A Brief History of Hecke algebras

Peng Shan

YMSC, Tsinghua University

AMSS, 2018

Disclaimer:

This talk does not, by any means, attempt to give an account for the history of Hecke algebras.

It is merely a personal attempt to understand why we should study Hecke algebras, and how does it influenced the development of representation theory.

What is a Hecke algebra ?

Coxeter system

is a pair (W, S) such that

$$W = \langle s \in S \mid \underbrace{sts\ldots}_{m_{st} \text{ terms}} = \underbrace{tst\ldots}_{m_{ts} \text{ terms}},\ s^2 = 1 \rangle,$$

where $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$.

Coxeter system

is a pair (W, S) such that

$$W = \langle s \in S \mid \underbrace{sts\ldots}_{m_{st}\ \text{terms}} = \underbrace{tst\ldots}_{m_{ts}\ \text{terms}},\ s^2 = 1 \rangle,$$

where $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$.

Example

The symmetric group

$$\mathfrak{S}_n = \mathrm{Perm}\{1, 2, \ldots, n\}$$

and $S = \{s_i = (i, i+1),\ 1 \leqslant i \leqslant n-1\}$ form a Coxeter system.

Coxeter system

is a pair (W, S) such that

$$W = \langle s \in S \mid \underbrace{sts\ldots}_{m_{st}\text{ terms}} = \underbrace{tst\ldots}_{m_{ts}\text{ terms}},\ s^2 = 1 \rangle,$$

where $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$.

Example

The symmetric group

$$\mathfrak{S}_n = \mathrm{Perm}\{1, 2, \ldots, n\}$$

and $S = \{s_i = (i, i+1),\ 1 \leqslant i \leqslant n-1\}$ form a Coxeter system.

Other examples: Weyl groups, affine Weyl groups, reflection groups...

Hecke algebra – Definition

The group algebra

$$\mathbb{Z}W = \frac{\mathbb{Z}\langle s \in S \rangle}{\Big(\underbrace{sts\ldots}_{m_{st} \text{ terms}} = \underbrace{tst\ldots}_{m_{ts} \text{ terms}} \;, \quad s^2 = 1 \Big)}.$$

Hecke algebra – Definition

The group algebra

$$\mathbb{Z}W = \frac{\mathbb{Z}\langle s \in S\rangle}{\left(\underbrace{sts\ldots}_{m_{st}\text{ terms}} = \underbrace{tst\ldots}_{m_{ts}\text{ terms}}, \quad s^2 = 1\right)}.$$

Hecke algebra $\mathcal{H}(W, S)$

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{\mathbb{Z}[v^{\pm 1}]\langle H_s \in S\rangle}{\left(\underbrace{H_s H_t H_s\ldots}_{m_{st}\text{ terms}} = \underbrace{H_t H_s H_t\ldots}_{m_{ts}\text{ terms}}, \quad (H_s + v)(H_s - v^{-1}) = 0\right)}.$$

Hecke algebra is a deformation of $\mathbb{Z}W$

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{\mathbb{Z}[v^{\pm 1}]\langle H_s \in S\rangle}{\left(\underbrace{H_s H_t H_s \ldots}_{m_{st} \text{ terms}} = \underbrace{H_t H_s H_t \ldots}_{m_{ts} \text{ terms}}, \quad (H_s + v)(H_s - v^{-1}) = 0\right)}.$$

- $\mathcal{H}\mid_{v=1} = \mathbb{Z}W$

Hecke algebra is a deformation of $\mathbb{Z}W$

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{\mathbb{Z}[v^{\pm 1}]\langle H_s \in S\rangle}{\left(\underbrace{H_s H_t H_s \ldots}_{m_{st} \text{ terms}} = \underbrace{H_t H_s H_t \ldots}_{m_{ts} \text{ terms}}, \quad (H_s + v)(H_s - v^{-1}) = 0\right)}.$$

- $\mathcal{H}\mid_{v=1} = \mathbb{Z}W$
- For $w \in W$, choose a reduced expression $w = s_1 \ldots s_n$, then

$$H_w = H_{s_1} \ldots . H_{s_n} \in \mathcal{H}$$

  is independent of the choice of expression.

$$\mathcal{H} = \bigoplus_{w \in W} \mathbb{Z}[v^{\pm 1}] H_w$$

  is a free $\mathbb{Z}[v^{\pm 1}]$-module of rank $|W|$.

Two questions

1. Why deform ?

Two questions

1. Why deform ?
2. Why $(H_s + v)(H_s - v^{-1}) = 0$?

Hecke algebra from a geometric perspective...

$W = \mathfrak{S}_n$
$G = GL_n \supset B = \{$ upper triangular matrices$\}$

Hecke algebra from a geometric perspective...

