Teichmüller space and its applications

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For the technique standpoint, the most original creation of the 19th century was the theory of functions of a complex variable....

... The theory of functions, a most fertile branch of mathematics, has been called the mathematical joy of the century. It has also been claimed as one of the most harmonic theories in the abstract sciences.

—— *M. Kline "古今数学思想"*

Recall that:

- A local holomorphic homeomorphism preserves angles.
- A holomorphic function maps each infinitesimal circle to an infinitesimal circle.

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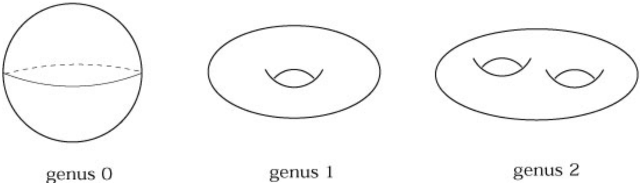
- Σ_g: closed surfaces of genus g;
- $\Sigma_{g,n} = \Sigma_g \setminus \cup_{1\leq i\leq n} D_i$: surfaces of genus g with n boundary components.



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■ Length L

$$L = \int_I \sqrt{E(\frac{du}{dt})^2 + 2F(\frac{du}{dt})(\frac{dv}{dt}) + G(\frac{dv}{dt})^2}$$

■ Angle C

$$\cos \angle(v_1, v_2) = \frac{v_1 \cdot v_2}{|v_1||v_2|}$$

where v_1, v_2 are tangent vectors.

Riemann Surface

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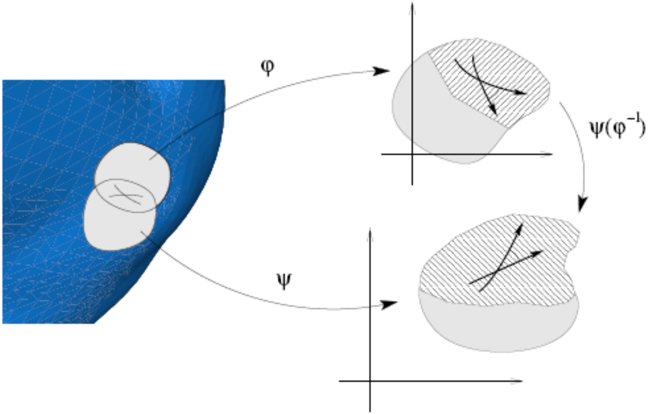
(Σ, C): Riemann introduced angle structure C in a smooth surface Σ

■ Angle Preserving = Holomorphic.

Therefore, we can talk of angles in a Riemann surface.



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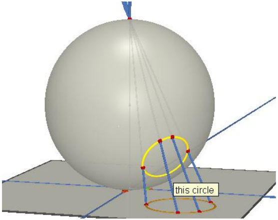
Definition

Two Riemann surfaces $(\Sigma, C), (\Sigma', C')$ are holomorphic (or conformal) equivalent if there is a angle preserving homeomorphism $h : \Sigma \to \Sigma'$.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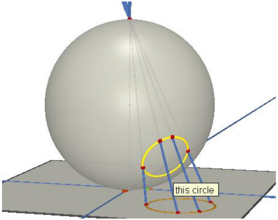
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Denote $\mathbb{D} = \{|z| < 1\}$.

Riemann mapping theorem

Let $U \subsetneq \mathbb{C}$ be a simply-connected domain and $z_0 \in U$. Then there is a unique conformal map $f : \mathbb{D} \to U$ such that $f(0) = z_0$ and $f'(0) > 0$.

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Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. $V \subset \hat{\mathbb{C}}$ is called a circular domain if each component of $\hat{\mathbb{C}} \backslash V$ is a round closed disk or a point.

Köebe uniformization theorem

Suppose that $U \subset \hat{\mathbb{C}}$ is an n-connected domain. Then there is a circular domain V and a conformal map $f : U \to V$. Moreover, f is unique up to Möbius transformations.

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This result was generalized to countably-connected domains (He-Schramm, Ann. Math. 1993).

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- There is a complete conformal metric (spherical metric)

$$\lambda(z)|dz| = \frac{2|dz|}{1 + |z|^2}$$

on the Riemann sphere $\hat{\mathbb{C}}$ with Gaussian curvature $+1$.

