Quasi-local mass and isoperimetric inequality in General Relativity

April 5， 2017

Isoperimetric problem

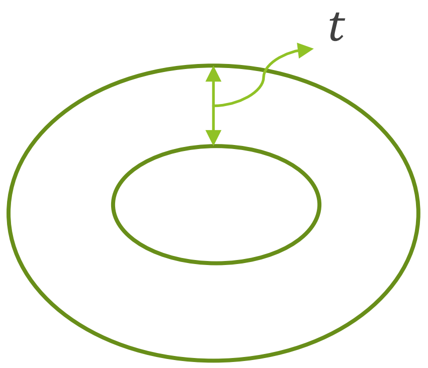
Quasi-local mass in GR

Uniqueness of isoperimetric surfaces with large area in AF manifolds

- Isoperimetric Problem on 2-dim case: Among all closed curves in the plane enclosing a fixed area, which curve (if any) minimizes the perimeter?

- The **isoperimetric inequality** states that: $4\pi A \leq L^2$ the equality holds iff the curve is a circle.
- The isoperimetric inequality in higher dim case: a sphere has the smallest surface area per given volume
- $Area(\partial S) \geq n\, Vol(S)^{\frac{n-1}{n}} Vol(B)^{\frac{1}{n}}$
- Steiner Symmetrisation (1838); Curve shorting flow (1990's), curvature flow

- Various inequalities in convex geometry (Alexandrov-Fenchel inequality)

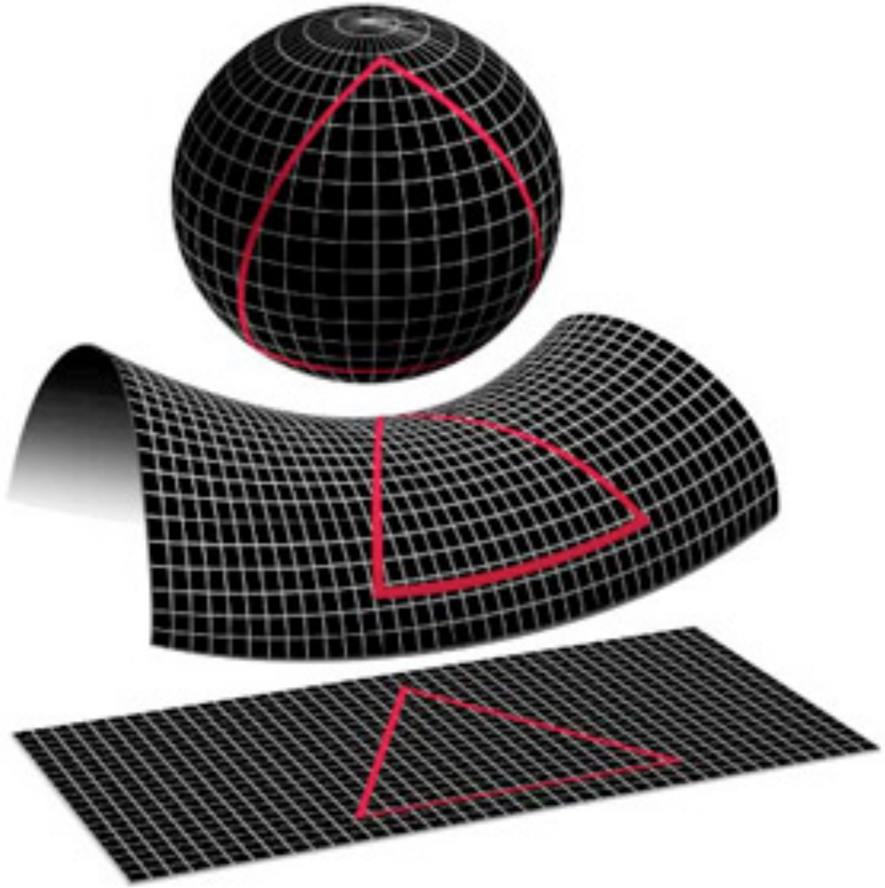


Calculate the certain terms of $Area^{\frac{3}{2}}(\partial\Omega_t) - 6\pi^{\frac{1}{2}}(Vol(\Omega_t))$

$\Longrightarrow \int H \geq 4\pi^{\frac{1}{2}} Area^{\frac{1}{2}}(\partial\Omega)$

- Sobolev inequality : $\left(\int |u|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq n^{-1}\omega_n^{-\frac{1}{n}} \int |\nabla u|$

- Curvature and geometry :



- Higher dim cases:

1. Sectional curvature : R_{ijij}
2. Ricci curvature: $R_{ij} = \sum_k R_{ikjk}$
3. Scalar curvature: $\mathrm{R} = \sum_k R_{kk}$

- Curvature and isoperimetric profile

- Isoperimetric profile: $A(v) = \inf\{Area(\partial\,\Omega)|Vol\,(\Omega) = v\}$
- Example: in R^3, $A(v) = (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}}$

- In a general 3-dim Riemannian manifold, and v is small enough,
- $A(v) = (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}} (1 - \frac{s}{30}\left(\frac{3v}{4\pi}\right)^{\frac{2}{3}} + o(v)^{\frac{2}{3}}), \quad s = \max R$

- Observation: the larger curvature is, the smaller $A(v)$ will be

- Isoperimetric comparison theorem under assumptions of the sectional curvature
- Observation: $A_\kappa(v)$ decreases as function of κ for any fixed v.

1. Generalized Cartan-Hadamard Conjecture : *If M is a complete, simply connected n-manifold with* $K \leq \kappa$, $\kappa \leq 0$, *then every domain* $\Omega \subseteq$ *M satisfies* $Area(\Omega) \geq A_\kappa(Vol(\Omega))$

2. A.Weil(1926), independently by Beckenbach and Radó : Conjecture is true for 2-dim and $\kappa = 0$ case;
3. Conjecture was mentioned by Aubin, and Burago- Zalgaller for $\kappa = 0$ and by Gromov
4. Conjecture was proved in dim=4, $\kappa = 0$ by C.B.Croke, 1984; and in dim=3, $\kappa \leq 0$ by B.Kleiner, 1992.
5. Conjecture is still open for other cases

- Isoperimetric comparison theorem under assumptions of Ricci curvature

- Theorem (Levy(1922), Gromov(1980)): Let (M^{n+1}, g) be a compact Riemannian manifold with $Ric(g) \geq ng$, for any $\Omega \subset M$, with $\frac{Vol(\Omega)}{Vol(M)} = \frac{Vol(B)}{Vol(S^{n+1})}$, here B is a geodesic ball in S^{n+1}, then $\frac{Area(\partial\Omega)}{Vol(M)} \geq \frac{Area(\partial B)}{Vol(S^{n+1})}$
- Remark: $Vol(M) \leq Vol(S^{n+1})$

- Question: Are there any isoperimetric comparison theorem under assumptions of scalar curvature ?