$W = \mathfrak{S}_n$
$G = GL_n \supset B = \{$ upper triangular matrices$\}$

Hecke algebra $\mathcal{H}(G, B)$

$$\mathcal{H}(G,B) \stackrel{\text{def}}{=} \left\{ f : G(\mathbb{F}_q) \to \mathbb{Q} \;\middle|\; \begin{array}{l} f(b_1 z b_2) = f(z), \\ \forall b_1, b_2 \in B(\mathbb{F}_q),\ z \in G(\mathbb{F}_q) \end{array} \right\}$$

Hecke algebra from a geometric perspective...

$W = \mathfrak{S}_n$
$G = GL_n \supset B = \{$ upper triangular matrices$\}$

Hecke algebra $\mathcal{H}(G, B)$

$$\mathcal{H}(G, B) \stackrel{\text{def}}{=} \left\{ f : G(\mathbb{F}_q) \to \mathbb{Q} \,\middle|\, \begin{array}{l} f(b_1 z b_2) = f(z), \\ \forall b_1, b_2 \in B(\mathbb{F}_q),\ z \in G(\mathbb{F}_q) \end{array} \right\}$$

equipped with the convolution product:

$$f * g(z) = \frac{1}{B(\mathbb{F}_q)} \sum_{xy=z} f(x)g(y)$$

Hecke algebra from a geometric perspective...

$W = \mathfrak{S}_n$
$G = GL_n \supset B = \{$ upper triangular matrices$\}$

Hecke algebra $\mathcal{H}(G, B)$

$$\mathcal{H}(G,B) \stackrel{\text{def}}{=} \left\{ f : G(\mathbb{F}_q) \to \mathbb{Q} \,\middle|\, \begin{array}{l} f(b_1 z b_2) = f(z), \\ \forall b_1, b_2 \in B(\mathbb{F}_q),\ z \in G(\mathbb{F}_q) \end{array} \right\}$$

equipped with the convolution product:

$$f * g(z) = \frac{1}{B(\mathbb{F}_q)} \sum_{xy=z} f(x)g(y)$$

NB: This definition applies to any Weyl group.

Isomorphism

View elements in $\mathfrak{S}_n$ as permutation matrices $\Rightarrow$

$$W \xrightarrow{\sim} B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q).$$

Hence $\mathcal{H}(G,B) = \bigoplus_{w\in W} \mathbb{Q} T_w$.

Isomorphism

View elements in $\mathfrak{S}_n$ as permutation matrices $\Rightarrow$

$$W \xrightarrow{\sim} B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q).$$

Hence $\mathcal{H}(G, B) = \bigoplus_{w \in W} \mathbb{Q} T_w$.

Theorem

There is an algebra isomorphism

$$\mathcal{H}(G, B) \xrightarrow{\sim} \mathcal{H}(W, S) \otimes_{\mathbb{Z}} \mathbb{Q}\,|_{v=q^{-1/2}}$$
$$T_w \mapsto v^{-\ell(w)} H_w.$$

Example SL_2

Let $\mathbf{k}$ be a field.

$$W = \mathfrak{S}_2 = \left\{ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$G(\mathbf{k}) = SL_2(\mathbf{k}) \curvearrowright \mathbf{k}^2 = \mathbf{k}v_1 \oplus \mathbf{k}v_2$$

Example SL_2

Let $\mathbf{k}$ be a field.

$$W = \mathfrak{S}_2 = \left\{ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$G(\mathbf{k}) = SL_2(\mathbf{k}) \curvearrowright \mathbf{k}^2 = \mathbf{k}v_1 \oplus \mathbf{k}v_2$$

$$B(\mathbf{k}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = Stab(\langle v_1 \rangle)$$

$$G(\mathbf{k})/B(\mathbf{k}) = \mathbb{P}^1(\mathbf{k})$$

Example SL_2

Let $\mathbf{k}$ be a field.

$W = \mathfrak{S}_2 = \left\{ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

$G(\mathbf{k}) = SL_2(\mathbf{k}) \curvearrowright \mathbf{k}^2 = \mathbf{k}v_1 \oplus \mathbf{k}v_2$

$B(\mathbf{k}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = Stab(\langle v_1 \rangle)$

$$G(\mathbf{k})/B(\mathbf{k}) = \mathbb{P}^1(\mathbf{k})$$

$T(\mathbf{k}) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \curvearrowright G(\mathbf{k})/B(\mathbf{k})$ with 2 fixed points:

$$e = \langle v_1 \rangle, \quad s = \langle v_2 \rangle = s\langle v_1 \rangle$$

Example SL_2

Let $\mathbf{k}$ be a field.