- There is a complete conformal metric (flat metric) $|dz|$ on the complex plane $\mathbb{C}$ with Gaussian curvature 0.

- There is a complete conformal metric (hyperbolic metric)

$$\lambda(z)|dz| = \frac{2|dz|}{1 - |z|^2}$$

on the unit disk $\mathbb{D}$ with Gaussian curvature -1.

Theorem

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Let S be a Riemann surface. Then S is conformal equivalent to one of the following:

- The Riemann sphere $\hat{\mathbb{C}}$;
- The complex plane $\mathbb{C}$;
- The cylinder $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$;
- The torus $\mathbb{T} = \mathbb{C}/\Lambda$, where Λ is a lattice generated by $z \to z + 1$ and $z \to z + \omega$, where $\Im\omega > 0$;
- The quotient space $\mathbb{D}/\Gamma$, where $\Gamma \subset Aut(\mathbb{D})$ is a torsion-free Fuchsian group isomorphic to its fundamental group.

Measurable Riemann Mapping

In formulating the measurable Riemann mapping theorem, now we write the metric $Edx^2 + 2Fdxdy + Gdy^2$ in another form:

$$Edx^2 + 2Fdxdy + Gdy^2 = \lambda(z)|dz + \mu d\bar{z}|^2, \quad z = x + yi,$$

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Measurable Riemann Mapping Theorem (Gauss, Morrey, Ahlfors-Bers)

Suppose that $\mu \in L^\infty(\hat{\mathbb{C}})$ and $\|\mu\|_\infty < 1$, Then there exists a unique homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that f fixes $0, 1$ and ∞. It is differentiable a.e., with partial derivatives locally $\partial_z f, \partial_{\bar{z}} f \in L^2$, and

$$\partial_{\bar{z}} f = \mu(z) \cdot \partial_z f.$$

That is, for some function $\lambda(z) > 0$, $f^*(\lambda_0(w)|dw|^2) = \lambda|dz + \mu d\bar{z}|^2$, where $\lambda_0(w)|dw|^2$ is the spherical metric in $\hat{\mathbb{C}}$. f depends continuously and holomorphically on μ.

Such a homeomorphism f is quasi-conformal. Its maximal dilatation is

$$K[f] = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}.$$

Riemann's problem

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> *Riemann's classical problem of moduli is not a problem with a single aim, but rather a program to obtain maximum information about a whole complex of questions which can be viewed from several different angles.*
>
> —— *Lars V. Ahlfors*

This is the first sentence of the beautiful opening address at the International Congress of Mathematicians held in Stockholm in August, 1962. Lars V. Ahlfors was the first winner of the Fields medal (in 1936, together with J. Douglas).

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Definition

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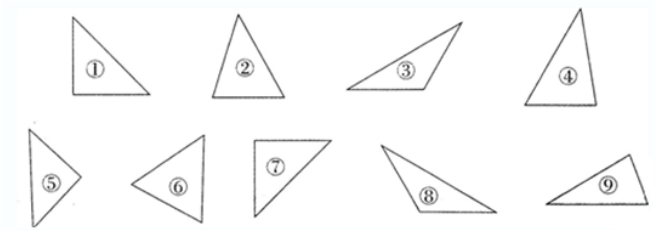
- Algebraic Geometry viewpoint . The set of embedding $\{\Sigma \hookrightarrow \mathbb{CP}^n\}/PGL(n+1, \mathbb{C})$.
- Complex Analysis viewpoint. O. Teichmüller, L. Ahlfors, $\cdots$.

Example

What is the set of isometry classes of all Euclidean triangles?



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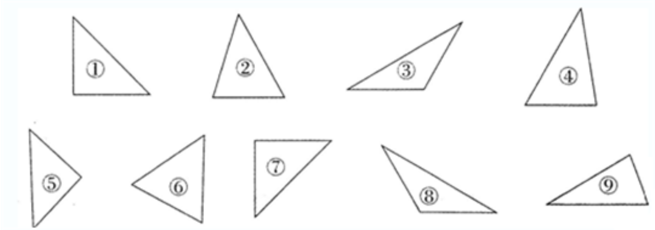

Answer. $\mathbb{R}^2/S_3 \times \mathbb{R}_+$, where S_3 is the order 3 symmetry group acing on $\mathbb{R}^2$.