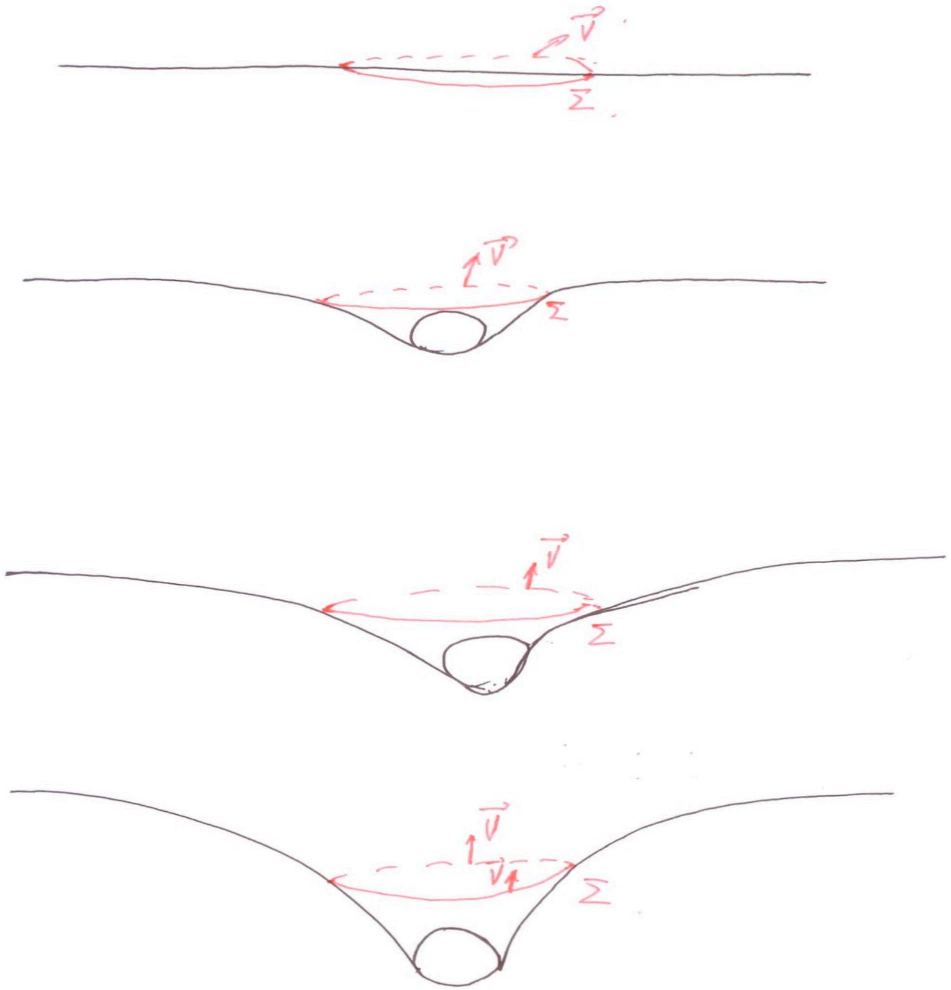
- General relativity: Scalar curvature R $\Longleftrightarrow$ energy density

Einstein 1915:

i) Matter ⟶ Curvature of spacetime ⟶

⟶ Gravity.

ii): $G(g) \triangleq Ric(g) - \frac{R(g)}{2} g = 8\pi T$



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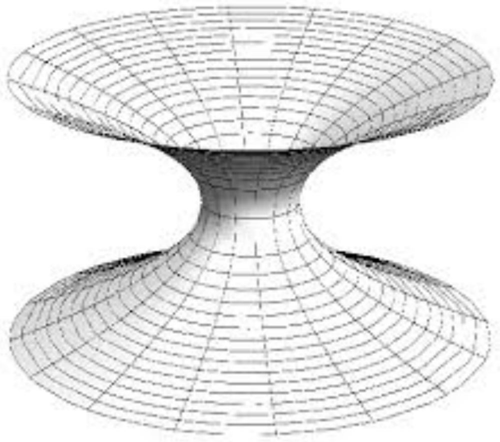
- (M^3, g) is AF if: $M \setminus \mathbf{K} \cong \{x \in R^3 : |x| > \frac{1}{2}\}$ with

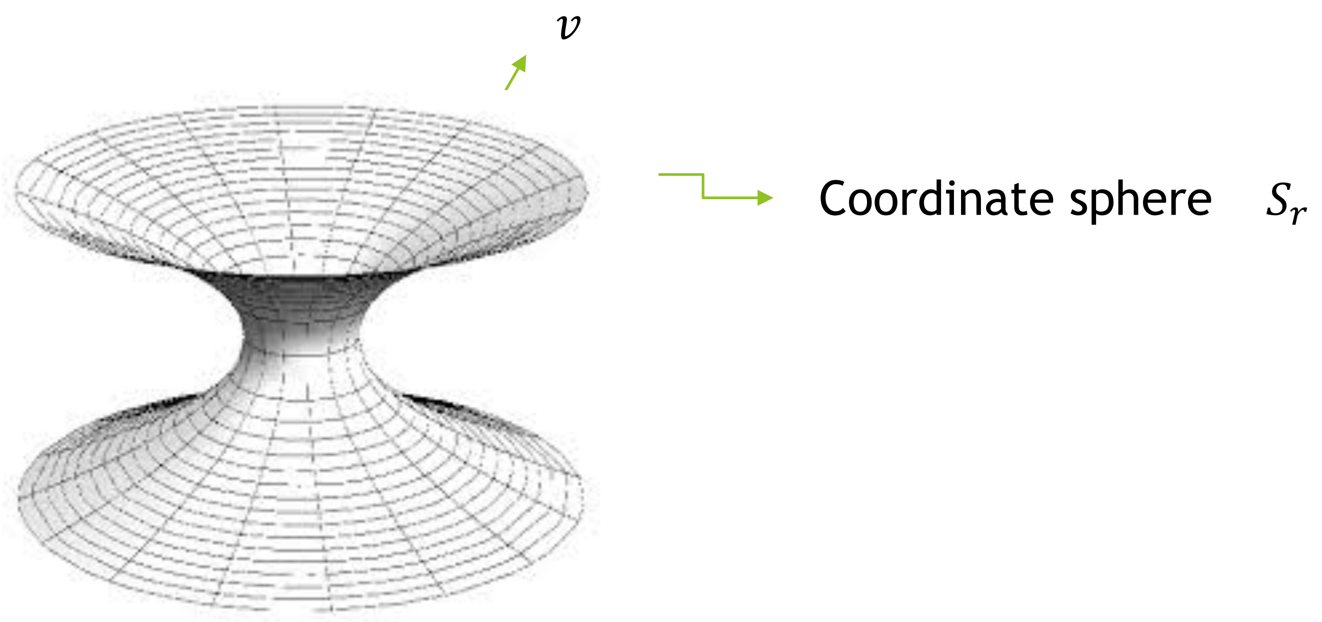
$$g_{ij} = \delta_{ij} + \sigma_{ij}, \qquad |x|^{|\alpha|}(\partial^\alpha \sigma_{ij})(x) = O(|x|^{-\tau}), \quad \tau > \frac{1}{2}$$