$W = \mathfrak{S}_2 = \left\{ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

$G(\mathbf{k}) = SL_2(\mathbf{k}) \curvearrowright \mathbf{k}^2 = \mathbf{k}v_1 \oplus \mathbf{k}v_2$

$B(\mathbf{k}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = Stab(\langle v_1 \rangle)$

$$G(\mathbf{k})/B(\mathbf{k}) = \mathbb{P}^1(\mathbf{k})$$

$T(\mathbf{k}) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \curvearrowright G(\mathbf{k})/B(\mathbf{k})$ with 2 fixed points:

$$e = \langle v_1 \rangle, \quad s = \langle v_2 \rangle = s\langle v_1 \rangle$$

$B(\mathbf{k})$-orbits on $G(\mathbf{k})/B(\mathbf{k})$ are:

$$B(\mathbf{k})e = \{\langle v_1 \rangle\}, \quad B(\mathbf{k})s = \{\langle av_1 + v_2 \rangle \mid a \in \mathbf{k}\} \simeq \mathbf{k}.$$

Example SL_2

We have $B(\mathbf{k})\backslash G(\mathbf{k})/B(\mathbf{k}) = \{B(\mathbf{k})e,\ B(\mathbf{k})sB(\mathbf{k})\}$.

Example SL_2

We have $B(\mathbf{k})\backslash G(\mathbf{k})/B(\mathbf{k}) = \{B(\mathbf{k})e,\ B(\mathbf{k})sB(\mathbf{k})\}$.

Convolution algebra

$\mathcal{H}(SL_2, B) = \mathbb{Q}T_e \oplus \mathbb{Q}T_s$,

Example SL_2

We have $B(\mathbf{k})\backslash G(\mathbf{k})/B(\mathbf{k}) = \{B(\mathbf{k})e,\ B(\mathbf{k})sB(\mathbf{k})\}$.

Convolution algebra

$\mathcal{H}(SL_2, B) = \mathbb{Q}T_e \oplus \mathbb{Q}T_s$,
recall $f * g(z) = \frac{1}{B(\mathbb{F}_q)} \sum_{xy=z} f(x)g(y)$

- $B(\mathbb{F}_q)$ is the orbit of identity
  $\Rightarrow$ T_e is the identity for $*$.
- Counting fibres of the surjective map

$$B(\mathbb{F}_q)sB(\mathbb{F}_q) \times B(\mathbb{F}_q)sB(\mathbb{F}_q) \to G(\mathbb{F}_q), \quad (x, y) \mapsto xy,$$

$\Rightarrow$ $T_s * T_s = (q-1)T_s + qT_e$.
$\Rightarrow$ $(T_s + 1)(T_s - q) = 0$.

Change $q \mapsto v^{-2}$, $T_s \mapsto v^{-1} H_s$,

$$\Rightarrow \quad (H_s + v)(H_s - v^{-1}) = 0.$$

So we have checked

$$\mathcal{H}(SL_2, B) \xrightarrow{\sim} \mathcal{H}(\mathfrak{S}_2) \otimes_{\mathbb{Z}} \mathbb{Q}\,|_{v=q^{-1/2}}$$
$$T_w \mapsto v^{-\ell(w)} H_w.$$

Change $q \mapsto v^{-2}$, $T_s \mapsto v^{-1} H_s$,

$$\Rightarrow \quad (H_s + v)(H_s - v^{-1}) = 0.$$

So we have checked

$$\mathcal{H}(SL_2, B) \xrightarrow{\sim} \mathcal{H}(\mathfrak{S}_2) \otimes_{\mathbb{Z}} \mathbb{Q}\,|_{v=q^{-1/2}}$$
$$T_w \mapsto v^{-\ell(w)} H_w.$$

Answer for Question 2

Convolution on functions on $B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)$ provides a natural explication for the relation $(H_s + v)(H_s - v^{-1}) = 0$.

Other perspectives

Number-theoretic perspective

Hecke operators on modular forms
$\rightsquigarrow$ convolution product
$\rightsquigarrow$ spherical affine Hecke algebra

Other perspectives

Number-theoretic perspective

Hecke operators on modular forms
$\rightsquigarrow$ convolution product
$\rightsquigarrow$ spherical affine Hecke algebra

Topological perspective

W complex reflection group with reflection representation V
$\mathcal{H}$ is a quotient of the braid group $\pi_1(V^{reg}/W)$
The quadratic relation arises from monodromy of certain KZ equations...

Categorifications of Hecke algebras

By categorification, we mean...

A monoidal category $\mathcal{C}$,
equipped with an automorphims ϑ,
such that

$$K_0(\mathcal{C}) \cong \mathcal{H} \qquad \text{(as algebras,)}$$
$$[\vartheta] \leftrightarrow v$$

Three categorifications of Hecke algebras

1979	Kazhdan-Lusztig	Perverse sheaves on G/B
$\downarrow$		
1990	Soergel	Special bimodules
$\downarrow$		
2013	Elias-Williamson	Diagram categories.