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Denote by Mod_g the space of homotopy classes of orientation preserving homeomorphism of Σ. It is called the mapping class group (or modular group) of Σ. Then

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Theorem (Teichmüller)

For any two Riemann surfaces $(\Sigma, C), (\Sigma, C')$, there is a unique quasiconformal mapping $h_0 \simeq id : (\Sigma, C) \to (\Sigma, C')$ such that $K[h_0] = \inf_h \log K[h]$, where h runs over all quasiconformal mapping $h \simeq id : (\Sigma, C) \to (\Sigma, C')$.

Using quasiconformal mappings, Teichmüuller introduced a natural metric (Teichmüller metric) d_T on $\mathcal{T}_g$ by

$$d_T((\Sigma, C), (\Sigma, C')) = K[h_0] = \inf_h \log K[h].$$

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■ $\overline{\mathcal{M}}_g^{DM} \backslash \mathcal{M}_g = \{\text{noded Riemann surfaces}\}$.

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Theorem (Nielsen, Thurston, Bers)

Let $\phi \in Mod_g$, either

(1) ϕ has finite order and fixes a point in $\mathcal{T}_g$ (elliptic); or

(2) $\exists$ a set of disjoint simple closed curves $\{\gamma_1, \gamma_2, \cdots, \gamma_n\} \subset \Sigma$ such that $\phi(\{\gamma_i\}) = \{\gamma_i\}$ (reducible); or

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Proof. Define $L(\phi) = \inf_{x \in \mathcal{T}_g} d_T(x, \phi(x))$. Pick a sequence $x_n \in \mathcal{T}_g$ such that $d_T(x_n, \phi(x_n)) \to L(\phi)$. Then
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Theorem (S. Kerckhoff, Nielsen Realization Problem)

Each finite subgroup $G_0 \subset Mod_g$ can be realized as a group of conformal automorphism of a closed Riemann surface (Σ, C).

M is a hyperbolic 3-manifold if it has a complete metric with constant sectional curvature -1. Then $M \cong \mathbb{H}^3/\Gamma$, where Γ is a Kleinian group (discrete subgroup of $PSL(2, \mathbb{C})$) isomorphic to its fundamental group $\pi_1(M)$.

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Given $\phi \in Mod_g$, consider the 3-manifold

$$\Sigma_\phi = \Sigma \times [0,1]/\sim, \quad (z,0) \sim (w,1) \text{ if } w = \phi(z).$$

Its fundamental group is $\pi_1(\Sigma_g) =< t, \pi_1(\Sigma) : t \cdot a \cdot t^{-1} = \phi_*(a), \forall a \in \pi_1(\Sigma) >$.

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Theorem (I. Agol, virtual fibering conjecture)

For any closed hyperbolic 3-manifold M, there is a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M} = \tilde{\Sigma}_\phi$, where $\tilde{\Sigma}$ is a closed surface and $\phi : \tilde{\Sigma} \to \tilde{\Sigma}$ is Pseudo-Anasov.

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In 2015, I. Agol was awarded the 2016 Breakthrough Prize in Mathematics, ”for spectacular contributions to low dimensional topology and geometric group theory, including work on the solutions of the tameness, virtually Haken and virtual fibering conjectures.”

Application-Circle Packing (圆堆积)

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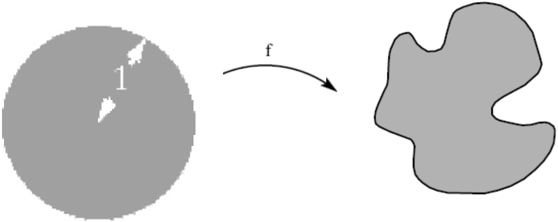
Recall that for any simply connected domain of the complex plane with at least 2 boundary points, we have the following classical Riemann mapping.