- Example: Schwarzschild manifold: $(\mathbf{R}^3 \setminus \{o\}, g)$ with

$$g_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij}\ , \quad r = \frac{m}{2}$$ is horizon



$$m_{ADM}(M) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \left(g_{ii,j} - g_{ij,i}\right) \nu^j d\Sigma_r^0,$$



- Gravitational radiation always carries positive energy from the isolated system, and cannot take away infinity energy, the energy of system must have a fixed lower bound $\Longleftrightarrow$ positive mass theorem

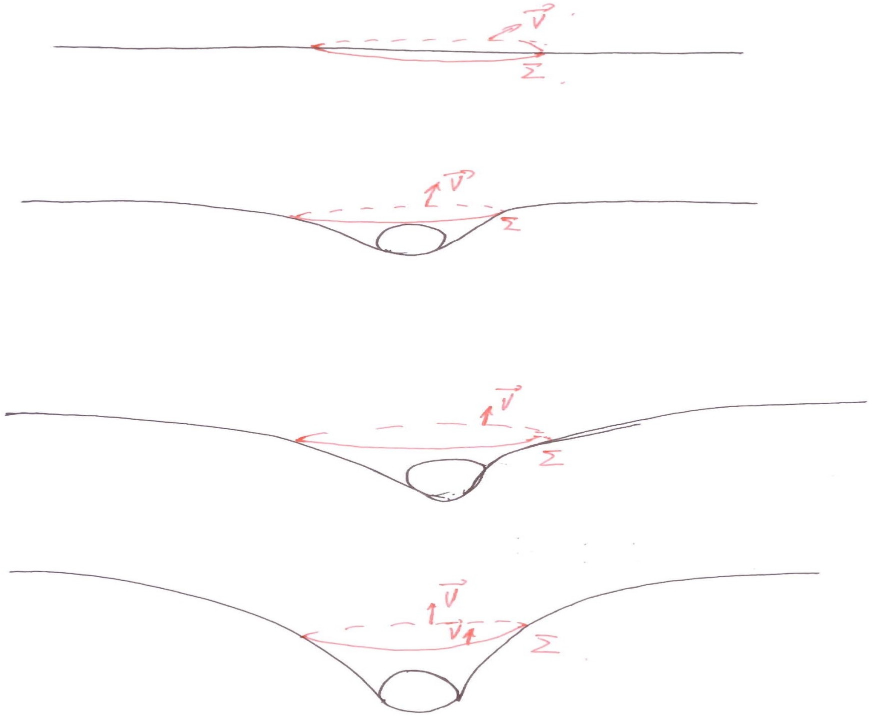
- Intuitive explanations of PMT: If energy density of an isolated system is nonnegative then the total mass is nonnegative, and the total mass is zero if and only if it has no gravitation.

- Scalar curvature R $\Longleftrightarrow$ energy density

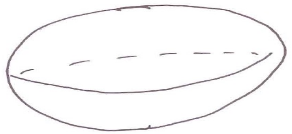
- Positive mass theorem(Schoen-Yau, Witten): Let (M^3, g) be an AF manifold with scalar curvature $R \geq 0$, then $m_{ADM}(M) \geq 0$, equality holds if and only if $(M^3, g) = R^3$

- Observation: When scalar curvature $R \geq 0$, $m_{ADM}(M)$ can be regarded as a kind of deviation of (M^3, g) and R^3.

- How to measure total energy of a isolated system:



- How to measure Curvature of a surface in spacetime ?



$\Sigma \hookrightarrow M$.

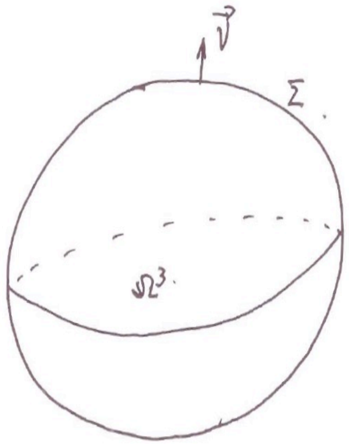
i). Gauss Curvature $\leftarrow$ instrinsic Curvature

ii) Mean Curvature $\leftarrow$ extrinsic Curvature.