1st Categorification

Kazhdan-Lusztig Theory

Kazhdan-Lusztig Theory

Grothendieck sheaf-function dictionary

$X/\mathbb{F}_q$ algebraic variety, $\mathrm{Frob} \curvearrowright X$

$$\mathrm{Tr} : Sh(X) \to Fun\big(X(\mathbb{F}_q)\big), \quad \mathcal{F} \mapsto \sum_{x \in X(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Frob}, \mathcal{F}_x)$$

$\rightsquigarrow \mathrm{Tr} : D^b_c(X) \to Fun\big(X(\mathbb{F}_q)\big).$

Kazhdan-Lusztig Theory

Grothendieck sheaf-function dictionary

$X/\mathbb{F}_q$ algebraic variety, $\mathrm{Frob} \curvearrowright X$

$$\mathrm{Tr} : Sh(X) \to Fun\big(X(\mathbb{F}_q)\big), \quad \mathcal{F} \mapsto \sum_{x \in X(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Frob}, \mathcal{F}_x)$$

$\rightsquigarrow \mathrm{Tr} : D^b_c(X) \to Fun\big(X(\mathbb{F}_q)\big)$.

Apply to $X = G/B$

$$\mathrm{Tr} : D^b_B(G/B) \to \mathcal{H}(G, B)$$

compatible with convolution product.

Geometry of G/B

Bruhat decomposition

$$X = \bigsqcup_{w \in W} X_w, \quad X_w = BwB/B \cong \mathbb{A}^{\ell(w)}$$

Schubert variety $\overline{X_w} = \bigsqcup_{y \preccurlyeq w} X_y$, projective, singular in general.

Geometry of G/B

Bruhat decomposition

$$X = \bigsqcup_{w \in W} X_w, \quad X_w = BwB/B \cong \mathbb{A}^{\ell(w)}$$

Schubert variety $\overline{X_w} = \bigsqcup_{y \preccurlyeq w} X_y$, projective, singular in general.

Distinguish objects

For each $w \in W$, let $j_w : X_w \hookrightarrow X$, $L_w = \underline{\overline{\mathbb{Q}}_\ell}[\ell(w)] \in D^b_B(X_w)$.

$$\Delta_w = (j_w)_! L_w, \quad IC_w = (j_w)_{!*} L_w, \quad \nabla_w = (j_w)_* L_w$$

are three distinguished objects in $D^b_B(X_w)$, whose cohomology compute H^*_c, IH^*, H^* of $\overline{X_w}$.

The map Tr

$$\mathrm{Tr} : D^b_B(G/B) \to \mathcal{H}(G, B)$$

The map Tr

$$\mathrm{Tr} : D^b_B(G/B) \to \mathcal{H}(G, B)$$

not hard to check

$$\mathrm{Tr}(\Delta_w) = H_w, \quad \mathrm{Tr}(\nabla_w) = H^{-1}_{w^{-1}},$$

The map Tr

$$\mathrm{Tr} : D^b_B(G/B) \to \mathcal{H}(G, B)$$

not hard to check

$$\mathrm{Tr}(\Delta_w) = H_w, \quad \mathrm{Tr}(\nabla_w) = H^{-1}_{w^{-1}},$$

Question

$$\mathrm{Tr}(IC_w) = ??$$

Canonical bases

Bar involution

is the ring homomorphism $\mathcal{H} \to \mathcal{H}$, $x \mapsto \bar{x}$, define by

$$\bar{v} = v^{-1}, \quad \overline{H_s} = H_s^{-1} = H_s + (v - v^{-1}).$$

We have $\overline{H_w} = H_{w^{-1}}^{-1}$.

Canonical bases

Bar involution

is the ring homomorphism $\mathcal{H} \to \mathcal{H}$, $x \mapsto \bar{x}$, define by

$$\bar{v} = v^{-1}, \quad \overline{H_s} = H_s^{-1} = H_s + (v - v^{-1}).$$

We have $\overline{H_w} = H_{w^{-1}}^{-1}$.

Theorem (Kazhdan-Lusztig, 1979)

For any Coxeter system (W, S), there exists unique $\mathbb{Z}[v^{\pm 1}]$-basis $\{C_w\}_{w \in W}$ such that

- $\overline{C_w} = C_w$
- $C_w = H_w + \sum_{y \prec w} h_{y,w} H_y$ *with* $h_{y,w} \in v\mathbb{Z}[v]$.

Canonical bases

Bar involution

is the ring homomorphism $\mathcal{H} \to \mathcal{H}$, $x \mapsto \bar{x}$, define by

$$\bar{v} = v^{-1}, \quad \overline{H_s} = H_s^{-1} = H_s + (v - v^{-1}).$$

We have $\overline{H_w} = H_{w^{-1}}^{-1}$.

Theorem (Kazhdan-Lusztig, 1979)

For any Coxeter system (W, S), there exists unique $\mathbb{Z}[v^{\pm 1}]$-basis $\{C_w\}_{w \in W}$ such that

- $\overline{C_w} = C_w$
- $C_w = H_w + \sum_{y \prec w} h_{y,w} H_y$ *with* $h_{y,w} \in v\mathbb{Z}[v]$.

This basis is called canonical basis, or Kazhdan-Lusztig basis. The coefficients $h_{y,w}$ are called Kazhdan-Lusztig polynomials.

Why deformation?

As we will see later, the canonical basis is a remarkable object. Its characterisation is only possible in the Hecke algebra, not in the group algebra of W. So deformation is crucial here.