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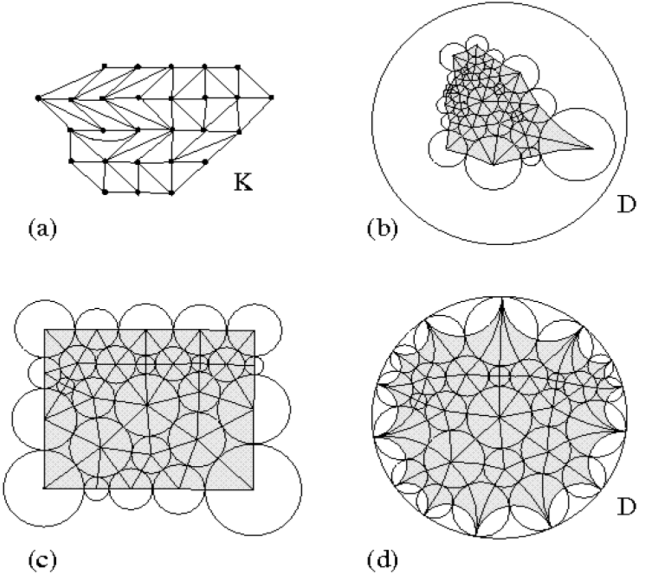
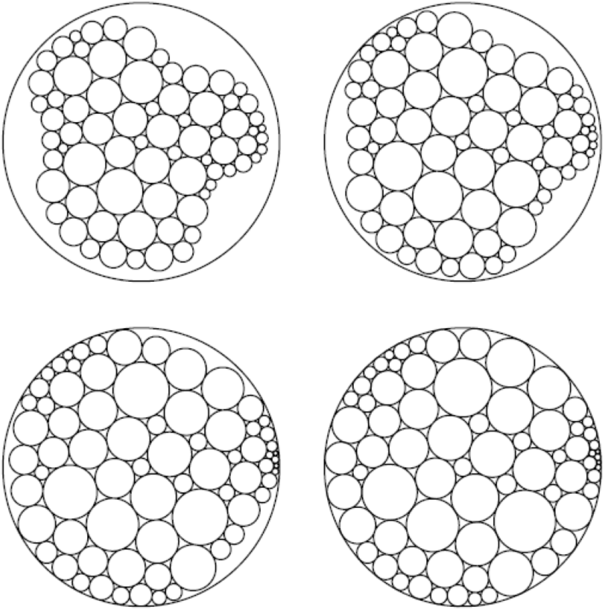
A finite circle packing P on the Riemann sphere $\hat{\mathbb{C}}$ is a configuration of circles with specified patterns of tangency.

The contact graph $G_P = (V, E)$ of such a circle packing P is a graph whose vertices correspond to the circles in the packing, and an edge appears in G_P if and only if the corresponding circles are tangent to each other.

The following are circle packings and their contact graph. Please see some of the illustrations from the web-page of Prof. K. Stenphenson.





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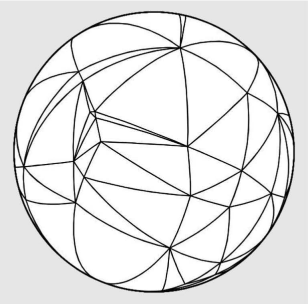
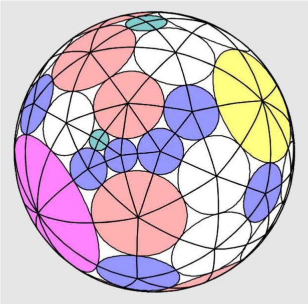
Theorem

For every triangulation G of the Riemann sphere, there is a circle packing P with graph (isomorphic to) G.

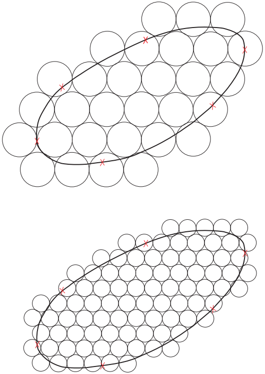
Moreover, P is unique up to Möbius transformation.