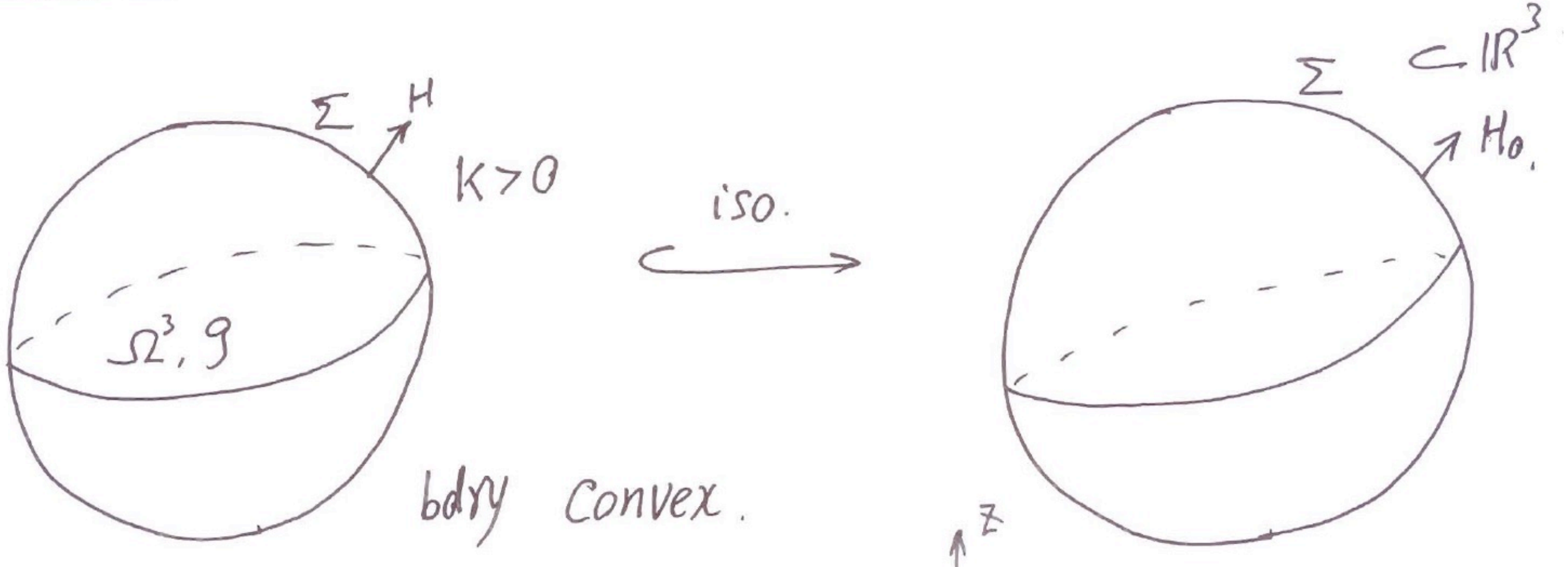
Brief descriptions of quasi local mass

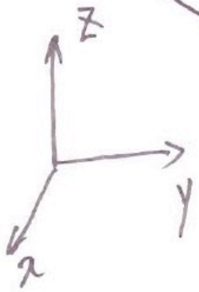
- QL mass : A geometric quantity of 2-dim surface Σ that measures the mass contained in the domain that enclosed by Σ

- QL mass usually depends only on the geometry of Σ , like mean curvature area etc.



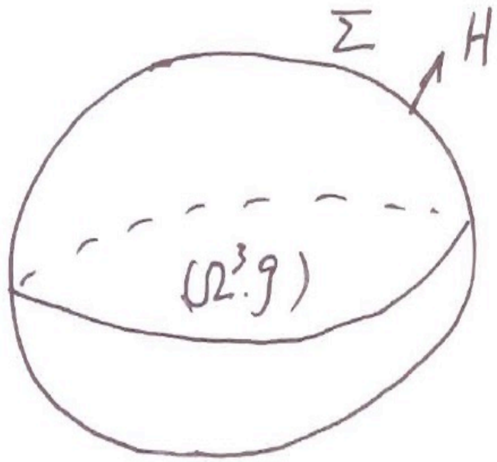
- Brown-York mass





$$\mathcal{M}_{BY}(\Sigma) \triangleq \frac{1}{8\pi}\int_{\Sigma} (H_0 - H)\, d\sigma$$

- Hawking Mass.

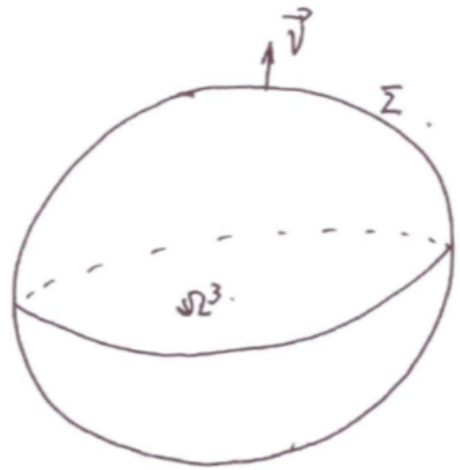


bdry may not convex.

$$m_H(\Sigma) \triangleq \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \int_\Sigma H^2 da\right)$$

- Huisken' Observation(1997): QL Isoperimetric mass

$$\mathcal{M}_{ISO}(\Sigma) = \frac{2}{|\Sigma|}(Vol(\Omega) - \frac{|\Sigma|^{\frac{3}{2}}}{6\sqrt{\pi}})$$



- For any $\Omega \subset R^3$, $M_{ISO}(\Omega) \leq 0$;
- Equality holds if and only if Ω is a ball

- **Positivity of Quasi-Local mass.**

  Mass density $\geq 0 \Longrightarrow$ Quasi-Local mass ≥ 0.

  "=" holds iff domain is "flat"

- Mass density $\longleftrightarrow$ Scalar curvature R.

- **Geometric version of Positivity of Quasi-local mass.**

If the scalar curvature of a compact mfd is nonnegative, then the quasi local mass is nonnegative, and quasi local mass is zero iff the mfd is open domain of $\mathbb{R}^3$.

Positivity of Brown-York mass (Riemannian Case):

Thm (Shi & Tam, 2002): Suppose (Ω^3, g) is a compact mfd with $K>0$, $H>0$. If $R\geq 0$, then

$$\mathcal{M}_{BY}(\partial\Omega) \triangleq \frac{1}{8\pi}\int_{\Sigma}(H_0 - H)\,d\sigma \geq 0$$

$\mathcal{M}_{BY}(\partial\Omega) = 0$ iff $(\Omega^3, g) \xhookrightarrow{iso} \mathbb{R}^3$.

$$\int_{\partial D} H\,ds + \iint_{D} K\,d\sigma = 2\pi\cdot(1-g)$$

$$\int_{\partial \mathcal{D}} H_0\,ds = 2\pi$$

$$\Longrightarrow \int_{\partial D} (H_0 - H)\,ds = 2\pi g + \iint_{D} K\,d\sigma \geq 0.$$

"=" iff $(D^2, g) \subset \mathbb{R}^2$. #.

- Proof of 3-dim case.