Why deformation?

As we will see later, the canonical basis is a remarkable object. Its characterisation is only possible in the Hecke algebra, not in the group algebra of W. So deformation is crucial here.

Example

For $s \in S$, we have

$$C_s = H_s + v.$$

Check $\overline{C_s} = H_s^{-1} + v^{-1} = H_s + v = C_s$.

Theorem (Kazhdan-Lusztig, 1980)

If W is a Weyl group, then

$$\mathrm{Tr}(IC_w) = C_w.$$

Theorem (Kazhdan-Lusztig, 1980)

If W is a Weyl group, then

$$\mathrm{Tr}(IC_w) = C_w.$$

Corollary (Positivity)

If W is a Weyl group, then

$$h_{y,w} \in \mathbb{N}[v].$$

because $h_{y,w} = \mathrm{Tr}(\mathrm{Frob}, (IC_w)_y)$, eigenvalues of Frob are powers of v.

Theorem (Kazhdan-Lusztig, 1980)

If W is a Weyl group, then

$$\mathrm{Tr}(IC_w) = C_w.$$

Corollary (Positivity)

If W is a Weyl group, then

$$h_{y,w} \in \mathbb{N}[v].$$

because $h_{y,w} = \mathrm{Tr}(\mathrm{Frob}, (IC_w)_y)$, eigenvalues of Frob are powers of v.

Kazhdan-Lusztig positivity conjecture

For any Coxeter system, we have

$$h_{y,w} \in \mathbb{N}[v].$$

Impact on representation theory

Impact on representation theory

Characters of simple modules in BGG category $\mathcal{O}$

$\mathfrak{g} = Lie(G_{\mathbb{C}})$ complex semi-simple Lie algebra
$\mathcal{O} = \{$highest weight $\mathfrak{g}$-modules$\}$.

$$\forall \lambda \in \mathfrak{t}^*, \ M(\lambda) = \text{ Verma module}, \quad L(\lambda) = \text{ simple module.}$$

Problem: compute character of $L(\lambda)$.
$\Longleftrightarrow$ compute multiplicity $[M(\mu) : L(\lambda)]$.

Impact on representation theory

Characters of simple modules in BGG category $\mathcal{O}$

$\mathfrak{g} = Lie(G_{\mathbb{C}})$ complex semi-simple Lie algebra
$\mathcal{O} = \{$highest weight $\mathfrak{g}$-modules$\}$.

$$\forall \lambda \in \mathfrak{t}^*, \; M(\lambda) = \text{ Verma module}, \quad L(\lambda) = \text{ simple module.}$$

Problem: compute character of $L(\lambda)$.
$\iff$ compute multiplicity $[M(\mu) : L(\lambda)]$.

Weyl group $W \curvearrowright \mathfrak{t}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Kazhdan-Lusztig conjecture, 1979

For any $y, w \in W$,

$$[M(w \cdot 0) : L(y \cdot 0)] = h_{yw_0, ww_0}|_{v=1}$$

Other $[M(\mu) : L(\lambda)]$ can be deduced from this crucial case.

Kazhdan-Lusztig conjecture, 1979

For any $y, w \in W$,

$$[M(w \cdot 0) : L(y \cdot 0)] = h_{yw_0, ww_0}|_{v=1}$$

This conjecture was solved by Beilinson-Bernstein, Brylinski-Kashiwara, around 1981, by establishing an equivalence

$$Perv_{(B)}(G/B) \cong \mathcal{O}_0.$$

Kazhdan-Lusztig conjecture, 1979

For any $y, w \in W$,

$$[M(w \cdot 0) : L(y \cdot 0)] = h_{yw_0, ww_0}|_{v=1}$$

This conjecture was solved by Beilinson-Bernstein, Brylinski-Kashiwara, around 1981, by establishing an equivalence

$$Perv_{(B)}(G/B) \cong \mathcal{O}_0.$$

Kazhdan-Lusztig conjecture is really remarkable...

- It tells us the structure of Hecke algebras controls representations of Lie algebras
- This conjecture has a lot of variations, including representations of Kac-Moody algebras, quantum groups,...

Impact on representation theory

Modular representations of reductive groups

$G/\mathbf{k}$ reductive group defined over $\mathbf{k} = \overline{\mathbb{F}_p}$
$\mathrm{Rep}_{\mathbf{k}}(G) = \{$ finite dim. algebraic G-representations$\}$

$\forall \lambda$ dominant, $W(\lambda) =$ Weyl module, $L(\lambda) =$ simple module.

Problem: compute character of $L(\lambda)$.
$\iff$ compute multiplicity $[W(\mu) : L(\lambda)]$.

Impact on representation theory

Modular representations of reductive groups

$G/\mathbf{k}$ reductive group defined over $\mathbf{k} = \overline{\mathbb{F}_p}$
$\mathrm{Rep}_{\mathbf{k}}(G) = \{$ finite dim. algebraic G-representations$\}$

$\forall \lambda$ dominant, $W(\lambda) =$ Weyl module, $\quad L(\lambda) =$ simple module.

