytic extension. Roughly speaking, our method shows

$$\text{a lifting property} \;\Rightarrow\; \text{positive transversality.}$$

Theorem (Levin-Shen-van Strien)

Let $g : I \to I$ be a unimodal map with critical point 0 with $g(0) = 0$. Assume that $g|_{I \setminus \{0\}}$ has extension to a holomorphic map $g : U \to V$ where

- *U, V are bounded open sets in $\mathbb{C}$;*
- *$g : U \setminus \{0\} \to V \setminus \{0\}$ is an unbranched covering.*

If the separation property

$$V \supset B(0; diam(U)) \supset U \tag{1}$$

holds, then the entropy of $g_t(x) = g(x) + t$ is monotone in t.

Application of the theorem

- A new proof for the family $g_t(x) = t + x^{2d}$, $d \in \mathbb{Z}^+$: Take V to be large disk centered at zero.
- Families

$$g_t(x) = be^{-1/|x|^\ell} + t, \ \ t \in [0, \beta],$$

  where $\ell \geq 1$, $b > 2(e\ell)^{1/\ell}$ and $2\beta = e^{1/\beta^\ell}$. of unimodal maps. (with flat critical points)

Main steps of proof

- Consider $f := g_{t_0}$ satisfying $f^q(0) = 0$, $f^j(0) \in U \setminus \{0\}$, $1 \le j < q$. It suffices to show

$$\frac{\frac{d}{dt} g_t^q(0)\big|_{t=t_0}}{Df^{q-1}(f(0))} > 0.$$

- Let $P = \{f^j(0) : 0 \le j < q\}$. A *holomorphic motion* of P over $\mathbb{D}_r$ is a family of injections $h_\lambda : P \to \mathbb{C}$, $\lambda \in \mathbb{D}_r$, such that $h_\lambda(0) \equiv 0$, $h_0 = id_P$, and $\lambda \mapsto h_\lambda(p)$ is holomorphic for each $p \in P$.
- A holomorphic motion of P is called *admissible* if $h_\lambda(p) \in U$ for all $p \in P \setminus \{0\}$. Note that any holomorphic motion, when restricted to a small disk, is admissible.

- (*The lifting property*) If h_λ is an admissible holomorphic motion of P over $\mathbb{D}_r$, then it has a *lift* $\widehat{h}_\lambda$, which is again a holomorphic motion of P over $\mathbb{D}_r$,

$$g(\widehat{h}_\lambda(p)) = h_\lambda(f(p)) - h_\lambda(f(0)) \in V \setminus \{0\}$$

holds for all $p \in P \setminus \{0\}$ and $\lambda \in \mathbb{D}_r$.

- An admissible holomorphic motion is called *asymptotic invariant of order m*, if

$$\widehat{h}_\lambda(p) - h_\lambda(p) = O(\lambda^{m+1}) \text{ as } \lambda \to 0.$$

- $\exists$ a non-trivial holomorphic motion asymptotically invariant of oder $m \Leftrightarrow g_t^q(0) = O(|t - t_0|^{m+1})$ as $t \to t_0$.

- (*Averaging process* improving asymptotical invariance order) If there is an admissible holomorphic motion h_λ asymptotically invariant of order m, then there is another one which is asymptotically invariant of order $m+1$.
  Indeed, letting h_λ^n be the successive lifts of $h_\lambda^0 = h_\lambda$, it suffices to choose a convergent subsequence

$$H_\lambda := \lim_{n_k \to \infty} \frac{1}{n_k} \sum_{j=1}^{n_k} h_\lambda^j.$$

- As $g_t^q(0)$ is a holomorphic function not-identically zero, we obtain a contradiction if transversality fails.
- A deformation argument shows the sign in the inequality.

- The lift process is a variation of Thurston's pull back algorithm, modified for maps which are locally defined. The lift property means that the σ^n is well-defined in a neighborhood of $[id]$ in the Teichmüller space $\mathcal{T}_P$.
- Adam Epstein seems to have a different approach, using some variation of the Thurston's pull-back algorithm, to obtain transversality. It is not clear how to apply his method to locally defined maps.

The hyperbolic case

Recently, we are able to extend the argument to the hyperbolic and parabolic case (with infinite a post-critical orbit), extending a classical result of Douady-Hubbard-Sullivan.

Theorem (Levin-Shen-van Strien)

Let g be as in the theorem above. Assume that for some t_0, $f = g_{t_0}$ has an attracting cycle attracting the critical point 0. Let $\lambda(t)$ denote the multiplier of the attracting periodic orbits of g_t for t close to t_0. Then $\lambda'(t_0) \neq 0$.

- The main challenge in the post-critically infinite case is the Averaging Process. While H_λ exists, it is not clear whether it is a holomorphic motion (for $|\lambda|$ small).
- The solution is as follows: If transversalilty fails, then the coholomogous equation

$$\alpha(f(z)) - f'(z)\alpha(z) = v(z)$$

  has a solution holomorphic near the attracting periodic orbit.
- So we can construct a holomorphic motion $h_\lambda(z)$, which is also holomorphic in z near the attracting periodic orbit. The corresponding $H_\lambda(z)$ will be a holomorphic motion by the Koebe distortion principle.

Unfortunately, for the interesting family $x \mapsto t + |x|^\ell$, we only have

Theorem

For each positive integer q, there exists $\ell_0(q) > 1$ such that if $\ell > \ell_0(q)$ and if 0 is of period q under the map $f_{a,\ell}(x) = a - |x|^\ell$, then

$$\sum_{n=0}^{q-1} \frac{1}{Df_a^n(a)} > 0.$$

Corollary

There exists $\varepsilon(\ell) > 0$ such that $\varepsilon(\ell) \to 0$ as $\ell \to \infty$ such that

$$\sup_{a>b} h_{top}(f_{b,\ell}) - h_{top}(f_{a,\ell}) > -\varepsilon(\ell).$$

Thank you for your attention!