<u>Main Idea:</u> Use QS metric introduced by R. Bartnik

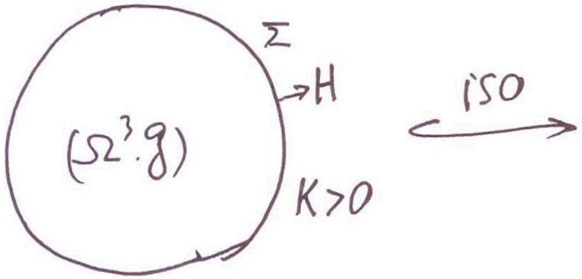
- Brown-York mass and Minkowski inequality, A-F inequality

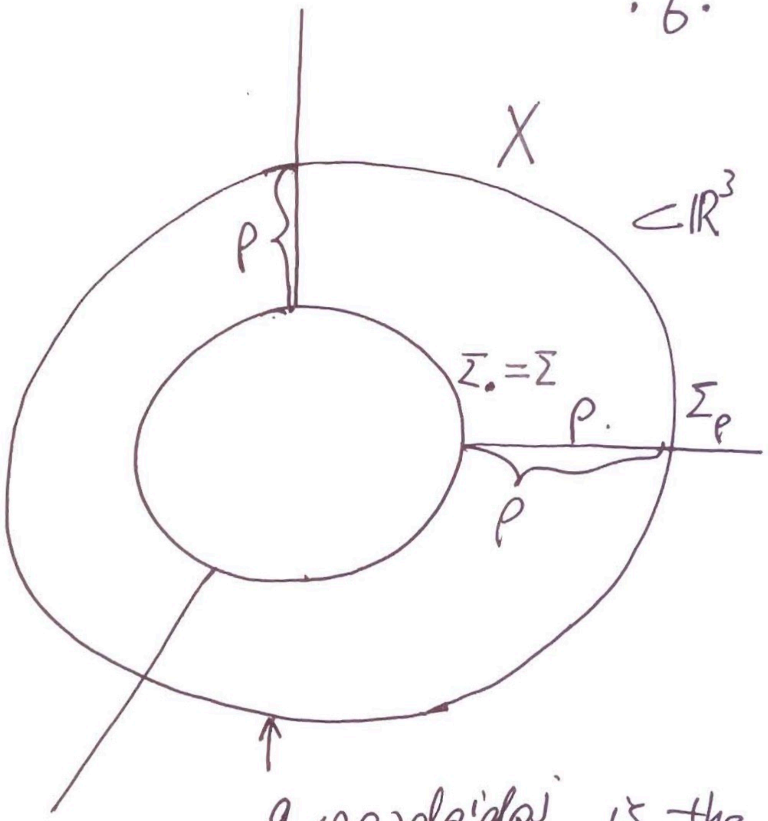
- Positivity of Brown-York mass ⟹ $\int H_0 \geq \int H$

- Minkowski inequality ⟹ $\int H_0 \geq 4\pi^{\frac{1}{2}} Area^{\frac{1}{2}}(\partial\Omega)$

- Both results give a lower bound of $\int H_0$; the arguments of proof are also quite similar.

1.





$$\Sigma_\rho \triangleq \{x \in \mathbb{R}^3 \mid \rho(x, \Sigma_0) = \rho\}$$

$$ds_0^2 = d\rho^2 + g_{ij}(\rho, \theta)\, d\theta^i d\theta^j$$

$g_{ij}(\rho, \theta)\, d\theta^i d\theta^j$ is the restriction metric on Σ_ρ.

2. Find a smooth function $u: X \longrightarrow \mathbb{R}$ s.t. the new metric

$$ds^2 = u^2 d\rho^2 + g_{ij}(\rho,\theta)\, d\theta^i d\theta^j$$ is.

- AF
- $R(ds^2) = 0$.
- The mean curvature of Σ w.r.t. ds^2 and outward unit normal vector is H.

· **Key observation:** The function:

$$m(\rho) = \int_{\Sigma_\rho} H_0 \cdot (1 - u^{-1}) \, d\sigma_\rho .$$

is non increasing in ρ, where H_0 is the mean curvature of Σ_ρ w.r.t. Euclidean metric.

RK: Brown–York mass is non increasing along QS metric !!

- Question: what is the relation between $\int H$ and $4\pi^{\frac{1}{2}}Area^{\frac{1}{2}}(\partial\Omega)$?

- Answer:

1. If the scalar curvature $R \geq 0$, Σ is an isoperimetric surface of the AF manifold, then we have: $\int H \leq 4\pi^{\frac{1}{2}}Area^{\frac{1}{2}}(\partial\Omega)$

2. (Miao 2009) If the scalar curvature $R \geq 0$, Σ is a metrically round sphere, then we have: $\int H \leq 4\pi^{\frac{1}{2}}Area^{\frac{1}{2}}(\partial\Omega)$

3. Question: What happen if the equalities holds in above inequalities ? Unknown!

- What can we say on $M(\partial\Omega) = 4\pi^{\frac{1}{2}} Area^{\frac{1}{2}}(\partial\Omega) - \int H$?

- Brendle, Hung, Wang's work(2016) on Ads-Schwarzchild manifold

- Wei Yong's recent work (2017) on Schwarzchild manifold

2 YONG WEI

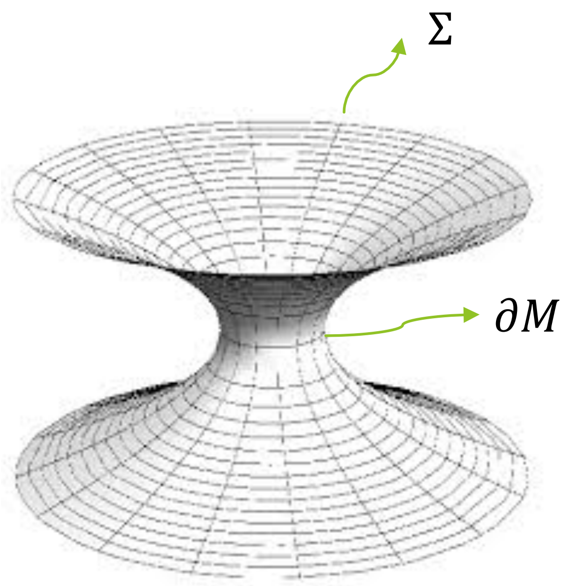
Theorem 1.1. *Let Ω be a bounded domain with smooth outward minimizing boundary in the Schwarzschild space (M^n, g). Assume either*

(1) $n < 8$, *or*
(2) $n \geq 8$, $\Sigma = \partial\Omega \setminus \partial M$ *is homologous to the horizon and has* $H > 0$.