Problem: compute character of $L(\lambda)$.
$\iff$ compute multiplicity $[W(\mu) : L(\lambda)]$.

Lusztig's conjecture, 1980

Under appropriate assumption on p, the multiplicities $[W(\mu) : L(\lambda)]$ are given by values of Kazhdan-Lusztig polynomials for affine Weyl group at $v = 1$.

Recap

Kazhdan-Lusztig theory provides a successful categorification of $\mathcal{H}$ when W is a Weyl group.

- it is given by perverse sheaves on G/B

Recap

Kazhdan-Lusztig theory provides a successful categorification of $\mathcal{H}$ when W is a Weyl group.

- it is given by perverse sheaves on G/B
- reveals a remarkable basis and a family of important polynomials

Recap

Kazhdan-Lusztig theory provides a successful categorification of $\mathcal{H}$ when W is a Weyl group.

- it is given by perverse sheaves on G/B
- reveals a remarkable basis and a family of important polynomials
- provides deep links to singularities of Schubert varieties and representations of reductive Lie algebras and algebraic groups

Recap

Kazhdan-Lusztig theory provides a successful categorification of $\mathcal{H}$ when W is a Weyl group.

- it is given by perverse sheaves on G/B
- reveals a remarkable basis and a family of important polynomials
- provides deep links to singularities of Schubert varieties and representations of reductive Lie algebras and algebraic groups
- gives rise to the question:

How much of the theory holds for general Coxeter system ?

Soergel Bimodules

Soergel bimodules

Q: How to categorify $\mathcal{H}(W, S)$ without using G/B ?

Soergel bimodules

Q: How to categorify $\mathcal{H}(W,S)$ without using G/B ?

A new look on $D^b_B(G/B)$

$R = H^*_B(\mathrm{pt}) = \overline{\mathbb{Q}}_\ell[\mathfrak{t}] \curvearrowleft W$

$$\begin{aligned} \mathbb{H}^* : \quad D^b_B(G/B) &\to R\mathrm{-gmod-}R, \\ * &\mapsto \otimes \\ [1] &\mapsto \langle 1 \rangle \\ IC_e &\mapsto H^*_B(\{e\}) = R \\ IC_w &\mapsto ??. \end{aligned}$$

The bimodule B_s

For $s \in S$, we have $IC_s = \pi_s^* \circ \pi_{s*}(IC_e)[1]$, for $\pi_s : G/B \to G/P_s$.

$$\mathbb{H}^*(IC_s) = R \otimes_{R^s} R\langle 1\rangle := B_s.$$

The bimodule B_s

For $s \in S$, we have $IC_s = \pi_s^* \circ \pi_{s*}(IC_e)[1]$, for $\pi_s : G/B \to G/P_s$.

$$\mathbb{H}^*(IC_s) = R \otimes_{R^s} R\langle 1 \rangle := B_s.$$

Bott-Samelson resolution

$\forall w \in W$, let $w = s_1 ... s_n$ be a reduced expression,

$$\pi_w : P_{s_1} \times_B ... \times_B P_{s_n}/B \to \overline{BwB/B}$$

is a resolution of singularity.

$$\pi_{w!}(\underline{\overline{\mathbb{Q}}_\ell}[n]) = IC_{s_1} * ... * IC_{s_n}.$$

The bimodule B_s

For $s \in S$, we have $IC_s = \pi_s^* \circ \pi_{s*}(IC_e)[1]$, for $\pi_s : G/B \to G/P_s$.

$$\mathbb{H}^*(IC_s) = R \otimes_{R^s} R\langle 1 \rangle := B_s.$$

Bott-Samelson resolution

$\forall w \in W$, let $w = s_1 ... s_n$ be a reduced expression,

$$\pi_w : P_{s_1} \times_B ... \times_B P_{s_n}/B \to \overline{BwB/B}$$

is a resolution of singularity.

$$\pi_{w!}(\underline{\overline{\mathbb{Q}}_\ell}[n]) = IC_{s_1} * ... * IC_{s_n}.$$

Decomposition Theorem

$$\Rightarrow \qquad \pi_!(\underline{\overline{\mathbb{Q}}_\ell}[n]) = IC_w \oplus \left(\bigoplus_{y \prec w} IC_y \otimes V_y^\bullet \right)$$

In the equality

$$IC_{s_1} * \ldots * IC_{s_n} = IC_w \oplus \left(\bigoplus_{y \prec w} IC_y \otimes V_y^\bullet \right)$$

IC_w is the unique direct factor which does not appear in products of IC_s of smaller length.

In the equality

$$IC_{s_1} * \dots * IC_{s_n} = IC_w \oplus \Big(\bigoplus_{y \prec w} IC_y \otimes V_y^\bullet \Big)$$

IC_w is the unique direct factor which does not appear in products of IC_s of smaller length.