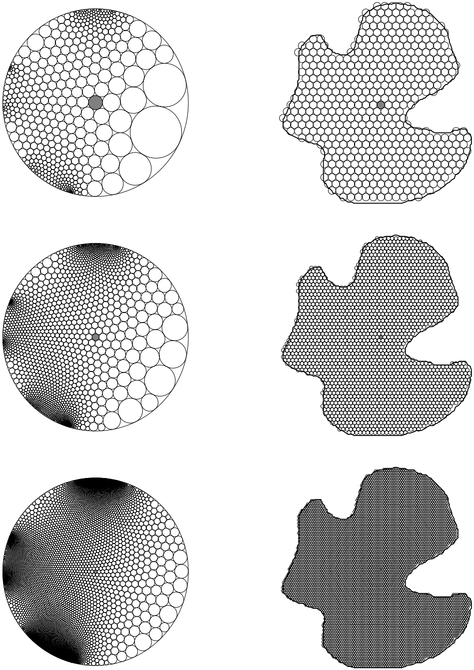




For any domain $D \subset \mathbb{C}$ with at least 2 boundary points, lay down a regular hexagonal packing of circles in $\mathbb{C}$, say each of radius $1/n$.

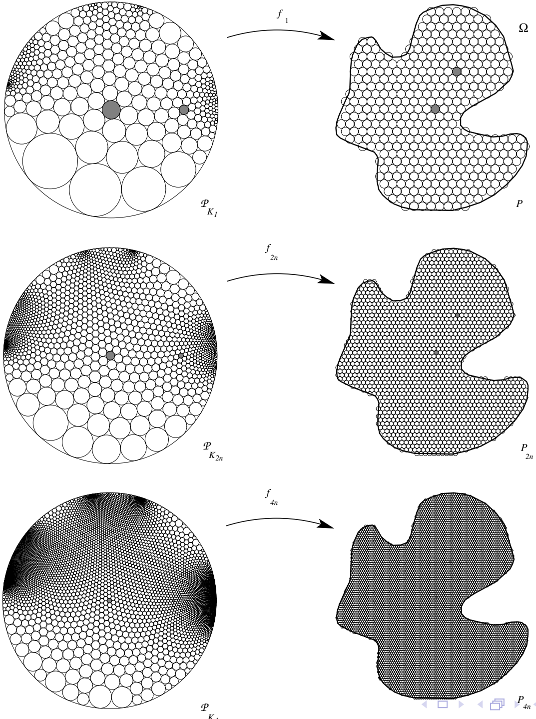
By using the boundary component ∂D like a cookie-cutter, we obtain a circle packing which consists of all the circles intersecting the closed region $\bar{D}$.





By using the results of Thurston, we obtain a circle packing of the Riemann sphere.

Therefore, we obtain a map between the unit disk and the given domain.



f_1
Ω
$\mathcal{P}_{K_1}$
P
f_{2n}
$\mathcal{P}_{K_{2n}}$
P_{2n}
f_{4n}
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Rodin & Sullivan proved this result.

Two circle packings $P,\ Q$ are called equivalent if and only if there exists a Mobiüs transformation $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $T(P) = Q$.

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One of the major problems concerning circle packing is to find necessary and sufficient conditions for two circle packings to be Möbius equivalent.

Now we consider the case that the planar graph $G = (V, E)$ is not a triangulation of $\hat{\mathbb{C}}$.

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Fix a circle packing P_0 with contact graph $G = (V, E)$. For $v \in V$, denote by $D(v)$ the open disk bounded by the circle $P(v)$. For any component I of $\hat{\mathbb{C}} - \cup_{v \in V} D(v)$, the region I has only finitely many boundary components. And each boundary component is a piecewise smooth curve formed by finitely many circular arcs.

The region I is called an interstice of P_0. For each interstice I of P_0, two quasiconformal homeomorphisms $h_1, h_2 : I \to \hat{\mathbb{C}}$ are called equivalent, if $h_2 \circ (h_1)^{-1} : h_1(I) \to h_2(I)$ is isotopic to a conformal homeomorphism g such that for each circular arc or circle $\gamma \subset \partial I$, the homeomorphism g maps $h_1(\gamma)$ onto $h_2(\gamma)$.

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If the interstice I is a k-sided polygon, it follows from the classical Teichmüller theory that the Teichmüller space of I is diffeomorphic to the euclidean space $\mathbb{R}^{k-3}$.