Then

$$\frac{1}{(n-1)\omega_{n-1}} \int_{\Sigma} f H d\mu \geq \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} - 2m, \quad (1.5)$$

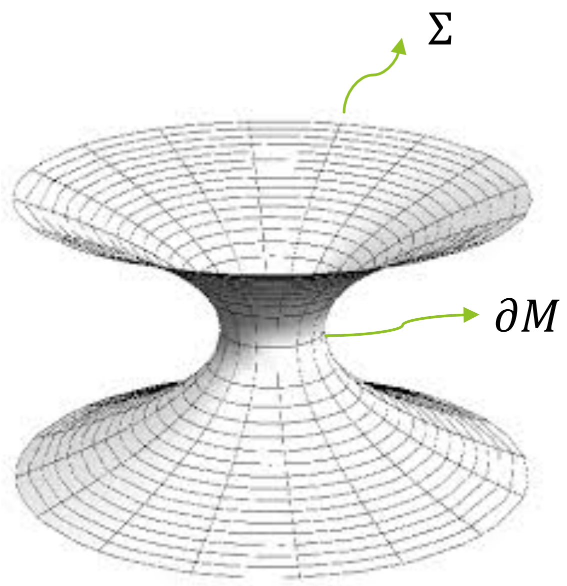
where ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $|\Sigma|$ is the area of Σ with respect to the induced metric. Moreover, equality holds in (1.5) if and only if Σ is a slice $\{s\} \times \mathbb{S}^{n-1}$.



- Remark: A domain in the Schwarzchild (Static?) manifold always has the least quasi-local mass under certain boundary conditions.

- Conjecture: For an AF manifold with nonnegative scalar curvature, we have

$$M_{ADM}(M,g) \geq \left(\frac{Area(\Sigma)}{16\pi}\right)^{\frac{1}{2}} - \frac{1}{16\pi}\int_{\Sigma} fH\,d\mu$$



- f is the harmonic function with zero on ∂M and 1 at
- the infinity of the manifold.

Thm (Christodoulou, Yau, 1988): (M^3, g) is AF with $R \geq 0$.

If Σ is "round" surface then

$$m_H(\Sigma) \geq 0.$$

here Σ is round ~~means~~ if. 1) Σ is topological sphere.

2) has least area in M among all surfaces in M which enclose the same volume as Σ does.

Positivity of isoperimetric mass——A heuristic proof:

$$\mathcal{M}_H(\Sigma) \geq 0 \Longrightarrow H^2 A(\Sigma) \leq 16\pi \Longrightarrow A'(v) A^{\frac{1}{2}}(v) \leq 4\pi^{\frac{1}{2}}$$

$$\Longrightarrow (A^{\frac{3}{2}}(v))' \leq 6\pi^{\frac{1}{2}} \Longrightarrow v - (6\pi^{\frac{1}{2}})^{-1} A^{\frac{3}{2}}(v) \geq 0$$

$$\Longrightarrow \mathcal{M}_{ISO}(\Sigma) = \frac{2}{|\Sigma|}(Vol(\Omega) - \frac{|\Sigma|^{\frac{3}{2}}}{6\sqrt{\pi}}) \geq 0$$

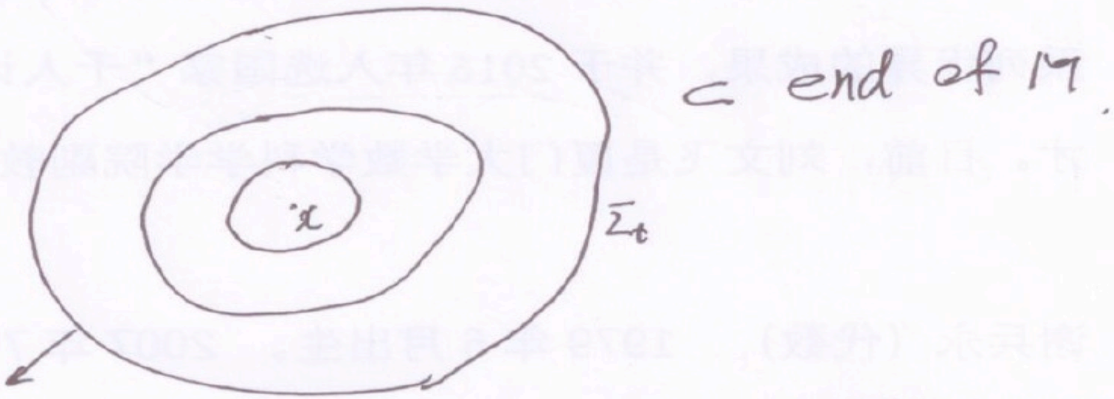
Non negativity of isoperimetric mass

- Theorem (Shi, 2016): Suppose (M^3, g) is an AF manifold with $R \geq 0$, then we have

$$A^{\frac{3}{2}}(v) \leq (6\pi^{\frac{1}{2}}) \int_0^v (1 - (16\pi)^{\frac{1}{2}} B^{-\frac{1}{2}}(t) m(t) dt \leq (6\pi^{\frac{1}{2}}) v$$

$$\Longrightarrow \qquad \mathcal{M}_{ISO}(v) = \frac{2}{A(v)}\left(v - \frac{A(v)^{\frac{3}{2}}}{6\sqrt{\pi}}\right) \geq 0$$

Moreover. $\mathcal{M}_{ISO}(v) = \frac{2}{A(v)}(v - \frac{A(v)^{\frac{3}{2}}}{6\sqrt{\pi}}) = 0$ if and only if $(M^3, g) = R^3$



Slice of IMCF. $\Sigma_t = \partial K_t$. $Vol(K_t) = V$.

$m(V) \triangleq m_H(\Sigma_t)$. $B(t) \triangleq Area(\Sigma_t)$.

$A^{\frac{3}{2}}(\gamma) = (6\pi^{\frac{1}{2}})V_0$ for some $V_0 > 0 \Longrightarrow m(V) = 0 \quad \forall V \in (0, V_0)$.

$\Longrightarrow |Ric|(x) = 0$ for all x near the infinity of (M^3, g)

$\Longrightarrow m_{ADM}(M^3, g) = 0. \Longrightarrow (M^3, g) = \mathbb{R}^3$.