Bott-Samelson bimodules

$$BS(w) := B_{s_1} \otimes_R \dots \otimes_R B_{s_n} = \mathbb{H}^*(IC_{s_1} * \dots * IC_{s_n}).$$

In the equality

$$IC_{s_1} * \ldots * IC_{s_n} = IC_w \oplus \left(\bigoplus_{y \prec w} IC_y \otimes V_y^\bullet \right)$$

IC_w is the unique direct factor which does not appear in products of IC_s of smaller length.

Bott-Samelson bimodules

$$BS(w) := B_{s_1} \otimes_R \ldots \otimes_R B_{s_n} = \mathbb{H}^*(IC_{s_1} * \ldots * IC_{s_n}).$$

Faithfulness of $\mathbb{H}^* \Rightarrow$
$B_w := \mathbb{H}^*(IC_w)$ is the unique direct factor of $BS(w)$ which does not appear in $BS(y)$ for $y \prec w$.

Soergel bimodules

Given any Coxeter system (W, S), and V a faithful real W-rep,

$$W \curvearrowright R = \mathbb{R}[V].$$

Definition

- $B_s = R \otimes_{R^s} R\langle 1 \rangle \in R\mathrm{-gmod-}R$,
- $\mathrm{SBim} = \langle B_s \mid s \in S \rangle_{\simeq,\ \langle \pm 1 \rangle,\ \oplus,\ \otimes,\ \mathrm{Kar}} \subset R\mathrm{-gmod-}R$

Soergel bimodules

Given any Coxeter system (W, S), and V a faithful real W-rep,

$$W \curvearrowright R = \mathbb{R}[V].$$

Definition

- $B_s = R \otimes_{R^s} R\langle 1\rangle \in R\mathrm{-gmod-}R$,
- $\mathrm{SBim} = \langle B_s \mid s \in S\rangle_{\simeq,\ \langle\pm 1\rangle,\ \oplus,\ \otimes,\ \mathrm{Kar}} \subset R\mathrm{-gmod-}R$

The split Grothendieck group $K_0(\mathrm{SBim})$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra.

Soergel bimodules

Given any Coxeter system (W, S), and V a faithful real W-rep,

$$W \curvearrowright R = \mathbb{R}[V].$$

Definition

- $B_s = R \otimes_{R^s} R\langle 1 \rangle \in R\mathrm{-gmod-}R$,
- $\mathrm{SBim} = \langle B_s \mid s \in S \rangle_{\simeq,\ \langle \pm 1 \rangle,\ \oplus,\ \otimes,\ \mathrm{Kar}} \subset R\mathrm{-gmod-}R$

The split Grothendieck group $K_0(\mathrm{SBim})$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra.

Theorem (Soergel, 1990)

There is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$*-algebras*

$$\mathcal{H}(W, S) \xrightarrow{\sim} K_0(\mathrm{SBim}), \qquad C_s \mapsto [B_s].$$

Soergel conjecture

- The inverse isomorphism is given by

$$\begin{aligned} \mathrm{Ch}: \quad K_0(\mathrm{SBim}) &\xrightarrow{\sim} \mathcal{H}(W,S) \\ [M] &\mapsto \sum_{w \in W} \mathrm{gdim}\big(M_{\preccurlyeq w}/M_{\prec w}\big) H_w \end{aligned}$$

Soergel conjecture

- The inverse isomorphism is given by

$$\begin{aligned} \mathrm{Ch}: \quad K_0(\mathrm{SBim}) &\xrightarrow{\sim} \mathcal{H}(W,S) \\ [M] &\mapsto \sum_{w \in W} \mathrm{gdim}\big(M_{\preccurlyeq w}/M_{\prec w}\big) H_w \end{aligned}$$

- Bott-Samelson bimodules $BS(w)$ are defined in the same way.

Soergel conjecture

- The inverse isomorphism is given by

$$\begin{aligned} \mathrm{Ch}: \quad & K_0(\mathrm{SBim}) \xrightarrow{\sim} \mathcal{H}(W,S) \\ & [M] \mapsto \sum_{w \in W} \mathrm{gdim}\big(M_{\preccurlyeq w}/M_{\prec w}\big) H_w \end{aligned}$$

- Bott-Samelson bimodules $BS(w)$ are defined in the same way.
- Soergel proved that $BS(w)$ contains a unique factor B_w which does not appear in $BS(y)$ for $y \prec w$.

Soergel conjecture

- The inverse isomorphism is given by

$$\begin{aligned} \mathrm{Ch}: \quad & K_0(\mathrm{SBim}) \xrightarrow{\sim} \mathcal{H}(W,S) \\ & [M] \mapsto \sum_{w \in W} \mathrm{gdim}\big(M_{\preccurlyeq w}/M_{\prec w}\big) H_w \end{aligned}$$

- Bott-Samelson bimodules $BS(w)$ are defined in the same way.
- Soergel proved that $BS(w)$ contains a unique factor B_w which does not appear in $BS(y)$ for $y \prec w$.