Then we have

Theorem

Assume G is a graph embedded in the Riemann sphere $\hat{\mathbb{C}}$*. Then the space of equivalence classes of circle packings realizing the contact graph* (G, Θ) *can be naturally identified with the Teichmuller space* $\Pi_1^p \mathcal{T}_{I_i}$*, where* $\{I_1, I_2, \cdots, I_p\}$ *are the interstices of* P_0*.*

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■ On the Teichmuller theory of circle patterns, Trans. AMS. 2013 (with Zhengxu He).

Midscribable Problem

Recall Koebe-Andreev-Thurston Theorem.

We can view the unit sphere $\mathbb{S}^2$ as the boundary of the closed unit ball $\mathbb{B}^3 \subset \mathbb{R}^3$. Then Koebe-Andreev-Thurston Theorem is equivalent that the closed unit ball is midscrible.

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That is, given a convex polyhedron, there exists a polyhedron $Q \subset \mathbb{R}^3$ which is combinatorial equivalent to the given one and midscribes the unit ball $\mathbb{B}^3$. Here the word "midscribe" means that all it's edges are tangent to the boundary surface $\partial\mathbb{B}^3$ of the unit ball $\mathbb{B}^3$.

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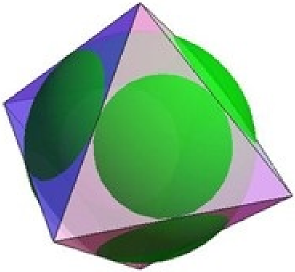


Schulte proposed the question of replacing the unit ball by any other smooth convex body $K \subset \mathbb{R}^3$.

For example, an egg.

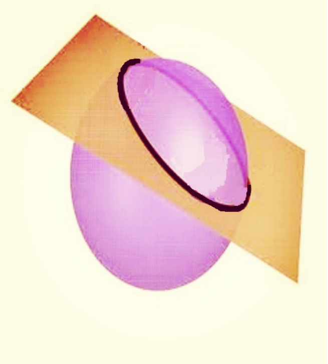
O.Schramm: The answer is ”yes”.

In the paper ”How to cage an egg. Invent Math, 1992”, he proved the existence part of the Midscribable Problem by using a combinatorial method.

Theorem

Given P, K, there exists at least one polyhedron $Q \subset \mathbb{R}^3$ combinatorial equivalent to P and midscribes K.

Let $K \subset R^3$ be a compact strictly convex surface and let $H \subset R^3$ be an affine half space. The intersection $K \cap H$ is then either a topological disk, or a point, or an empty set. For the first case, we call it a K-disk(or circle). The set of all K-disks (or circles) together consists of a packing set.



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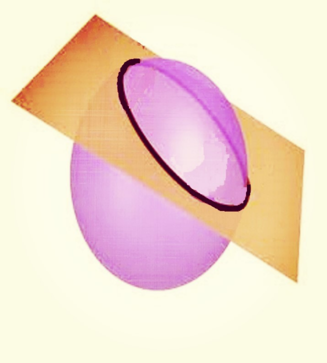

Theorem

The space of equivalence classes of normalized K-circle packings realizing the contact graph G can be naturally identified with the Teichmuller space $\Pi_1^p \mathcal{T}_{I_K,i}$, where $\{I_{K,1}, I_{K,2}, \cdots, I_{K,p}\}$ are the interstices of a fixed K-circle packings $P_{K,0}$.

By using Teichmüller theory, together with method of Differential topology (intersection number theory, transversality, etc), we give an alternative proof of the existence part of the Midscribable Problem.
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Theorem

Let $K \subset \mathbb{R}^3$ be a strictly convex body with smooth boundary. Given a convex polyhedron P, for any triple (p_1, p_2, p_3) with distinct points on ∂K, then there exists a unique normalized convex K-midscribing P-type polyhedron $Q \subset \mathbb{RP}^3$.

■ How many cages midscribe an egg, Invent Math. 2016 (with Ze Zhou).

Other Applications

- Existence of complete hyperbolic metric on closed irreducible, atoroidal, Haken 3-manifolds (W. Thurston, C. McMullen).

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Thanks for your attention!