Isoperimetric mass and ADM mass

- Theorem (Huisken, 2006): Suppose (M^3, g) is an AF manifold with $R \geq 0$, let $A(v)$ be the isoperimetric profile on (M^3, g) , then

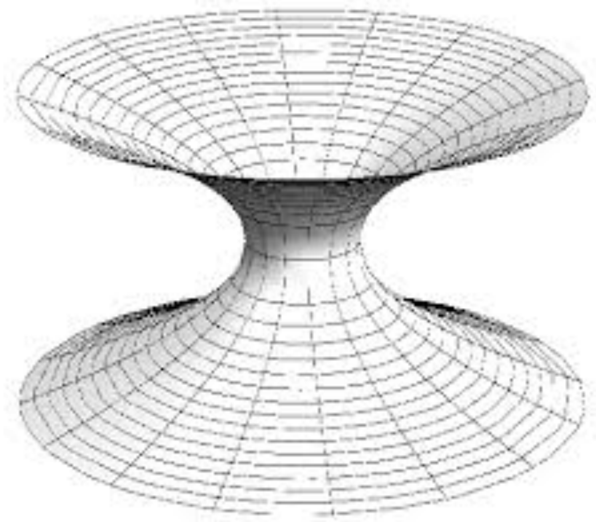
$$\lim_{v \to \infty} \mathcal{M}_{ISO}(v) = \mathcal{M}_{ADM}(M^3, g)$$

$$\mathcal{M}_{ISO}(v) = \frac{2}{A(v)}(v - \frac{A(v)^{\frac{3}{2}}}{6\sqrt{\pi}})$$

- Fan-Miao-Shi-Tam's result

$$\mathcal{M}_{ISO}(\Sigma_r) = \frac{2}{|\Sigma_r|}(Vol(\Omega_r) - \frac{|\Sigma_r|^{\frac{3}{2}}}{6\sqrt{\pi}}) \to \mathcal{M}_{ADM}(M, g)$$

$$\Longrightarrow \lim_{v\to\infty} \mathcal{M}_{ISO}(v) \geq \mathcal{M}_{ADM}(M, g)$$



- On other hand, we have

$$\mathcal{M}_H(v) = \frac{|\Sigma_v|^{\frac{1}{2}}}{16\pi^{\frac{3}{2}}}\left(16\pi - \int_{\Sigma_v} H^2\right) \leq \mathcal{M}_{ADM}(M, g)$$

$$A(v) = |\Sigma_v|$$

$$H \leq A'_-(v)$$

$$\Longrightarrow$$

$$16\pi A^{\frac{1}{2}}(v) - (A'_-(v))^2(v)A^{\frac{3}{2}}(v) \leq 16\pi^{\frac{3}{2}}\mathcal{M}_{ADM}$$

$$\Longrightarrow \quad \mathcal{M}_{ISO}(v) \leq \mathcal{M}_{ADM}, \qquad v >> 1 \quad \Longrightarrow$$

$$\mathcal{M}_{ISO}(v) \to \mathcal{M}_{ADM}, \qquad v \to \infty$$

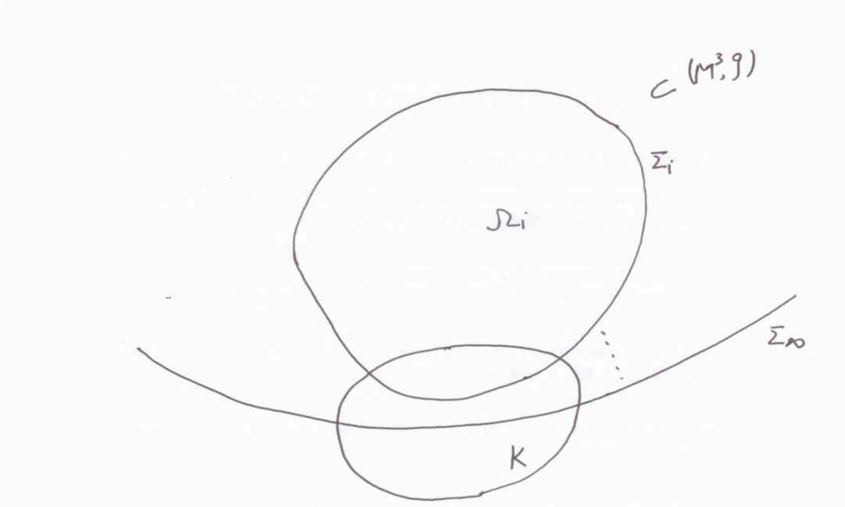
No drift off to infinity of isoperimetric regions

- Let $\{\Omega_i\}$ be a sequence of isoperimetric regions in M, we say Ω_i drift off to infinity, if for any compact set $\mathbf{D} \subset M$, $\{\Omega_i\}$ is disjoint with $\mathbf{D}$, for sufficiently large i.

Theorem(CE 2016). Isoperimetric regions with unbounded enclosed volume cannot pass through a fixed compact set

- Suppose $\{\Sigma_i\}$ always pass through a fixed compact set $\mathbf{K}$
- Σ_∞ is a complete, noncompact and properly embedding area-minimizing Surface in (M^3, g).
- A result due to , O.Chodosh, M.Eichmair

$\Longrightarrow$ No such area-minimizing Surface in (M^3, g)



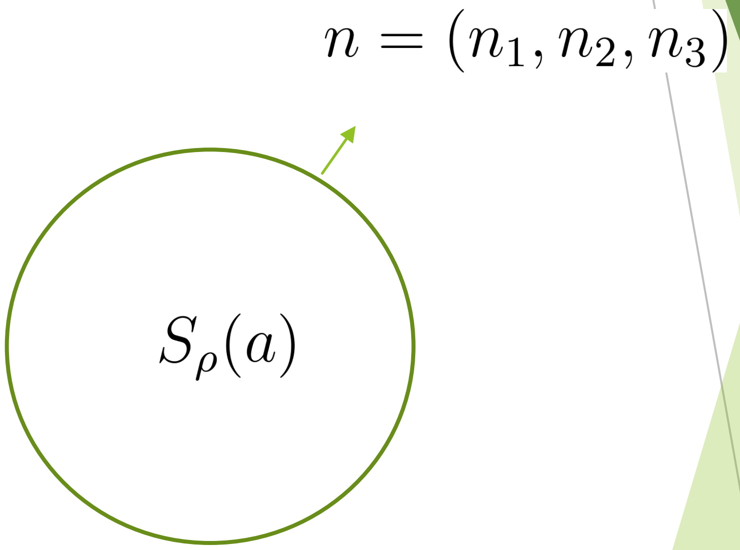
- By a result due to CE, $\{\Omega_i\}$ cannot always pass through a fixed compact set $\mathbf{D}$.
- $\{\Omega_i\}$ cannot drift off to the infinity
- $\Longrightarrow$ $\{\Omega_i\}$ is exhausting .