Conjecture (Soergel)

$$\mathrm{Ch}(B_w) = C_w$$

Soergel conjecture

- The inverse isomorphism is given by

$$\begin{aligned} \mathrm{Ch}: \quad K_0(\mathrm{SBim}) &\xrightarrow{\sim} \mathcal{H}(W,S) \\ [M] &\mapsto \sum_{w \in W} \mathrm{gdim}\big(M_{\preccurlyeq w}/M_{\prec w}\big) H_w \end{aligned}$$

- Bott-Samelson bimodules $BS(w)$ are defined in the same way.
- Soergel proved that $BS(w)$ contains a unique factor B_w which does not appear in $BS(y)$ for $y \prec w$.

Conjecture (Soergel)

$$\mathrm{Ch}(B_w) = C_w$$

$\Rightarrow$ Kazhdan-Lusztig positivity conjecture.

Impact on representation theory

- Discovery of Koszul duality for $\mathcal{O}$,

Impact on representation theory

- Discovery of Koszul duality for $\mathcal{O}$,
- Provides new proof for Kazhdan-Lusztig conjecture,

Impact on representation theory

- Discovery of Koszul duality for $\mathcal{O}$,
- Provides new proof for Kazhdan-Lusztig conjecture,
- Such ideas lead to a proof of Lusztig Conjecture for $p \gg 0$ by Andersen-Jantzen-Soergel.

Recap

Soergel bimodules provides a (conjectural) categorification of $\mathcal{H}$ for any Coxeter system

Recap

Soergel bimodules provides a (conjectural) categorification of $\mathcal{H}$ for any Coxeter system

- The formulation is purely algebraic.
  It uses only (W, S) and a faithful representation.

Recap

Soergel bimodules provides a (conjectural) categorification of $\mathcal{H}$ for any Coxeter system

- The formulation is purely algebraic.
  It uses only (W, S) and a faithful representation.
- Soergel only proved his conjecture for Weyl groups, the proof uses geometry of G/B.

Recap

Soergel bimodules provides a (conjectural) categorification of $\mathcal{H}$ for any Coxeter system

- The formulation is purely algebraic.
  It uses only (W, S) and a faithful representation.
- Soergel only proved his conjecture for Weyl groups, the proof uses geometry of G/B.
- Soergel bimodules provides combinatoric model for category $\mathcal{O}$ and other representation categories.

Recap

Soergel bimodules provides a (conjectural) categorification of $\mathcal{H}$ for any Coxeter system

- The formulation is purely algebraic.
  It uses only (W, S) and a faithful representation.
- Soergel only proved his conjecture for Weyl groups, the proof uses geometry of G/B.
- Soergel bimodules provides combinatoric model for category $\mathcal{O}$ and other representation categories.
- Applications to knot invariants.

3rd Categorification

Elias-Williamson Diagram Category

Elias-Williamson Diagram Category

Presentation of SBim by generators and relations

Key point: Compute $\bigoplus_{w,y} \mathrm{Hom}_{\mathrm{SBim}}(BS(w), BS(y))$.

Elias-Williamson Diagram Category

Presentation of SBim by generators and relations

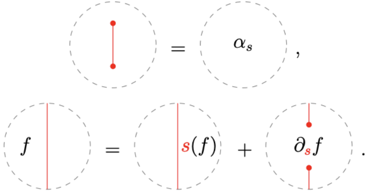
Key point: Compute $\bigoplus_{w,y} \mathrm{Hom}_{\mathrm{SBim}}(BS(w), BS(y))$.

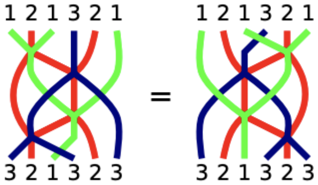
Generators

	deg 1	$B_s \to R$	$f \otimes g \mapsto fg$
	deg 1	$R \to B_s$	$1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$
	deg -1	$B_s B_s \to B_s$	$1 \otimes g \otimes 1 \mapsto \partial_s g \otimes 1$
	deg -1	$B_s \to B_s B_s$	$1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1$
	deg f	$R \to R$	$1 \mapsto f$
	deg 0	$\underbrace{B_s B_t \ldots}_{m_{st}} \to \underbrace{B_t B_s \ldots}_{m_{st}}$	

Diagram Category

Example of relations:





Impact

- Proof of Soergel conjecture (Elias-Williamson, 2014)

Impact

- Proof of Soergel conjecture (Elias-Williamson, 2014)
- Counter examples for Lusztig Character Formula (Williamson, 2015)

Impact

- Proof of Soergel conjecture (Elias-Williamson, 2014)
- Counter examples for Lusztig Character Formula (Williamson, 2015)
- Discovery of p-canonical basis.

Impact

- Proof of Soergel conjecture (Elias-Williamson, 2014)
- Counter examples for Lusztig Character Formula (Williamson, 2015)
- Discovery of p-canonical basis.

...many mysteries remain to be unravelled

Thanks for listening