Proof of no drift

- Key observation: If $\{\Omega_i\}$ drift off to the infinity, then $\mathcal{M}_{ISO}(\Omega_i)$ tends to zero, contradiction to $\mathcal{M}_{ISO}(\Omega_i) \geq \mathcal{M}_{ADM}(M, g) > 0$.
- How to estimate $\mathcal{M}_{ISO}(\Omega_i)$?
- Key observation: If a Ω_i drifts off to the infinity, then the boundary looks like an Euclidean sphere

- What happen if a Euclidean sphere slides off to the infinity of an AF manifold ?
- We can get its area and volume expansion in an explicit way.

$$n^j(x) = \frac{x^j - a^j}{|x - a|}.$$



Proposition 4.2. *Let* $S_\rho(a) = \{x \in \mathbb{R}^3 : |x-a| = \rho\}$ *be Euclidean coordinate spheres with* $\rho \to \infty$ *and* $|a| - \rho \to \infty$. *Then*

$$\text{area}(S_\rho(a)) = 4\pi\rho^2 + \frac{1}{2}\int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho) \tag{14}$$

and

$$\text{vol}(B_\rho(a)) = \frac{4\pi\rho^3}{3} + \frac{\rho}{4}\int_{S_\rho(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + o(\rho^2). \tag{15}$$

$$\Longrightarrow \quad \mathcal{M}_{ISO}(S_\rho(a)) = o(1)$$

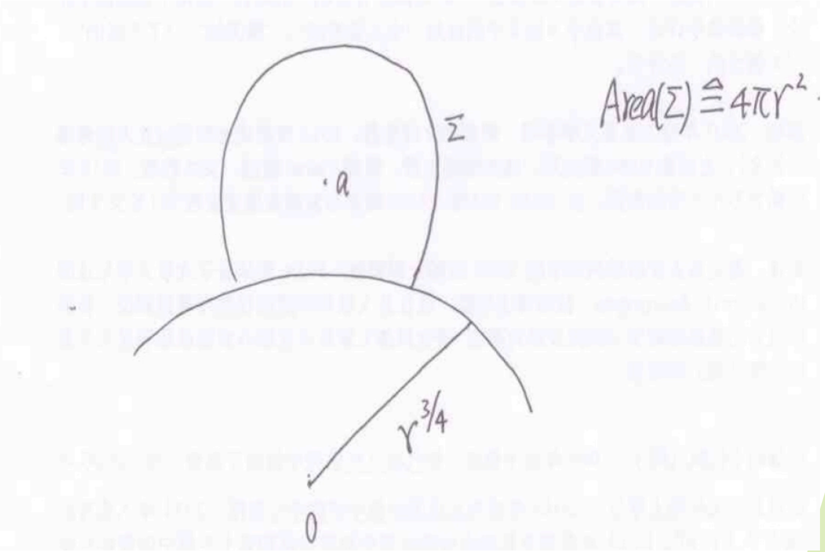
Estimate for isoperimetric surfaces

- Observation: Isoperimetric surfaces in AF manifolds look like Euclidean spheres outside a large compact set.

- $$0 \leq \mathcal{M}_H(\Sigma) = \frac{|\Sigma|^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}}\left(16\pi - \int_\Sigma H^2\right) \leq m$$

$\Rightarrow$

$$\frac{2}{r} - \frac{8m}{r^2} + O(r^{-3}) \leq H \leq \frac{2}{r}$$



- ⟹ If $\sum$ is an isoperimetric surface with the topology of sphere and drift off to the infinity, then $\mathcal{M}_{ISO}(\Sigma) = o(1)$
- Huisken: $M_{ISO}(\Sigma) \to M_{ADM} > 0$
- Conclusion: Suppose (M^3, g) is an AF manifold with $R \geq 0$, $\{\Sigma_i\}$ is a sequence of isoperimetric surfaces with the topology of spheres, and areas approach to infinity, then
- $\{\Sigma_i\}$ cannot drift off to the infinity.

⟹ $\{\Sigma_i\}$ exhausting

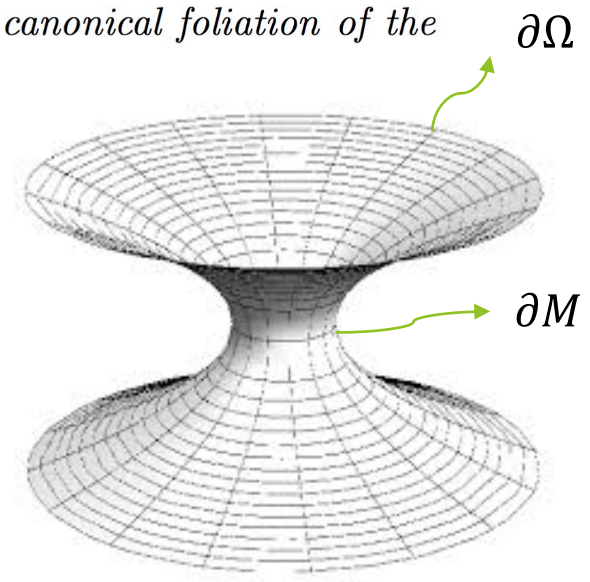
Canonical foliations in AF manifolds and isoperimetric surfaces with large enclosed volume

Theorem 1.1. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat at rate $\tau > 1/2$ and which has non-negative scalar curvature and positive mass. There is $V_0 > 0$ with the following property. Let $V \geq V_0$. There is a unique region $\Omega_V \in \mathcal{R}_V$ such that*

$$\text{area}(\partial\Omega_V) \leq \text{area}(\partial\Omega)$$

for all $\Omega \in \mathcal{R}_V$. The boundary of Ω_V consists of ∂M and a leaf of the canonical foliation of the end of M.

$$\mathcal{R}_V = \{\Omega : \Omega \subset M \text{ is a compact region with } \partial M \subset \partial\Omega \text{ and } \text{vol}(\Omega) = V\}$$



- Example(A. Carlotto and R.Schoen) There is an AF Riemannian metric $g = g_{ij}dx_i dx_j$ on $\mathbf{R}^3$ that has non-negative scalar curvature and positive mass such that $g_{ij} = \delta_{ij}$ on $\mathbf{R}^2 \times (0, \infty)$

- There is no uniqueness for stable CMC surfaces in an AF manifold with asymptotical order $\tau < 1$